MA322-007 Apr 6 Quiz Name:________________

6.1a Find a vector that is orthogonal to $\vec{\mathbf{v}} = (3,4)$. Can you find one that is unit length?

Old Way: $\vec{u} = (x,y)$ $\quad \vec{v}\cdot\vec{u} = 3x+4y$ set it equal to 0

$[3\ 4]\begin{bmatrix} x \\ y \end{bmatrix} = [0]$ $\quad \left[\begin{array}{cc|c} 3 & 4 & 0 \end{array}\right] \xrightarrow{R_1/3} \left[\begin{array}{cc|c} 1 & 4/3 & 0 \end{array}\right]$ $\quad x = -4/3\, y$, y is free

$\vec{u} = \left(-\frac{4}{3}y, y\right)$ Solve $1 = \|\vec{u}\| = \sqrt{(-\frac{4}{3}y)^2 + y^2} = \sqrt{(\frac{16}{9}+1)y^2} = \sqrt{\frac{25}{9}}|y| = \frac{5}{3}|y|$

so $y = \pm\frac{3}{5}$ $\quad \vec{u} = \pm\left(\frac{-4}{5}, \frac{3}{5}\right)$

New Way: $\vec{u} = (1,1)$'s correctness is undeniable. By undeniable, I mean completely deniable.

$\vec{u}\cdot\vec{v} = (1,1)\cdot(3,4) = 3+4 = 7$

$\vec{v}\cdot\vec{v} = (3,4)\cdot(3,4) = 9+16 = 25$.

Step1: Remove $\vec{u}$ $\quad$ but take $\vec{w} = \vec{u} - \frac{7}{25}\vec{v}$ and see what happens. (its twice as good as $\vec{u}$)

$\vec{w}\cdot\vec{v} = (\vec{u} - \frac{7}{25}\vec{v})\cdot\vec{v} = \vec{u}\cdot\vec{v} - \frac{7}{25}\vec{v}\cdot\vec{v} = 7 - \frac{7}{25}25 = 0$ ☺

Step2: Fix Length $\quad$ but $\vec{w}$ is too big so take $\vec{x} = \frac{\vec{w}}{\|\vec{w}\|}$ then $\|\vec{x}\| = \frac{\|\vec{w}\|}{\|\vec{w}\|} = 1$ ☺ $\quad \vec{x}\cdot\vec{v} = \frac{1}{\|\vec{w}\|}\vec{w}\cdot\vec{v} = 0$ ☺

6.2a Find a vector that is orthogonal to $\vec{\mathbf{u}} = (4,4,7)$ and $\vec{\mathbf{w}} = (8,-1,-4)$. Can you find one that is unit length?

Old Way $\quad \vec{v} = (x,y,z)$ $\quad \vec{u}\cdot\vec{v} = 4x+4y+7z = 0$, $\quad \vec{w}\cdot\vec{v} = 8x - 1y - 4z = 0$ $\quad \begin{bmatrix} 4 & 4 & 7 \\ 8 & -1 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 4 & 4 & 7 & 0 \\ 8 & -1 & -4 & 0 \end{array}\right] \xrightarrow{R_2-2R_1} \left[\begin{array}{ccc|c} 4 & 4 & 7 & 0 \\ 0 & -9 & -18 & 0 \end{array}\right] \xrightarrow{R_2/-9} \left[\begin{array}{ccc|c} 4 & 4 & 7 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow{R_1/4} \left[\begin{array}{ccc|c} 1 & 1 & 7/4 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow{R_1-R_2} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{4} & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

$x = \frac{1}{4}z$, $y = -2z$, z is free $\quad \vec{v} = \left(\frac{1}{4}z, -2z, z\right)$ $\quad 1 = \|\vec{v}\| = \sqrt{(\frac{1}{4}z)^2 + (-2z)^2 + z^2} = \sqrt{(-\frac{1}{4})^2 + (-2)^2 + 1}\,|z| = \frac{9}{4}|z|$

$z = \pm\frac{4}{9}$

$\vec{v} = \left(\frac{1}{9}, \frac{-8}{9}, \frac{4}{9}\right)$

New Way: $\vec{v} = (1,1,1)$ $\quad \vec{v}\cdot\vec{u} = 4+4+7 = 15$, $\vec{u}\cdot\vec{u} = 16+16+49 = 81$ $\quad \vec{v}\cdot\vec{w} = 8-1-4 = 3$, $\vec{w}\cdot\vec{w} = 64+1+16 = 81$ $\quad \vec{u}\cdot\vec{w} = 32-4-28 = 0$

$\vec{x} = \vec{v} - \frac{15}{81}\vec{u} - \frac{3}{81}\vec{w}$ $\quad$ then $\vec{x}\cdot\vec{u} = \vec{v}\cdot\vec{u} - \frac{15}{81}\vec{u}\cdot\vec{u} - \frac{3}{81}\vec{w}\cdot\vec{u}$

$= 15 - \frac{15}{81}81 - \frac{3}{81}0 = 0$ ☺

$\|x\| = \sqrt{\vec{x}\cdot\vec{x}} = \frac{1}{3}$ $\quad \vec{x}\cdot\vec{w} = \vec{v}\cdot\vec{w} - \frac{15}{81}\vec{u}\cdot\vec{w} - \frac{3}{81}\vec{w}\cdot\vec{w}$

$\vec{y} = 3\vec{x}$ works $\quad = 3 - \frac{15}{81}0 - \frac{3}{81}81 = 0$ ☺

Hint: $\vec{\mathbf{x}} = (1,1,1)$ is wrong, since it points in both the $\vec{\mathbf{u}}$ and the $\vec{\mathbf{w}}$ directions. Could remove the wrongness?

MA322-007 Apr 6 — 6.2 - orthogonal bases

The most interesting things happen when we have a bunch of unit length vectors that are all orthogonal to each other. For example, let's look at this basis of $\mathbb{R}^4$:

$$\vec{\mathbf{a}} = \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix},\quad \vec{\boldsymbol{\ell}} = \frac{1}{2}\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix},\quad \vec{\mathbf{m}} = \frac{1}{2}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix},\quad \vec{\mathbf{h}} = \frac{1}{2}\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}$$

If $A = \begin{bmatrix}\uparrow & \uparrow & \uparrow & \uparrow\\ a & \ell & m & h\\ \downarrow & \downarrow & \downarrow & \downarrow\end{bmatrix}$ then $A^TA = I$

so $\|a\| = \|\ell\| = \|m\| = \|h\| = 1$

$\vec a\cdot\vec\ell = \vec a\cdot\vec m = \vec a\cdot\vec h = \vec\ell\cdot\vec m = \vec\ell\cdot\vec h = \vec m\cdot\vec h = 0$

Orthonormal

Can you write $\vec{\mathbf{v}} = \begin{bmatrix}3\\4\\5\\6\end{bmatrix}$ as a linear combination of $\vec{\mathbf{a}}$, $\vec{\boldsymbol{\ell}}$, $\vec{\mathbf{m}}$, $\vec{\mathbf{h}}$?

If $\vec v = x\cdot\vec a + y\cdot\vec\ell + z\cdot\vec m + t\cdot\vec h$

then $\vec v\cdot\vec a = (x\vec a + y\vec\ell + z\vec m + t\vec h)\cdot\vec a = x(\vec a\cdot\vec a) + y(\vec\ell\cdot\vec a) + z(\vec m\cdot\vec a) + t(\vec h\cdot\vec a)$

$= x(1) + y(0) + z(0) + t(0)$

$= x$

so $x = \vec v\cdot\vec a = \frac12(3+4+5+6) = 9$

$y = \vec v\cdot\vec\ell = \frac12(3+4-5-6) = -2$

$z = \vec v\cdot\vec m = \frac12(3-4-5+6) = 0$

$t = \vec v\cdot\vec h = \frac12(3-4+5-6) = -1$

so $\boxed{\vec v = 9\vec a - 2\vec\ell - \vec h}$

What if you weren't allowed to use $\vec{\mathbf{h}}$? What is the closest you could get to being a linear combination of $\vec{\mathbf{a}}$, $\vec{\boldsymbol{\ell}}$, $\vec{\mathbf{m}}$?

$\text{dist}(x\vec a + y\vec\ell + z\vec m, \vec v) = \text{dist}(x\vec a + y\vec\ell + z\vec m, 9\vec a - 2\vec\ell - \vec h) = \|(x-9)\vec a + (y+2)\vec\ell + z\vec m + \vec h\|$

but $\|b\vec a + c\vec\ell + d\vec m + e\vec h\| = \sqrt{(b\vec a + c\vec\ell + d\vec m + e\vec h)\cdot(b\vec a + c\vec\ell + d\vec m + e\vec h)}$

$$= \sqrt{\begin{array}{llll} b^2\vec a\cdot\vec a & + bc\,\vec a\cdot\vec\ell & + bd\,\vec a\cdot\vec m & + be\,\vec a\cdot\vec h\\ + cb\,\vec\ell\cdot\vec a & + c^2\,\vec\ell\cdot\vec\ell & + cd\,\vec\ell\cdot\vec m & + ce\,\vec\ell\cdot\vec h\\ + db\,\vec m\cdot\vec a & + dc\,\vec m\cdot\vec\ell & + d^2\,\vec m\cdot\vec m & + de\,\vec m\cdot\vec h\\ + eb\,\vec h\cdot\vec a & + ec\,\vec h\cdot\vec\ell & + ed\,\vec h\cdot\vec m & + e^2\,\vec h\cdot\vec h\end{array}} = \sqrt{\begin{array}{llll} b^2(1) & +0 & +0 & +0\\ 0 & + c^2 & +0 & +0\\ 0 & +0 & +d^2 & +0\\ 0 & +0 & +0 & +e^2\end{array}}$$

so we want to minimize $\sqrt{(x-9)^2 + (y+2)^2 + z^2 + 1}$

so $x = 9$, $y = -2$, $z = 0$

NO CHANGE!

What if you were only allowed to use $\vec{\mathbf{a}}$? What is the closest you could get to being a linear combination of $\vec{\mathbf{a}}$?

Same deal! $\text{dist}(x\vec a, 9\vec a - 2\vec\ell - \vec h) = \sqrt{(x-9)^2 + (-2)^2 + (0)^2 + (-1)^2}$

best choice is $x = 9 = \vec a\cdot\vec v$ error/dist is $\sqrt{\text{sum of squares of other coeffs}}$

$= \sqrt{(-2)^2 + (0)^2 + (1)^2} = \sqrt5$

Formula version $\dfrac{\vec a\cdot\vec v}{\vec a\cdot\vec a}\vec a$ and $\dfrac{\vec a\cdot\vec v}{\vec a\cdot\vec a}\vec a + \dfrac{\vec\ell\cdot\vec v}{\vec\ell\cdot\vec\ell}\vec\ell + \dfrac{\vec m\cdot\vec v}{\vec m\cdot\vec m}\vec m$ etc

These are called the **projections** onto the span of the vectors you are allowed to use, and they can be computed as in theorem 5, page 339 in the book.

as long as $\vec a\cdot\vec\ell = 0$

$\vec a\cdot\vec m = 0$

and $\vec\ell\cdot\vec m = 0$

HW 6.2 # 1,3,5,7,9