MA322-007 Mar 24 Quiz Name: Key

(5.1a) What are the eigenvalues of $A = \begin{bmatrix} 2 & 0 \\ 0 & \sqrt{7} \end{bmatrix}$? $2, \sqrt{7}$

$$\underbrace{\begin{bmatrix} 2 & 0 \\ 0 & \sqrt{7} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}}_{\begin{bmatrix} 2x \\ \sqrt{7}y \end{bmatrix}} = \underbrace{c \begin{bmatrix} x \\ y \end{bmatrix}}_{\begin{bmatrix} cx \\ cy \end{bmatrix}}$$

Solve for c...

$\left\{ \begin{array}{l} 2x = cx \\ \sqrt{7}y = cy \end{array} \right\}$ has 2 solutions

① $c = 2, y = 0, x$ free

② $c = \sqrt{7}, x = 0, y$ free

(5.1b) What are the eigenvectors of $A = \begin{bmatrix} 2 & 0 \\ 0 & \sqrt{7} \end{bmatrix}$?

For $c = 2$, $\begin{bmatrix} x \\ 0 \end{bmatrix}$ for any $x \neq 0$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ for example

For $c = \sqrt{7}$, $\begin{bmatrix} 0 \\ y \end{bmatrix}$ for any $y \neq 0$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ for example

(5.1c) Give a different matrix with the same eigenvalues:

Only easy ones are $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax \\ by \end{bmatrix}$ has ① $c = a, y = 0, x$ free ② $c = b, x = 0, y$ free

but we want $\{a, b\} = \{2, \sqrt{7}\}$

only other option is $\begin{bmatrix} \sqrt{7} & 0 \\ 0 & 2 \end{bmatrix}$

(5.1d) Give a different matrix with the same eigenvectors:

Only easy ones are $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ and they <u>all</u> work

(well $a \neq b$ if we don't want any extra eigenvectors)

(5.1e) What are the eigenvalues of $A^3 + 3A + 5I$?

$$\begin{aligned} A^3 + 3A + 5I &= \begin{bmatrix} 2 & 0 \\ 0 & \sqrt{7} \end{bmatrix}^3 + 3 \begin{bmatrix} 2 & 0 \\ 0 & \sqrt{7} \end{bmatrix} + 5 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2^3 & 0 \\ 0 & \sqrt{7}^3 \end{bmatrix} + \begin{bmatrix} 3(2) & 0 \\ 0 & 3(\sqrt{7}) \end{bmatrix} + \begin{bmatrix} 5(1) & 0 \\ 0 & 5(1) \end{bmatrix} \\ &= \begin{bmatrix} 2^3 + 3(2) + 5(2^0) & 0 \\ 0 & \sqrt{7}^3 + 3(\sqrt{7}) + 5(\sqrt{7}^0) \end{bmatrix} \end{aligned}$$

MA322-007 Mar 24 Worksheet - Eigenvalues and eigenvectors

An **eigenpair** of a matrix A is a vector $\vec{v}$ and a number c so that $A\vec{v} = c\vec{v}$. The vector $\vec{v}$ is called the **eigenvector** and the number c is called the **eigenvalue**.

Note that only square matrices can have eigenvectors. Why?

If $A\vec{x} = \vec{b}$ then $\vec{x}$ has one entry per column of A
and $\vec{b}$ has one entry per row of A
if $\vec{b} = c\vec{x}$, then they have the same number of entries

Eigenvectors are very handy: they change matrices into numbers.

Let's consider $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

What is $A\vec{v}$ in terms of $\vec{v}$?

$$\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ -2 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\vec{v}$$

What is $A\vec{w}$ in terms of $\vec{w}$?

$$\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1\begin{bmatrix} 1 \\ -2 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3\vec{w}$$

What is $A(5\vec{v} + \sqrt{7}\vec{w})$ in terms of $\vec{v}$ and $\vec{w}$?

$$= A(5\vec{v}) + A(\sqrt{7}\vec{w})$$
$$= 5A(\vec{v}) + \sqrt{7}A(\vec{w}) = 5(2\vec{v}) + \sqrt{7}(3\vec{w})$$

What is $A(x\vec{v} + y\vec{w})$? Call it $b\vec{v} + c\vec{w}$.

$$2x\vec{v} + 2y\vec{w}, \quad b = 2x, \quad c = 3y$$

What matrix takes $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ to $\vec{b} = \begin{bmatrix} b \\ c \end{bmatrix}$?

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 3y \end{bmatrix} = \begin{bmatrix} b \\ c \end{bmatrix}, \text{ so } \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \text{ works.}$$

$$\left(A \text{ "="} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \text{ if we use } \vec{i} = \vec{v} \text{ and } \vec{j} = \vec{w} \right)$$