MA322-007 Apr 21 - Exam Name: ____________

1. $\vec{\mathbf{a}} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \vec{\mathbf{b}} = \begin{bmatrix} 7 \\ 5 \\ -5 \\ -1 \end{bmatrix}, \vec{\mathbf{c}} = \begin{bmatrix} 14 \\ 10 \\ -10 \\ -2 \end{bmatrix}$.

(a) Compute $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}$

$$1(7) + 2(5) + 3(-5) + 4(-1)$$
$$= 7 + 10 - 15 - 4$$
$$= -2$$

(b) Compute $\vec{\mathbf{a}} \cdot \vec{\mathbf{c}}$

$$1(14) + 2(10) + 3(-10) + 4(-2)$$
$$= 14 + 20 - 30 - 8$$
$$= -4$$

(c) Compute $\|\vec{\mathbf{b}}\|$

$$\sqrt{7^2 + 5^2 + (-5)^2 + (-1)^2} = \sqrt{49 + 25 + 25 + 1} = \sqrt{100} = 10$$

(d) Compute $\|\vec{\mathbf{c}}\|$ $\sqrt{14^2 + 10^2 + (-10)^2 + (-2)^2} = \sqrt{196 + 100 + 100 + 4} = \sqrt{400} = 20$

(e) Compute $\frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}}{\vec{\mathbf{b}} \cdot \vec{\mathbf{b}}}\vec{\mathbf{b}}$ $= \frac{-2}{100}\begin{bmatrix} 7 \\ 5 \\ -5 \\ -1 \end{bmatrix} = \begin{bmatrix} -0.14 \\ -0.10 \\ 0.10 \\ 0.02 \end{bmatrix}$

(f) Compute $\frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{c}}}{\vec{\mathbf{c}} \cdot \vec{\mathbf{c}}}\vec{\mathbf{c}}$ $= \frac{-4}{400}\begin{bmatrix} 14 \\ 10 \\ -10 \\ -2 \end{bmatrix} = \begin{bmatrix} -0.14 \\ -0.10 \\ 0.10 \\ 0.02 \end{bmatrix}$

(g) Explain what happened for (e) versus (f)

$\{t\vec{b}\} = \{t\vec{c}\}$ is the same line

2. $G = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \vec{\mathbf{g}}_1 & \vec{\mathbf{g}}_2 & \vec{\mathbf{g}}_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 & 0.7 \\ 0.5 & 0.5 & -0.7 \\ 0.5 & -0.5 & 0.1 \\ 0.5 & -0.5 & -0.1 \end{bmatrix}$ and $\vec{\mathbf{v}} = \begin{bmatrix} 0 \\ 100 \\ 100 \\ 100 \end{bmatrix}$ and $G^T G = I_3$.

(a) Find $\vec{\mathbf{y}}$ so that $G\vec{\mathbf{y}}$ is as close as possible to $\vec{\mathbf{v}}$.

$$\vec{y} = G^T \vec{v} = \begin{bmatrix} 0+50+50+50 \\ 0+50-50-50 \\ 0-70+10-10 \end{bmatrix} = \begin{bmatrix} 150 \\ -50 \\ -70 \end{bmatrix}$$

(b) How close is that?

$$G\vec{y} = 150\begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix} - 50\begin{bmatrix} 0.5 \\ 0.5 \\ -0.5 \\ -0.5 \end{bmatrix} - 70\begin{bmatrix} 0.7 \\ -0.7 \\ 0.1 \\ -0.1 \end{bmatrix}$$

$$= \begin{bmatrix} 75 \\ 75 \\ 75 \\ 75 \end{bmatrix} + \begin{bmatrix} -25 \\ -25 \\ 25 \\ 25 \end{bmatrix} + \begin{bmatrix} -49 \\ +49 \\ -7 \\ +7 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 99 \\ 93 \\ 107 \end{bmatrix} \text{ is off by } \vec{e} = \begin{bmatrix} -1 \\ +1 \\ +7 \\ -7 \end{bmatrix}$$

$$\|\vec{e}\| = \sqrt{1^2+1^2+7^2+7^2} = \sqrt{100} = 10$$

but $\|\vec{v}\| \approx 170$ and $G^T\vec{e} = \begin{bmatrix} -\frac{1}{2}+\frac{1}{2}+\frac{7}{2}-\frac{7}{2} \\ -\frac{1}{2}+\frac{1}{2}-\frac{7}{2}+\frac{7}{2} \\ -0.7-0.7+0.7+0.7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

So we cannot fix $\vec{e}$ using G

3. $A = \begin{bmatrix} \uparrow & \uparrow & \uparrow & \uparrow \\ \vec{\mathbf{a}}_1 & \vec{\mathbf{a}}_2 & \vec{\mathbf{a}}_3 & \vec{\mathbf{a}}_4 \\ \downarrow & \downarrow & \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} 1 & 1 & 7 & 1 \\ 1 & 1 & -7 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ $\quad G = \begin{bmatrix} \frac{1}{2} & 1/2 & 0.7 & 0.1 \\ \frac{1}{2} & 1/2 & -0.7 & -0.1 \\ \frac{1}{2} & -1/2 & 0.1 & -0.7 \\ \frac{1}{2} & -1/2 & -0.1 & +0.7 \end{bmatrix}$

Find orthonormal vectors $\vec{\mathbf{g}}_1, \vec{\mathbf{g}}_2, \vec{\mathbf{g}}_3, \vec{\mathbf{g}}_4$ that span Col(A). In other words, span$(\vec{\mathbf{g}}_1, \vec{\mathbf{g}}_2, \vec{\mathbf{g}}_3, \vec{\mathbf{g}}_4) =$ span$(\vec{\mathbf{a}}_1, \vec{\mathbf{a}}_2, \vec{\mathbf{a}}_3, \vec{\mathbf{a}}_4)$, $\vec{\mathbf{g}}_i \cdot \vec{\mathbf{g}}_i = 1$, and $\vec{\mathbf{g}}_i \cdot \vec{\mathbf{g}}_j = 0$ if $i \neq j$.

$$\vec{g}_1 = \frac{\vec{a}_1}{\|\vec{a}_1\|} = \frac{(1,1,1,1)}{\sqrt{1+1+1+1}} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$$

$$\vec{g}_2 = \frac{\vec{a}_2 - \frac{\vec{a}_2 \cdot \vec{g}_1}{\vec{g}_1 \cdot \vec{g}_1}\vec{g}_1}{\| \sim \|} = \frac{\begin{bmatrix}1\\1\\0\\0\end{bmatrix} - \frac{\frac{1}{2}+\frac{1}{2}}{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}}\begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix}}{\| \sim \|} = \begin{bmatrix}1/2\\1/2\\-1/2\\-1/2\end{bmatrix}$$

$\| \sim \| = 1$ already

$$\vec{g}_3 = \vec{a}_3 - \frac{\vec{a}_3 \cdot \vec{g}_1}{\vec{g}_1 \cdot \vec{g}_1}\vec{g}_1 - \frac{\vec{a}_3 \cdot \vec{g}_2}{\vec{g}_2 \cdot \vec{g}_2}\vec{g}_2 = \begin{bmatrix}7\\-7\\2\\0\end{bmatrix} - \frac{\frac{7}{2} - \frac{7}{2} + \frac{2}{2} + 0}{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}} \text{ etc}$$

$$= \begin{bmatrix}7\\-7\\2\\0\end{bmatrix} - \vec{g}_1 + \vec{g}_2 = \begin{bmatrix}7 - 1/2 + 1/2 \\ -7 - 1/2 + 1/2 \\ 2 - 1/2 - 1/2 \\ 0 - 1/2 - 1/2\end{bmatrix} = \begin{bmatrix}7\\-7\\1\\-1\end{bmatrix} \qquad \| \sim \| = 10$$

$$\vec{g}_3 = \begin{bmatrix}0.7\\-0.7\\0.1\\-0.1\end{bmatrix}$$

$$\vec{g}_4 = \vec{a}_4 - (\vec{a}_4 \cdot \vec{g}_1)\vec{g}_1 - (\vec{a}_4 \cdot \vec{g}_2)\vec{g}_2 - (\vec{a}_4 \cdot \vec{g}_3)\vec{g}_3$$

$$= \begin{bmatrix}1\\0\\0\\0\end{bmatrix} - \frac{1}{2}\vec{g}_1 - \frac{1}{2}\vec{g}_2 - 0.7\vec{g}_3$$

$$= \begin{bmatrix}1 - 1/4 - 1/4 - 0.49 \\ 0 - 1/4 - 1/4 + 0.49 \\ 0 - 1/4 + 1/4 - 0.07 \\ 0 - 1/4 + 1/4 + 0.07\end{bmatrix} = \begin{bmatrix}0.01 \\ -0.01 \\ -0.07 \\ 0.07\end{bmatrix} \qquad \| e \| = 0.1$$

$$\vec{g}_4 = (0.1, -0.1, -0.7, 0.7)$$

4. $A = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \vec{\mathbf{a}}_1 & \vec{\mathbf{a}}_2 & \vec{\mathbf{a}}_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 7 \\ 1 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$ and $\vec{\mathbf{b}} = \begin{bmatrix} 0 \\ 100 \\ 100 \\ 100 \end{bmatrix}$

(a) Use RREF to show that there is no $\vec{\mathbf{x}}$ with $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$.

$$\begin{array}{c} x_1 \; x_2 \; x_3 \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 0 & 7 & 100 \\ 1 & 1 & 3 & 100 \\ 1 & 1 & 4 & 100 \end{array}\right] \end{array} \leftarrow x_1 = 0 \qquad x_3 = \frac{100}{7}$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 7 & 100 \\ 0 & 1 & 3 & 100 \\ 0 & 1 & 4 & 100 \end{array}\right] \xrightarrow{R_4 - R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 100/7 \\ 0 & 1 & 3 & 100 \\ 0 & 0 & 1 & 0 \end{array}\right] \leftarrow x_3 = 0$$

Contradiction

$$x_2 = \frac{4}{7}100$$

$$A\begin{bmatrix} 0 \\ 4/7 \cdot 100 \\ 4/7 \cdot 100 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 4/7 \cdot 100 \\ 4/7 \cdot 100 \end{bmatrix} + \begin{bmatrix} 0 \\ 100 \\ 3/7 \cdot 100 \\ 4/7 \cdot 100 \end{bmatrix} = \begin{bmatrix} 0 \\ 100 \\ 100 \\ \frac{8}{7}100 \end{bmatrix}$$

$\|e\|$ is $\frac{1}{7}100 \approx 14.3$

(b) Use G to find the $\vec{\mathbf{x}}$ with $A\vec{\mathbf{x}}$ as close $\vec{\mathbf{b}}$ as possible.

$G = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \vec{\mathbf{g}}_1 & \vec{\mathbf{g}}_2 & \vec{\mathbf{g}}_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 & 0.7 \\ 0.5 & 0.5 & -0.7 \\ 0.5 & -0.5 & 0.1 \\ 0.5 & -0.5 & -0.1 \end{bmatrix}$ Note that $G^TG = I_3$ and $G^TA = \begin{bmatrix} 2 & 1 & 7 \\ 0 & -1 & 0 \\ 0 & 0 & -5 \end{bmatrix}$

$$\left[G^TA \,|\, G^Tb\right] = \left[\begin{array}{ccc|c} 2 & 1 & 7 & 150 \\ 0 & -1 & 0 & -50 \\ 0 & 0 & -5 & -70 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 2 & 1 & 7 & 150 \\ 0 & 1 & 0 & 50 \\ 0 & 0 & 1 & 14 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & 50 \\ 0 & 0 & 1 & 14 \end{array}\right]$$

$$A\begin{bmatrix} 1 \\ 50 \\ 14 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 50 \\ 50 \end{bmatrix} + \begin{bmatrix} 0 \\ 98 \\ 42 \\ 56 \end{bmatrix} = \begin{bmatrix} 1 \\ 99 \\ 93 \\ 107 \end{bmatrix}, \quad \|e\| = \sqrt{(-1)^2 + (1)^2 + (7)^2 + (-7)^2} = \sqrt{100} = 10$$

Just like in #2

Smaller error than in (a)