Practice

MA322-007 Apr 21 - Exam Name: ______________

1. $\vec{\mathbf{a}} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, $\vec{\mathbf{b}} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \end{bmatrix}$, $\vec{\mathbf{c}} = \begin{bmatrix} -0.1 \\ -0.2 \\ 0.1 \\ 0.2 \end{bmatrix}$.

(a) Compute $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}$ $1(1) + 2(2) + 3(-1) + 4(-2) = 1 + 4 - 3 - 8 = -6$

(b) Compute $\vec{\mathbf{a}} \cdot \vec{\mathbf{c}}$ $1(-0.1) + 2(-0.2) + 3(0.1) + 4(0.2) = -0.1 - 0.4 + 0.3 + 0.8 = 0.6$

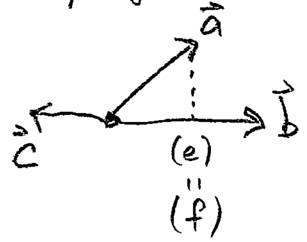
(c) Compute $\|\vec{\mathbf{b}}\|$ $\sqrt{1^2 + 2^2 + (-1)^2 + (-2)^2} = \sqrt{1+4+1+4} = \sqrt{10}$

(d) Compute $\|\vec{\mathbf{c}}\|$ $\sqrt{(-0.1)^2 + (-0.2)^2 + (0.1)^2 + 0.2^2} = \sqrt{0.01 + 0.04 + 0.01 + 0.4} = \sqrt{0.1}$

(e) Compute $\frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}}{\vec{\mathbf{b}} \cdot \vec{\mathbf{b}}} \vec{\mathbf{b}}$ $\frac{-6}{10} \begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} -0.6 \\ -1.2 \\ 0.6 \\ 1.2 \end{bmatrix}$

(f) Compute $\frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{c}}}{\vec{\mathbf{c}} \cdot \vec{\mathbf{c}}} \vec{\mathbf{c}}$ $\frac{0.6}{0.1} \begin{bmatrix} -0.1 \\ -0.2 \\ 0.1 \\ 0.2 \end{bmatrix} = \begin{bmatrix} -0.6 \\ -1.2 \\ 0.6 \\ 1.2 \end{bmatrix}$

(g) Explain what happened for (e) versus (f) $\vec{b}$ and $\vec{c}$ span the same line so the projection of $\vec{a}$ onto $\vec{b}$ and $\vec{c}$ is literally the same



2. $G = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \vec{g}_1 & \vec{g}_2 & \vec{g}_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} 0.0 & 0.4 & 0.2 \\ -0.2 & 0.0 & 0.8 \\ -0.4 & 0.8 & 0.0 \\ 0.8 & 0.2 & 0.4 \\ 0.4 & 0.4 & -0.4 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 7 \\ -8 \\ 14 \\ -15 \\ 5 \end{bmatrix}$

(a) Find $\vec{y}$ so that $G\vec{y}$ is as close as possible to $\vec{v}$.

$$\vec{y} = G^T G \vec{y} = G^T \vec{v} = \begin{bmatrix} \vec{g}_1 \cdot \vec{v} \\ \vec{g}_2 \cdot \vec{v} \\ \vec{g}_3 \cdot \vec{v} \end{bmatrix} = \begin{bmatrix} 0 + 1.6 - 5.6 - 12.0 + 2.0 \\ 2.8 + 0 + 11.2 - 3.0 + 2.0 \\ 1.4 - 6.4 + 0 - 6.0 - 2.0 \end{bmatrix} = \begin{bmatrix} -14 \\ 13 \\ -13 \end{bmatrix}$$

(b) How close is that? $\vec{e} = \vec{V} - G\vec{y}$

$$= \begin{bmatrix} 7 \\ -8 \\ 14 \\ -15 \\ 5 \end{bmatrix} - \left(-14 \begin{bmatrix} 0.0 \\ -0.2 \\ -0.4 \\ 0.8 \\ 0.4 \end{bmatrix} + 13 \begin{bmatrix} 0.4 \\ 0.0 \\ 0.8 \\ 0.2 \\ 0.4 \end{bmatrix} - 13 \begin{bmatrix} 0.2 \\ 0.8 \\ 0.0 \\ 0.4 \\ -0.4 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 7 \\ -8 \\ 14 \\ -15 \\ 5 \end{bmatrix} - \left(\begin{bmatrix} 0 \\ 2.8 \\ 5.6 \\ -11.2 \\ -5.6 \end{bmatrix} + \begin{bmatrix} 5.2 \\ 0 \\ 10.4 \\ 2.6 \\ 5.2 \end{bmatrix} + \begin{bmatrix} -2.6 \\ -10.4 \\ 0 \\ -5.2 \\ 5.2 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 7 \\ -8 \\ 14 \\ -15 \\ 5 \end{bmatrix} - \begin{bmatrix} 2.6 \\ -7.6 \\ 16.0 \\ -13.8 \\ +4.8 \end{bmatrix} = \begin{bmatrix} 4.4 \\ -0.4 \\ -2.0 \\ -1.2 \\ 0.2 \end{bmatrix}$$

$$\|\vec{e}\| = \sqrt{4.4^2 + 0.4^2 + 2.0^2 + 1.2^2 + 0.2^2} = \sqrt{19.36 + 0.16 + 4.00 + 1.44 + 0.04}$$

$$= \sqrt{25.00} = 5$$

You can use that $G^T G = I_3$ without checking it.

3. $A = \begin{bmatrix} \uparrow & \uparrow & \uparrow & \uparrow \\ \vec{\mathbf{a}}_1 & \vec{\mathbf{a}}_2 & \vec{\mathbf{a}}_3 & \vec{\mathbf{a}}_4 \\ \downarrow & \downarrow & \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} 4 & 1 & 1 & 32 \\ 3 & 32 & 2 & -1 \\ 0 & 0 & 3 & 32 \\ 0 & 0 & 4 & 1 \end{bmatrix}$ $\qquad G = \begin{bmatrix} 4/5 & -3/5 & 0 & 0 \\ 3/5 & 4/5 & 0 & 0 \\ 0 & 0 & 3/5 & 4/5 \\ 0 & 0 & 4/5 & -3/5 \end{bmatrix}$

Find orthonormal vectors $\vec{\mathbf{g}}_1, \vec{\mathbf{g}}_2, \vec{\mathbf{g}}_3, \vec{\mathbf{g}}_4$ that span Col(A). In other words, span$(\vec{\mathbf{g}}_1, \vec{\mathbf{g}}_2, \vec{\mathbf{g}}_3, \vec{\mathbf{g}}_4) =$ span$(\vec{\mathbf{a}}_1, \vec{\mathbf{a}}_2, \vec{\mathbf{a}}_3, \vec{\mathbf{a}}_4)$, $\vec{\mathbf{g}}_i \cdot \vec{\mathbf{g}}_i = 1$, and $\vec{\mathbf{g}}_i \cdot \vec{\mathbf{g}}_j = 0$ if $i \neq j$.

$$\vec{g}_1 = \frac{\vec{a}_1}{\|\vec{a}_1\|} = \frac{(4,3,0,0)}{\sqrt{16+9+0+0}} = \left(\frac{4}{5}, \frac{3}{5}, 0, 0\right)$$

$$\vec{g}_2 = \frac{\vec{a}_2 - \frac{\vec{a}_2 \cdot \vec{g}_1}{\vec{g}_1 \cdot \vec{g}_1}\vec{g}_1}{\left\| \quad \right\|} \qquad \vec{a}_2 - \frac{\vec{a}_2 \cdot \vec{g}_1}{\vec{g}_1 \cdot \vec{g}_1}\vec{g}_1 = (1,32,0,0) - \frac{\frac{4}{5} + \frac{96}{5} + 0 + 0}{1}\left(\frac{4}{5}, \frac{3}{5}, 0, 0\right) = (1-16, 32-12, 0, 0)$$

$$= \frac{(-15, 20, 0, 0)}{\sqrt{15^2 + 20^2 + 0 + 0}} = \frac{(-15,20,0,0)}{\sqrt{625}} = \left(\frac{-15}{25}, \frac{20}{25}, 0, 0\right) = \left(-\frac{3}{5}, \frac{4}{5}, 0, 0\right)$$

$$\vec{g}_3 = \vec{a}_3 - (\vec{a}_3 \cdot \vec{g}_1)\vec{g}_1 - (\vec{a}_3 \cdot \vec{g}_2)\vec{g}_2 = (1,2,3,4) - \left(\frac{4}{5} + \frac{6}{5} + 0 + 0\right)\begin{bmatrix} 4/5 \\ 3/5 \\ 0 \\ 0 \end{bmatrix} - \left(\frac{-3}{5} + \frac{8}{5} + 0 + 0\right)$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} - 2\begin{bmatrix} 4/5 \\ 3/5 \\ 0 \\ 0 \end{bmatrix} - 1\begin{bmatrix} -3/5 \\ 4/5 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 4 \end{bmatrix}, \qquad \vec{g}_3 = \frac{(0,0,3,4)}{\sqrt{0+0+9+16}} = \begin{bmatrix} 0 \\ 0 \\ 3/5 \\ 4/5 \end{bmatrix}$$

$$\vec{g}_4 = \frac{\vec{a}_4 - (\vec{a}_4 \cdot \vec{g}_1)\vec{g}_1 - (\vec{a}_4 \cdot \vec{g}_2)\vec{g}_2 - (\vec{a}_4 \cdot \vec{g}_3)\vec{g}_3}{\left\| \quad \right\|} \qquad = \begin{bmatrix} 32 \\ -1 \\ 32 \\ 1 \end{bmatrix} - \left(\frac{128}{5} - \frac{3}{5}\right)\begin{bmatrix} 4/5 \\ 3/5 \\ 0 \\ 0 \end{bmatrix} - \cdots$$

$$= \begin{bmatrix} 32 \\ -1 \\ 32 \\ 1 \end{bmatrix} - 25\begin{bmatrix} 4/5 \\ 3/5 \\ 0 \\ 0 \end{bmatrix} - \left(\frac{-96-4}{5} + 20\right)\begin{bmatrix} -3/5 \\ 4/5 \\ 0 \\ 0 \end{bmatrix} - \left(\frac{96+4}{5}\ 20\right)\begin{bmatrix} 0 \\ 0 \\ 3/5 \\ 4/5 \end{bmatrix} = \begin{bmatrix} 32 - 20 - 12 - 0 \\ -1 - 15 + 16 - 0 \\ 32 - 0 + 0 - 12 \\ 1 - 0 + 0 - 16 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 20 \\ -15 \end{bmatrix}$$

$$\vec{g}_4 = \frac{(0,0,20,-15)}{\sqrt{20^2 + 15^2}} = \frac{(0,0,20,-15)}{\sqrt{625}} = \frac{(0,0,20,-15)}{25} = \left(0, 0, \frac{4}{5}, -\frac{3}{5}\right)$$

4. $A = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \vec{\mathbf{a}}_1 & \vec{\mathbf{a}}_2 & \vec{\mathbf{a}}_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix} = \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 \\ 5 & 4 & 0 \\ 7 & 5 & 5 \\ 5 & 3 & -1 \end{bmatrix} \end{array}$ and $\vec{\mathbf{b}} = \begin{bmatrix} 6 \\ 0 \\ 2 \\ 10 \end{bmatrix}$

(a) Use RREF to show that there is no $\vec{\mathbf{x}}$ with $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$.

$1x_1 = 6, \quad x_1 = 6$

$5x_1 + 4x_2 = 0 \Rightarrow 30 + 4x_2 = 0, \quad x_2 = -7.5$

$7x_1 + 5x_2 + 5x_3 = 2 \Rightarrow 42 + (-37.5) + 5x_3 = 2, \quad x_3 = \frac{-2.5}{5} = -0.5$

$5x_1 + 3x_2 - 1x_3 = 10 \Rightarrow 30 - 22.5 + 0.5 = 8 \neq 10$

No solution, off by 2 at least

(b) Use G to find the $\vec{\mathbf{x}}$ with $A\vec{\mathbf{x}}$ as close $\vec{\mathbf{b}}$ as possible.

$G = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \vec{\mathbf{g}}_1 & \vec{\mathbf{g}}_2 & \vec{\mathbf{g}}_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} 0.1 & -0.7 & 0.1 \\ 0.5 & 0.5 & -0.5 \\ 0.7 & 0.1 & 0.7 \\ 0.5 & -0.5 & -0.5 \end{bmatrix}$

Replace $[A \mid b]$ $(A\vec{x} = \vec{b})$
with $[G^T A \mid G^T b]$ $(G^T A\vec{x} = G^T b)$

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 \end{matrix} \\ \left[\begin{array}{ccc|c} 10 & 7 & 3 & 0.6 + 0 + 1.4 + 5 \\ 0 & 1 & 1 & -4.2 + 0 + 0.2 - 5 \\ 0 & 0 & 4 & 0.6 + 0 + 1.4 - 5 \end{array}\right] \end{array}$$

$4x_3 = -3, \quad x_3 = -0.75$ (not -0.5)

$x_2 + x_3 = -9, \quad x_2 = -8.25$ (not -7.5)

$10x_1 + 7(-8.25) + 3(-0.75) = 7, \quad x_1 = 6.7$ (not 6)

$$A\vec{x} = \begin{bmatrix} 6.7 + 0 + 0 \\ 33.5 - 33 + 0 \\ 46.9 - 41.25 - 3.75 \\ 33.5 - 24 + .75 + 0.75 \end{bmatrix} = \begin{bmatrix} 6.7 \\ 0.5 \\ 1.9 \\ 9.5 \end{bmatrix} \text{ off by } \begin{bmatrix} 0.7 \\ -0.5 \\ 0.1 \\ 0.5 \end{bmatrix}$$

You can use that $G^T G = I_3$ and that $G^T A = \begin{bmatrix} 10 & 7 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}$

$\|\vec{e}\| = \sqrt{0.49 + 0.25 + 0.01 + 0.25}$
$= 1$ less than 2 from (a)