(Another explanation with chol(B) hidden)

Maximize $\vec{v}^T A \vec{v}$ subject to $\vec{v}^T B \vec{v} \leq 100$

is basically the same as

Maximize $\dfrac{\vec{v}^T A \vec{v}}{\vec{v}^T B \vec{v}}$

The answer to this is fairly simple:

☆ $\text{Max} = \lambda$, $\vec{v} = \vec{g}_1$ where $(\lambda, \vec{g}_1)$

is the eigenpair of $B^{-1}A$ with largest λ

We ~~can~~ must assume A, B are symmetric

(Replace A with $\dfrac{A+A^T}{2}$, B with $\dfrac{B+B^T}{2}$

without changing $v^T A v$ or $v^T B v$)

We also need that B is "positive definite"

to avoid any zero or negative cost vectors.

Now to show our answer is right

Write $\vec{v} = \sum a_i \vec{g}_i$

where $(\lambda_i, \vec{g}_i)$ are the eigenvectors of

$B^{-1}A$ (which matlab will use chol(B) to find)

Now $\vec{g}_i \cdot \vec{g}_j$ need <u>not</u> be 0, since $B^{-1}A$

is not symmetric. However we get something

(Lemma) close: $\vec{g}_i \cdot (B\vec{g}_j) = 0$

Proof: (If $\lambda_i \neq \lambda_j$)

$$B^{-1}A\vec{g}_i = \lambda_i \vec{g}_i \implies A\vec{g}_i = \lambda_i (B\vec{g}_i)$$

so $\vec{g}_i \cdot A\vec{g}_j = \vec{g}_i \cdot \lambda_j B\vec{g}_j = \lambda_j (\vec{g}_i^T B\vec{g}_j)$

but $(A\vec{g}_i) \cdot \vec{g}_j = \lambda_i B\vec{g}_i \cdot \vec{g}_j = \lambda_i (\vec{g}_i^T B\vec{g}_j)$

since $B = B^T$. But $(A\vec{g}_i)\vec{g}_j = \vec{g}_i^T A\vec{g}_j$ since

$A = A^T$. Hence $\lambda_i (g_i^T B g_j) = \lambda_j (g_i^T B g_j)$

and $\vec{g}_i^T B\vec{g}_j = 0$

So now we can calculate

$$\vec{v}^T A \vec{v} = \left(\sum a_i \vec{g}_i\right)^T A \left(\sum a_j \vec{g}_j\right)$$

$$= \sum_{i,j} a_i a_j \vec{g}_i^T A \vec{g}_j$$

$$= \sum \lambda_j a_i a_j \vec{g}_i^T B \vec{g}_i \qquad \text{(Lemma)}$$

$$= \sum_{i=1}^{n} \lambda_i |a_i|^2 \vec{g}_i^T B \vec{g}_i$$

But $\vec{v}^T B \vec{v} = \sum_{i=1}^{n} |a_i|^2 \vec{g}_i^T B \vec{g}_i$

So we have the same problem!

Maximize $\sum \lambda_i x_i^2$

subject to $\sum x_i^2 \leq$ whatever

Best choice is $x_1 =$ All of it, $x_2 = \cdots = x_n = 0$

where $\lambda_1 > \lambda_2 > \cdots > \lambda_n$

And that was the answer in ☆