MA322-007 Mar 26 Quiz Name:____________________

(5.1a) Is $c = 2$ an eigenvalue of $A = \begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}$? Why or why not?

Solve $A\vec{v} = 2\vec{v}$

$$\begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{Bmatrix} 3x + 2y = 2x \\ 3x + 8y = 2y \end{Bmatrix} \xrightarrow[\text{to LHS}]{\text{Move RHS}} \begin{Bmatrix} 1x + 2y = 0 \\ 3x + 6y = 0 \end{Bmatrix} \to \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 3 & 6 & 0 \end{array}\right]$$

$\uparrow R_2 - 3R_1$

$$\begin{Bmatrix} x = -2y \\ y = 1 \text{ say} \end{Bmatrix} \leftarrow \begin{Bmatrix} x + 2y = 0 \\ y \text{ is free} \end{Bmatrix} \leftarrow \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\hookrightarrow \left(2, \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right)$ is e-pair

(5.1b) Is $\vec{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ an eigenvector of $B = \begin{bmatrix} 1 & -1 \\ 6 & -4 \end{bmatrix}$? What about $\vec{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$?

Solve $B\vec{v} = c\vec{v}$

$$\begin{bmatrix} 1 & -1 \\ 6 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = c\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1-3 \\ 6-12 \end{bmatrix} = \begin{bmatrix} c \\ 3c \end{bmatrix}$$

so $\begin{Bmatrix} c = -2 \\ c = -2 \end{Bmatrix}$

when $c = -2$ works for both rows, so $\left(-2, \begin{bmatrix} 1 \\ 3 \end{bmatrix}\right)$ is an eigenpair.

$\left(-1, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$ also works

(5.1c) Find the eigenvalues of the matrix $C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & -2 \end{bmatrix}$

Already nearly in RREF

$$C - xI \Rightarrow \left[\begin{array}{ccc|c} -x & 0 & 0 & 0 \\ 0 & 3-x & 4 & 0 \\ 0 & 0 & -2-x & 0 \end{array}\right]$$ has free variables if

$x = 0$

$x = 3$

or $x = -2$

(MA214) Solve the system of differntial equations $y' = y + 6z$ and $z' = -y - 4z$. Hint: Let $v = y + 3z$. Then $v' = y' + 3z' = (y + 6z) + 3(-y - 4z) = -2y - 6z = -2v$. What about $w = y + 2z$? Since $z = v - w$, can you find y and z?

$$\begin{aligned} w' &= y' + 2z' \\ &= (y+6z) + 2(-y-4z) \\ &= -y - 2z \\ &= -w \end{aligned}$$

$w = e^{-t} \cdot w(0)$

$v = e^{-2t} \cdot v(0)$

$z = v - w = v_0 e^{-2t} - w_0 e^{-t}$

$y = 3w - 2v = 3w_0 e^{-t} - 2v_0 e^{-2t}$

Bonus: Rewrite using y_0, z_0

Solve $\left(x, \begin{bmatrix} a \\ b \end{bmatrix}\right)$

Fibonacci via eigenvalues

Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. What are its eigenpairs?

$$\begin{array}{cc} a & b \end{array}$$

$$\left[\begin{array}{cc|c} -x & 1 & 0 \\ 1 & 1-x & 0 \end{array}\right] \xrightarrow{R_1 + xR_2} \left[\begin{array}{cc|c} 0 & x - x^2 + 1 & 0 \\ 1 & 1-x & 0 \end{array}\right]$$

so $x = \frac{1 \pm \sqrt{5}}{2}$ to get b is free

$a = (x-1) \cdot b$ take $b = 1$

$\left(\frac{1+\sqrt{5}}{2}, \begin{bmatrix} (-1+\sqrt{5})/2 \\ 1 \end{bmatrix}\right)$ $\vec{v}_1$

$\left(\frac{1-\sqrt{5}}{2}, \begin{bmatrix} \frac{-1-\sqrt{5}}{2} \\ 1 \end{bmatrix}\right)$ $\vec{v}_2$

Write the vector $\vec{\mathbf{e}}_2$ as a linear combination of your eigenvectors.
(Hint: it is probably $\pm(\vec{\mathbf{v}}_1 - \vec{\mathbf{v}}_2)/\sqrt{5}$.)

$$\frac{\vec{v}_1 - \vec{v}_2}{\sqrt{5}} = \begin{bmatrix} -\frac{1}{2} - -\frac{1}{2} + \frac{\sqrt{5}}{2} - -\frac{\sqrt{5}}{2} \\ 1-1 \end{bmatrix} = \begin{bmatrix} \sqrt{5} \\ 0 \end{bmatrix} \frac{1}{\sqrt{5}} = \vec{e}_1$$

Consider the sequence $0, 1, 1, 2, 3, 5, 8, 13, \ldots, x_k, \ldots$ with $x_{k+2} = x_{k+1} + x_k$. What is $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix}$?

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix} = \begin{bmatrix} x_{k+1} \\ x_k + x_{k+1} \end{bmatrix} = \begin{bmatrix} x_{k+1} \\ x_{k+2} \end{bmatrix}$$ (the "next" one)

Suppose we start with $\vec{\mathbf{e}}_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. What do you get from $A^k \vec{\mathbf{e}}_2$?

$$A\vec{e}_2 = \begin{bmatrix} x_2 \\ x_3 \end{bmatrix}, \quad A^2\vec{e}_2 = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}, \quad A^k e_2 = \begin{bmatrix} x_{k+1} \\ x_{k+2} \end{bmatrix}$$ same

Suppose we start with a linear combination of your eigenvectors $a_1\vec{\mathbf{v}}_1 + a_2\vec{\mathbf{v}}_2$. What do we get from $A^k(a_1\vec{\mathbf{v}}_1 + a_2\vec{\mathbf{v}}_2)$?

$$a_1 \left(\frac{1+\sqrt{5}}{2}\right)^k \vec{v}_1 + a_2 \left(\frac{1-\sqrt{5}}{2}\right)^k \vec{v}_2$$

One of your eigenvalues should be about 0.6. What is $(0.6)^{35}$? Zippo, or nearly 0.

For me, $\vec{v}_1 + \vec{v}_2 = \begin{bmatrix} \frac{-1+\sqrt{5}}{2} + \frac{-1-\sqrt{5}}{2} \\ 1 + 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

So $\dfrac{\vec{v}_1 + \vec{v}_2 + \frac{\vec{v}_1 - \vec{v}_2}{\sqrt{5}}}{2} = \vec{e}_2$, $\frac{1+\sqrt{5}}{2\sqrt{5}} \vec{v}_1 + \frac{1-\sqrt{5}}{2\sqrt{5}} \vec{v}_2$

Approximately what is x_{35}? Can you find it exactly, knowing it has to be an integer?

Take $a_1 = \frac{1+\sqrt{5}}{2\sqrt{5}}$, $a_2 = \frac{1-\sqrt{5}}{2\sqrt{5}}$

$k = 33$

then $A^{33}(a_1\vec{v}_1 + a_2\vec{v}_2) = A^{33}\vec{e}_2 = \begin{bmatrix} x_{34} \\ x_{35} \end{bmatrix}$

but $\approx a_1 \left(\frac{1+\sqrt{5}}{2}\right)^{33} \vec{v}_1 \approx \left[\frac{(1+\sqrt{5})}{2\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^{33} \right.$

So $x_{35} \approx 9227465$

MA322-007 Mar 26 Worksheet - 5.2 Finding eigenvalues

Finding eigenvectors given an eigenvalue is just solving $A\vec{x} = c\vec{x}$. This is the same as solving $(A - cI)\vec{x} = \vec{0}$, a null space calculation.

Find the eigenvector of $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ associated to the eigenvalue $c = 2$.

Solve $A\vec{x} = 2\vec{x}$ $\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$ $\left(2, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

$\left\{ \begin{array}{l} x + y = 2x \\ 2y = 2y \end{array} \right\}$

(other one is $\left(1, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$) $\left\{ \begin{array}{l} -x + y = 0 \\ 0 = 0 \end{array} \right\} \to \left\{ \begin{array}{l} x = y \\ y \text{ is free} \end{array} \right\}$

Finding eigenvalues given only the matrix is hard in general. However, for up to 4×4 we can use quadratic (or cubic, or quartic) formulas to find the eigenvalues. The **characteristic polynomial** of a matrix is defined to be

$$\det(A - xI)$$

and is an nth degree polynomial if A is $n \times n$. The zeros of the characteristic polynomial are exactly the eigenvalues of A. In particular, if $n = 2$, then we just use quadratic formula to find the eigenvalues.

Find the egienvalues of $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$.

$$\det\left(\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} - x\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$
$$= \det\begin{pmatrix} 1-x & 1 \\ 0 & 2-x \end{pmatrix}$$
$$= (1-x)(2-x) - 0$$
$$= x^2 - 3x + 2 - 0 = (x-1)(x-2)$$

has roots $x = 1$, $x = 2$

Why does the characteristic polynomial trick work?

It figures exactly when there is a free variable

Reality note: If n is larger than about 10, then root finding algorithms are not very well suited. For the same cost as calculating the polynomial (about n^3 operations), we can get a fairly accurate estimate not only of the eigenvalue (nearest to any chosen number) but also the eigenvector that goes with it. See page 322 in section 5.8 for the easiest and fairly effective method (inverse iteration). In fact, for the same cost, we can find good approximations to the eigenvalues closest to any set of n numbers, and the approximations can be improved very quickly (QR iteration).

MA322-007 Mar 26 Worksheet - Diagonal matrices (like 5.3)

A **diagonal matrix** can be defined as a matrix with eigenvectors $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$. The corresponding eigenvalues $c_1, c_2, \ldots, c_n$ are called its diagonal entries. For example, the diagonal matrix with diagonal entries $4, 7, 13$ is the matrix

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 13 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4(0)+0(1)+0(0) \\ 0(0)+7(1)+0(0) \\ 0(0)+0(1)+13(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \\ 0 \end{bmatrix}$$

What is $A\vec{e}_2$ in terms of $\vec{e}_2$? $7\vec{e}_2$

Arithmetic with diagonal matrices is a little easier than with general matrices. What is A^2?

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 13 \end{bmatrix} = \begin{bmatrix} 4(4)+0(0)+0(0) & 4(0)+0(7)+0(0) & \ldots \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \end{bmatrix} = \begin{bmatrix} 4^2 & 0 & 0 \\ 0 & 7^2 & 0 \\ 0 & 0 & 13^2 \end{bmatrix}$$

What is A^3? $\begin{bmatrix} 4^3 & 0 & 0 \\ 0 & 7^3 & 0 \\ 0 & 0 & 13^3 \end{bmatrix}$

What is A^{10}? (No need to simplify.) $\begin{bmatrix} 4^{10} & 0 & 0 \\ 0 & 7^{10} & 0 \\ 0 & 0 & 13^{10} \end{bmatrix}$

What is $A^5 + 11A^2 + 17A + 5I$? (You don't need to simplify.)

$$\begin{bmatrix} 4^5 + 11\,4^2 + 17(4) + 5 & 0 & 0 \\ 0 & 7^5 + 11(7)^2 + 17(7) + 5 & 0 \\ 0 & 0 & 13^5 + 11(13)^2 + 17(13) + 5 \end{bmatrix}$$

If you know Taylor series or are good with patterns, what do you think $\exp(A)$ is?

$$\exp(A) = I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \cdots \qquad \begin{bmatrix} 1+4+\frac{1}{2}4^2+\cdots & 0 & 0 \\ 0 & \ldots & 0 \\ 0 & 0 & \ldots \end{bmatrix} = \begin{bmatrix} e^4 & 0 & 0 \\ 0 & e^7 & 0 \\ 0 & 0 & e^{13} \end{bmatrix}$$

Let B be the diagonal matrix with diagonal entries $\sqrt{2}, \sqrt{3}, \sqrt{5}$. What is AB?

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 13 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{5} \end{bmatrix} = \begin{bmatrix} 4\sqrt{2} & 0 & 0 \\ 0 & 7\sqrt{3} & 0 \\ 0 & 0 & 13\sqrt{5} \end{bmatrix}$$

What is BA?

Ditto

What is $A + B$?

$$\begin{bmatrix} 4+\sqrt{2} & 0 & 0 \\ 0 & 7+\sqrt{3} & 0 \\ 0 & 0 & 13+\sqrt{5} \end{bmatrix}$$