MA322-007 Apr 14 - Practice Exam Name: ______________________

1. $\vec{\mathbf{a}} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, $\vec{\mathbf{b}} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \end{bmatrix}$, $\vec{\mathbf{c}} = \begin{bmatrix} -0.1 \\ -0.2 \\ 0.1 \\ 0.2 \end{bmatrix}$.

(a) Compute $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}$

(b) Compute $\vec{\mathbf{a}} \cdot \vec{\mathbf{c}}$

(c) Compute $\|\vec{\mathbf{b}}\|$

(d) Compute $\|\vec{\mathbf{c}}\|$

(e) Compute $\frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}}{\vec{\mathbf{b}} \cdot \vec{\mathbf{b}}}\vec{\mathbf{b}}$

(f) Compute $\frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{c}}}{\vec{\mathbf{c}} \cdot \vec{\mathbf{c}}}\vec{\mathbf{c}}$

(g) Explain what happened for (e) versus (f)

2. $G = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \vec{\mathbf{g}}_1 & \vec{\mathbf{g}}_2 & \vec{\mathbf{g}}_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} 0.44 & -0.32 & -0.56 \\ -0.08 & 0.74 & -0.58 \\ -0.40 & -0.30 & 0.10 \\ -0.80 & -0.10 & -0.30 \\ 0.00 & 0.50 & 0.50 \end{bmatrix}$ and $\vec{\mathbf{v}} = \begin{bmatrix} 5 \\ 0 \\ 6 \\ 6 \\ -12 \end{bmatrix}$

(a) Find $\vec{\mathbf{y}}$ so that $G\vec{\mathbf{y}}$ is as close as possible to $\vec{\mathbf{v}}$.

(b) How close is that?

You can use that $G^T G = I_3$ without checking it.

3. $A = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \vec{\mathbf{a}}_1 & \vec{\mathbf{a}}_2 & \vec{\mathbf{a}}_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 4 & 1 \\ 1 & 2 & 8 \\ 3 & -4 & 9 \\ 2 & -5 & 5 \end{bmatrix}$

Find orthonormal vectors $\vec{\mathbf{g}}_1$, $\vec{\mathbf{g}}_2$, $\vec{\mathbf{g}}_3$ that span Col(A). In other words, span$(\vec{\mathbf{g}}_1, \vec{\mathbf{g}}_2, \vec{\mathbf{g}}_3) =$ span$(\vec{\mathbf{a}}_1, \vec{\mathbf{a}}_2, \vec{\mathbf{a}}_3)$, $\vec{\mathbf{g}}_i \cdot \vec{\mathbf{g}}_i = 1$, and $\vec{\mathbf{g}}_i \cdot \vec{\mathbf{g}}_j = 0$ if $i \neq j$.

4. $A = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \vec{\mathbf{a}}_1 & \vec{\mathbf{a}}_2 & \vec{\mathbf{a}}_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & -10 & 0 \\ -8 & 10 & 30 \\ 1 & 1 & -7 \\ -3 & 7 & 1 \end{bmatrix}$ and $\vec{\mathbf{b}} = \begin{bmatrix} 2 \\ 15 \\ 4 \\ -31 \\ 38 \end{bmatrix}$

(a) Use RREF to show that there is no $\vec{\mathbf{x}}$ with $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$.

(b) Use G to find the $\vec{\mathbf{x}}$ with $A\vec{\mathbf{x}}$ as close $\vec{\mathbf{b}}$ as possible.

$$G = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \vec{\mathbf{g}}_1 & \vec{\mathbf{g}}_2 & \vec{\mathbf{g}}_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} 0.1 & -0.3 & 0.7 \\ 0.5 & 0.5 & 0.5 \\ -0.8 & 0.4 & 0.4 \\ 0.1 & -0.5 & 0.3 \\ -0.3 & -0.5 & 0.1 \end{bmatrix}$$

You can use that $G^T G = I_3$ and that $G^T A = \begin{bmatrix} -10 & 15 & 25 \\ 0 & -5 & 15 \\ 0 & 0 & 10 \end{bmatrix}$

Answers:

2. $\vec{\mathbf{y}} = \begin{bmatrix} 5 \\ 10 \\ 10 \end{bmatrix}$, $\vec{\mathbf{e}} = \vec{\mathbf{v}} - G\vec{\mathbf{y}} = \begin{bmatrix} -1.6 \\ 1.2 \\ 2.0 \\ -2.0 \\ -2.0 \end{bmatrix}$, a distance of $\|\vec{\mathbf{e}}\| = 4$.

3. $G = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \vec{\mathbf{g}}_1 & \vec{\mathbf{g}}_2 & \vec{\mathbf{g}}_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} 0.25 & 0.30 & -0.85 \\ -0.25 & 0.50 & 0.25 \\ 0.25 & 0.70 & 0.35 \\ 0.75 & 0.10 & 0.05 \\ 0.50 & -0.40 & 0.30 \end{bmatrix}$

4. First solve $G\vec{\mathbf{y}} = \vec{\mathbf{b}}$ as $\vec{\mathbf{y}} = G^T\vec{\mathbf{b}}$. Then solve $G^T A\vec{\mathbf{x}} = G^T\vec{\mathbf{b}} = \vec{\mathbf{y}}$.

$$\vec{\mathbf{y}} = G^T\vec{\mathbf{b}} = \begin{bmatrix} \vec{\mathbf{g}}_1 \cdot \vec{\mathbf{b}} \\ \vec{\mathbf{g}}_2 \cdot \vec{\mathbf{b}} \\ \vec{\mathbf{g}}_3 \cdot \vec{\mathbf{b}} \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \\ 5 \end{bmatrix}$$

Solve $G^T A\vec{\mathbf{x}} = \vec{\mathbf{y}}$ as

$$\left[\, G^T A \mid \vec{\mathbf{y}} \,\right] = \left[\begin{array}{ccc|c} -10 & 15 & 25 & 10 \\ 0 & -5 & 15 & 5 \\ 0 & 0 & 10 & 5 \end{array}\right] \xrightarrow[R_2-1.5R_3]{R_1-2.5R_3} \left[\begin{array}{ccc|c} -10 & 15 & 0 & -2.5 \\ 0 & -5 & 0 & -2.5 \\ 0 & 0 & 10 & 5.0 \end{array}\right] \xrightarrow[R_3/10]{R_1+3R_2}$$

$$\left[\begin{array}{ccc|c} -10 & 0 & 0 & -10.0 \\ 0 & -5 & 0 & -2.5 \\ 0 & 0 & 1 & 0.5 \end{array}\right] \xrightarrow[R_2/-5]{R_1/-10} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1.0 \\ 0 & 1 & 0 & 0.5 \\ 0 & 0 & 1 & 0.5 \end{array}\right] \rightarrow \vec{\mathbf{x}} = \begin{bmatrix} 1.0 \\ 0.5 \\ 0.5 \end{bmatrix}$$

Note that $A\vec{\mathbf{x}} = \begin{bmatrix} 1 \\ 0 \\ 12 \\ -2 \\ 1 \end{bmatrix}$ is not all that close to $\vec{\mathbf{b}} = \begin{bmatrix} 2 \\ 15 \\ 4 \\ -31 \\ 38 \end{bmatrix}$ but that the error $\vec{\mathbf{e}} = \vec{\mathbf{b}} - A\vec{\mathbf{x}} = \begin{bmatrix} 1 \\ 15 \\ -8 \\ -29 \\ 37 \end{bmatrix}$ is perpendicular to every column in A:

$$A^T\vec{\mathbf{e}} = \begin{bmatrix} \vec{\mathbf{a}}_1 \cdot \vec{\mathbf{e}} \\ \vec{\mathbf{a}}_2 \cdot \vec{\mathbf{e}} \\ \vec{\mathbf{a}}_3 \cdot \vec{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} 1 + 75 + 64 - 29 - 111 \\ 0 - 150 - 80 - 29 + 259 \\ 0 + 0 - 240 + 203 + 37 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If we tried a new $\vec{\mathbf{x}} + \vec{\mathbf{z}}$ then we'd want $A(\vec{\mathbf{x}} + \vec{\mathbf{z}}) = \vec{\mathbf{b}}$ but $A(\vec{\mathbf{x}} + \vec{\mathbf{z}}) = A\vec{\mathbf{x}} + A\vec{\mathbf{z}} = (\vec{\mathbf{b}} - \vec{\mathbf{e}}) + A\vec{\mathbf{z}}$ so what we need is $A\vec{\mathbf{z}} = \vec{\mathbf{e}}$ to cancel, but we just said that $\vec{\mathbf{e}} \cdot A\vec{\mathbf{z}} = 0$. In other words our new $\vec{\mathbf{x}} + \vec{\mathbf{z}}$ may have changed things, but it changed them in completely the wrong direction.