The **projection** of a vector $\vec{\mathbf{v}}$ onto a vector $\vec{\mathbf{w}}$ is the multiple of $\vec{\mathbf{w}}$ that is nearest to $\vec{\mathbf{v}}$.

Calculus interlude: The multiples of $\vec{\mathbf{w}}$ are $t\vec{\mathbf{w}}$, so which value of t is best? Let $f(t) = \|\vec{\mathbf{v}} - t\vec{\mathbf{w}}\|$. Then

$$f(t)^2 = \langle \vec{\mathbf{v}} - t\vec{\mathbf{w}}, \vec{\mathbf{v}} - t\vec{\mathbf{w}} \rangle = \langle \vec{\mathbf{v}}, \vec{\mathbf{v}} \rangle - 2\langle \vec{\mathbf{v}}, \vec{\mathbf{w}} \rangle t + \langle \vec{\mathbf{w}}, \vec{\mathbf{w}} \rangle t^2$$

is quadratic, so its minimum (and the minimum of $f(t)$) occurs at "$-\frac{b}{2a}$", that is at $t = \frac{\vec{\mathbf{v}}\cdot\vec{\mathbf{w}}}{\vec{\mathbf{w}}\cdot\vec{\mathbf{w}}}$.

The **formula** for the projection of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{w}}$ is thus

$$\mathrm{proj}_{\vec{\mathbf{w}}}(\vec{\mathbf{v}}) = \frac{\vec{\mathbf{v}} \cdot \vec{\mathbf{w}}}{\vec{\mathbf{w}} \cdot \vec{\mathbf{w}}} \vec{\mathbf{w}}$$

Example: Define $\vec{\mathbf{g}}_1 = \begin{bmatrix} 4/9 \\ 4/9 \\ 7/9 \end{bmatrix}$, $\vec{\mathbf{g}}_2 = \begin{bmatrix} 1/9 \\ -8/9 \\ 4/9 \end{bmatrix}$, $\vec{\mathbf{g}}_3 = \begin{bmatrix} 8/9 \\ -1/9 \\ -4/9 \end{bmatrix}$ and $\vec{\mathbf{v}} = \begin{bmatrix} 1 \\ 6 \\ 5 \end{bmatrix}$.

By the way, $\vec{\mathbf{g}}_i \cdot \vec{\mathbf{g}}_j = 0$ if $i \neq j$ and $\vec{\mathbf{g}}_i \cdot \vec{\mathbf{g}}_i = 1$.

Find the projection of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{g}}_1$:
$\vec{\mathbf{g}}_1 \cdot \vec{\mathbf{g}}_1 =?$
$\vec{\mathbf{v}} \cdot \vec{\mathbf{g}}_1 =?$

Find the projection of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{g}}_2$:
$\vec{\mathbf{g}}_2 \cdot \vec{\mathbf{g}}_2 =?$
$\vec{\mathbf{v}} \cdot \vec{\mathbf{g}}_2 =?$

Find the projection of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{g}}_3$:
$\vec{\mathbf{g}}_3 \cdot \vec{\mathbf{g}}_3 =?$
$\vec{\mathbf{v}} \cdot \vec{\mathbf{g}}_3 =?$

What happens when you add them up?

Find x_1, x_2, and x_3 such that $\vec{\mathbf{v}} = x_1\vec{\mathbf{g}}_1 + x_2\vec{\mathbf{g}}_2 + x_3\vec{\mathbf{g}}_3$.

How does this magic work? Well, define the matrix $G = \begin{bmatrix} \vec{\mathbf{g}}_1 & \vec{\mathbf{g}}_2 & \vec{\mathbf{g}}_3 \end{bmatrix} = \begin{bmatrix} 4/9 & 1/9 & 8/9 \\ 4/9 & -8/9 & -1/9 \\ 7/9 & 4/9 & -4/9 \end{bmatrix}$.

How is G^TG related to $\langle \vec{\mathbf{g}}_i, \vec{\mathbf{g}}_j \rangle$?

So that means $G^{-1} = G^T$. Hence multiplying by G^T solves systems of equations, like $G\vec{\mathbf{x}} = \vec{\mathbf{v}}$.

Example 6.4 and 6.5 - Gram-Schmidt and Least Squares

Let $\vec{\mathbf{v}}_1 = \begin{bmatrix} 4 \\ 4 \\ 7 \end{bmatrix}$, $\vec{\mathbf{v}}_2 = \begin{bmatrix} 16 \\ 7 \\ 10 \end{bmatrix}$, and let $A = \begin{bmatrix} \uparrow & \uparrow \\ \vec{\mathbf{v}}_1 & \vec{\mathbf{v}}_2 \\ \downarrow & \downarrow \end{bmatrix}$. Set $\vec{\mathbf{b}} = \begin{bmatrix} 1 \\ 6 \\ 5 \end{bmatrix}$ and $\vec{\mathbf{x}}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

We want to find x_1 and x_2 so that $x_1\vec{\mathbf{v}}_1 + x_2\vec{\mathbf{v}}_2 = \vec{\mathbf{b}}$, that is, find $\vec{\mathbf{x}}$ so that $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$.

We cannot directly use dot products, because the columns of A are not orthogonal. We will replace A with a matrix G that does have orthogonal columns.

We'll keep the first column $\vec{\mathbf{g}}_1 = \vec{\mathbf{v}}_1$, but the second column of A points in the $\vec{\mathbf{g}}_1$ direction:

$\vec{\mathbf{v}}_2 \cdot \vec{\mathbf{g}}_1 = 162$ $\quad$ $\vec{\mathbf{g}}_1 \cdot \vec{\mathbf{g}}_1 = 81$ $\quad$ $\mathrm{proj}_{\vec{\mathbf{g}}_1}(\vec{\mathbf{v}}_2) = 162/81\vec{\mathbf{g}}_1 = 2\vec{\mathbf{g}}_1 = \begin{bmatrix} 8 \\ 8 \\ 14 \end{bmatrix}$.

If we set $\vec{\mathbf{g}}_2 = \vec{\mathbf{v}}_2 - \mathrm{proj}_{\vec{\mathbf{v}}_1}(\vec{\mathbf{v}}_2) = \begin{bmatrix} 8 \\ -1 \\ -4 \end{bmatrix}$ then

$\vec{\mathbf{g}}_1 \cdot \vec{\mathbf{g}}_2 = 0$ $\quad$ $\vec{\mathbf{g}}_2 \cdot \vec{\mathbf{g}}_2 = 81$ $\quad$ $\vec{\mathbf{v}}_2 = 2\vec{\mathbf{g}}_1 + \vec{\mathbf{g}}_2$

How does this relate to $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$? Well we can write $A\vec{\mathbf{x}}$ in terms of G:

$$A\vec{\mathbf{x}} = x_1\vec{\mathbf{v}}_1 + x_2\vec{\mathbf{v}}_2 = x_1\vec{\mathbf{g}}_1 + x_2(2\vec{\mathbf{g}}_1 + \vec{\mathbf{g}}_2) = (x_1 + 2x_2)\vec{\mathbf{g}}_1 + (x_2)\vec{\mathbf{g}}_2$$

And we can almost write $\vec{\mathbf{b}}$ in terms of G:

$\vec{\mathbf{b}} \cdot \vec{\mathbf{g}}_1 = 63$ $\quad$ $\vec{\mathbf{g}}_1 \cdot \vec{\mathbf{g}}_1 = 81$ $\quad$ $\mathrm{proj}_{\vec{\mathbf{g}}_1}(\vec{\mathbf{b}}) = 63/81\vec{\mathbf{g}}_1 = 7/9\vec{\mathbf{g}}_1 = 1/9 \begin{bmatrix} 28 \\ 28 \\ 49 \end{bmatrix}$.

$\vec{\mathbf{b}} \cdot \vec{\mathbf{g}}_2 = -18$ $\quad$ $\vec{\mathbf{g}}_2 \cdot \vec{\mathbf{g}}_2 = 81$ $\quad$ $\mathrm{proj}_{\vec{\mathbf{g}}_2}(\vec{\mathbf{b}}) = -18/81\vec{\mathbf{g}}_1 = -2/9\vec{\mathbf{g}}_2 = 1/9 \begin{bmatrix} -16 \\ 2 \\ 8 \end{bmatrix}$.

Now we get sneaky, and set $\vec{\mathbf{g}}_3 = \vec{\mathbf{b}} - \mathrm{proj}_{\vec{\mathbf{g}}_1}(\vec{\mathbf{b}}) - \mathrm{proj}_{\vec{\mathbf{g}}_2}(\vec{\mathbf{b}}) = 1/3 \begin{bmatrix} -1 \\ 8 \\ -4 \end{bmatrix}$

$\vec{\mathbf{g}}_1 \cdot \vec{\mathbf{g}}_3 = 0$ $\quad$ $\vec{\mathbf{g}}_2 \cdot \vec{\mathbf{g}}_3 = 0$ $\quad$ $\vec{\mathbf{g}}_3 \cdot \vec{\mathbf{g}}_3 = 9$ $\quad$ $\vec{\mathbf{b}} = 7/9\vec{\mathbf{g}}_1 + -2/9\vec{\mathbf{g}}_2 + \vec{\mathbf{g}}_3$

Calculate $\|A\vec{\mathbf{x}} - \vec{\mathbf{b}}\|$ using the $\vec{\mathbf{g}}$s:

$$\begin{aligned} \|A\vec{\mathbf{x}} - \vec{\mathbf{b}}\|^2 &= \|\left((x_1 + 2x_2)\vec{\mathbf{g}}_1 + (x_2)\vec{\mathbf{g}}_2\right) - \left(\tfrac{7}{9}\vec{\mathbf{g}}_1 + \tfrac{-2}{9}\vec{\mathbf{g}}_2 + \vec{\mathbf{g}}_3\right)\|^2 \\ &= \|\left(x_1 + 2x_2 - \tfrac{7}{9}\right)\vec{\mathbf{g}}_1 + \left(x_2 - \tfrac{-2}{9}\right)\vec{\mathbf{g}}_2 + \vec{\mathbf{g}}_3\|^2 \\ &= (x_1 + 2x_2 - \tfrac{7}{9})^2\|\vec{\mathbf{g}}_1\|^2 + (x_2 - \tfrac{-2}{9})^2\|\vec{\mathbf{g}}_2\|^2 + \|\vec{\mathbf{g}}_3\|^2 \end{aligned}$$

This is clearly minimized exactly when $x_1 + 2x_2 = \frac{7}{9}$ and $x_2 = \frac{-2}{9}$, that is, when $\vec{\mathbf{x}} = \frac{1}{9}\begin{bmatrix} 11 \\ -2 \end{bmatrix}$, but it can never be smaller than $\|\vec{\mathbf{g}}_3\|^2 = 9$.

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Let $\vec{\mathbf{v}}_1 = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$, $\vec{\mathbf{v}}_2 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$, and $\vec{\mathbf{b}} = \begin{bmatrix} 7 \\ 11 \\ 13 \end{bmatrix}$.

1. Find vectors $\vec{\mathbf{g}}_1$ and $\vec{\mathbf{g}}_2$ with the same span as $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$, except with $\vec{\mathbf{g}}_1 \cdot \vec{\mathbf{g}}_2 = 0$.

2. What is the projection of $\vec{\mathbf{b}}$ onto $\vec{\mathbf{g}}_1$

3. What is the projection of $\vec{\mathbf{b}}$ onto $\vec{\mathbf{g}}_2$

4. Define $\vec{\mathbf{g}}_3 = \vec{\mathbf{b}} - \text{proj}_{\vec{\mathbf{g}}_1}(\vec{\mathbf{b}}) - \text{proj}_{\vec{\mathbf{g}}_2}(\vec{\mathbf{b}})$

5. If you write $\vec{\mathbf{b}} = y_1\vec{\mathbf{g}}_1 + y_2\vec{\mathbf{g}}_2 + y_3\vec{\mathbf{g}}_3$, what are y_1, y_2, and y_3?

6. Write each $\vec{\mathbf{v}}_i$ in terms of the $\vec{\mathbf{g}}_j$s

7. Find the best x_1, x_2 so that $x_1\vec{\mathbf{v}}_1 + x_2\vec{\mathbf{v}}_2$ is as close to $\vec{\mathbf{b}}$ as possible.

8. How far from $\vec{\mathbf{b}}$ must it be?