MA322-007 May 4 - Practice Exam Name: ______________________

1. What are the eigenpairs of:

(a) $A = 26\vec{\mathbf{u}}\vec{\mathbf{u}}^T = \frac{1}{13}\begin{bmatrix} 50 & 120 \\ 120 & 288 \end{bmatrix}$ if $\vec{\mathbf{u}} = \frac{1}{13}\begin{bmatrix} 5 \\ 12 \end{bmatrix}$?

$$A\vec{u} = 26\vec{u}\vec{u}^T\vec{u} = 26\vec{u}\,\|u\|^2 \quad \text{so} \quad (26\|u\|^2, \vec{u})$$

$$\|u\|^2 = \frac{1}{13}[5\ 12]\begin{bmatrix}5\\12\end{bmatrix}\frac{1}{13} = \frac{1}{169}(5^2+12^2) = \frac{169}{169} = 1$$

but also $u_2 = \frac{1}{13}\begin{bmatrix}-12\\5\end{bmatrix}$ $\quad \vec{u}^T\vec{u}_2 = 0$ so $(0, \vec{u}_2)$

Explicitly $\left(26, \begin{bmatrix}5\\12\end{bmatrix}\right)$ and $\left(0, \begin{bmatrix}-12\\5\end{bmatrix}\right)$

(b) $B = \vec{\mathbf{v}}\vec{\mathbf{v}}^T = \begin{bmatrix} 16 & 20 \\ 20 & 25 \end{bmatrix}$ if $\vec{\mathbf{v}} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$?

$$B\vec{v} = \vec{v}\vec{v}^T\vec{v} = \vec{v}\left(\|\vec{v}\|^2\right) \quad \text{so} \quad \left(\|\vec{v}\|^2, \vec{v}\right) \qquad \|v\|^2 = 4^2+5^2 = 41$$

but also $\vec{v}_2 = \begin{bmatrix}-5\\4\end{bmatrix}$ $\quad \vec{v}^T\vec{v}_2 = 0$ so $(0, \vec{v}_2)$

Explicitly $\left(41, \begin{bmatrix}4\\5\end{bmatrix}\right)$ and $\left(0, \begin{bmatrix}-5\\4\end{bmatrix}\right)$

(c) $C = \vec{\mathbf{w}}\vec{\mathbf{w}}^T - \vec{\mathbf{p}}\vec{\mathbf{p}}^T = \begin{bmatrix} -12 & -16 & 30 \\ -16 & -21 & 36 \\ 30 & 36 & -27 \end{bmatrix}$ if $\vec{\mathbf{w}} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$ and $\vec{\mathbf{p}} = \begin{bmatrix} 4 \\ 5 \\ -6 \end{bmatrix}$?

$\|w\|^2 = 4+4+9 = 17 \qquad \vec{w}^T\vec{p} = \vec{p}^T\vec{w} = 8+10-18 = 0$

$\|p\|^2 = 16+25+36 = 77$

$$C\vec{w} = \vec{w}\,\|w\|^2 - \vec{p}(0) \quad \text{so} \quad \left(\|\vec{w}\|^2, \vec{w}\right)$$

$$C\vec{p} = \vec{w}(0) - \vec{p}\left(\|\vec{p}\|^2\right) \quad \text{so} \quad \left(-\|\vec{p}\|^2, \vec{p}\right)$$ but need a <u>third</u>

$$\vec{q} = \begin{bmatrix}1\\1\\1\end{bmatrix} - \frac{(1,1,1)\cdot(2,2,3)}{(2,2,3)\cdot(2,2,3)}(2,2,3) - \frac{(1,1,1)\cdot(4,5,-6)}{(4,5,-6)\cdot(4,5,-6)}(4,5,-6) \quad (\text{or } \vec{q} = \vec{w}\times\vec{p})$$

$$= \begin{bmatrix}1\\1\\1\end{bmatrix} - \frac{7}{17}(2,2,3) - \frac{3}{77}(4,5,-6) = \frac{1}{17(77)}\begin{bmatrix}27\\-24\\-2\end{bmatrix}$$

Explicitly $\left(17, \begin{bmatrix}2\\2\\3\end{bmatrix}\right)$, $\left(-77, \begin{bmatrix}4\\5\\-6\end{bmatrix}\right)$, and $\left(0, \begin{bmatrix}27\\-24\\-2\end{bmatrix}\right)$

2. The 4×4 matrix $A = \begin{bmatrix} 32 & -5 & 2 & 25 \\ -5 & 8 & -1 & -2 \\ 2 & -1 & 8 & 5 \\ 25 & -2 & 5 & 32 \end{bmatrix}$ has eigenpairs $\left(\overset{\lambda_1}{58}, \overset{10\vec{v}_1}{\begin{bmatrix} 7 \\ -1 \\ 1 \\ 7 \end{bmatrix}}\right)$, $\left(\overset{\lambda_2}{10}, \overset{2\vec{v}_2}{\begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}\right)$, $\left(\overset{\lambda_3}{8}, \overset{10\vec{v}_3}{\begin{bmatrix} 1 \\ 7 \\ -7 \\ 1 \end{bmatrix}}\right)$, and $\left(\overset{\lambda_4}{4}, \overset{2\vec{v}_4}{\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}}\right)$.

(a) Find a vector $\vec{\mathbf{v}}$ with length $\|\vec{\mathbf{v}}\| = 1$ that maximizes the energy $E = \vec{\mathbf{v}}^T A \vec{\mathbf{v}}$. What is that maximum energy E?

If $\vec{v} = \sum_{i=1}^{4} x_i \vec{v}_i$

$$\vec{v}^T A \vec{v} = \left(\sum_{i=1}^{4} x_i \vec{v}_i^{\,T}\right) A \left(\sum_{j=1}^{4} x_j \vec{v}_j\right)$$

$$= \sum_{i=1}^{4}\sum_{j=1}^{4} x_i x_j \left(\vec{v}_i^{\,T} A \vec{v}_j\right)$$

$$= \sum_{i=1}^{4}\sum_{j=1}^{4} x_i x_j \lambda_j \left(\vec{v}_i^{\,T} \vec{v}_j\right)$$

$\star$ but $\vec{v}_i^{\,T}\vec{v}_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$ (just check, or because A is symmetric)

$$= \sum_{i=1}^{4} x_i^2 \lambda_i$$

So Maximize $58x_1^2 + 10x_2^2 + 8x_3^2 + 4x_4^2$

Subject to $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$

Clear solution is $x_1^2 = 1$

$x_2^2 = x_3^2 = x_4^2 = 0$

So $\vec{v} = \pm\vec{v}_1 = \dfrac{\pm 1}{10}\begin{bmatrix} 7 \\ -1 \\ 1 \\ 7 \end{bmatrix}$

$E = 58$

(b) Find a vector $\vec{\mathbf{v}}$ with length $\|\vec{\mathbf{v}}\| = 1$ that minimizes the energy $E = \vec{\mathbf{v}}^T A \vec{\mathbf{v}}$. What is that minimum energy E?

opposite $\quad x_1^2 = x_2^2 = x_3^2 = 0, \quad x_4^2 = 1$

so $\vec{v} = \pm\vec{v}_4 = \dfrac{\pm 1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \quad E = 4$

Detail $\|v\|^2 = \left(\sum_{i=1}^{4} x_i \vec{v}_i^{\,T}\right)\left(\sum_{j=1}^{4} x_j \vec{v}_j\right)$

$$= \sum_{i=1}^{4}\sum_{j=1}^{4} x_i x_j \left(\vec{v}_i^{\,T}\vec{v}_j\right)$$

$$= \sum_{i=1}^{4} x_i^2 \quad \text{by } \star$$

$\vec{v}_i^{\,T} A \vec{v}_i = \vec{v}_i^{\,T}(\lambda_i \vec{v}_i)$

$= \lambda_i (1)$

$= \lambda_i$

So $E = \underset{\text{max}}{\lambda_1}$ or $\underset{\text{min}}{\lambda_4}$

3. The 5×6 matrix $A = \begin{bmatrix} -66.47 & 4.86 & 27.15 & 43.38 & 39.11 & 48.20 \\ 118.83 & 60.05 & -96.45 & -114.68 & -22.32 & -54.55 \\ 66.26 & -4.92 & -27.27 & -43.47 & -39.14 & -48.31 \\ 38.46 & 134.89 & -111.12 & -99.22 & 72.63 & 35.63 \\ -199.25 & 14.63 & 81.84 & 129.95 & 117.46 & 144.66 \end{bmatrix}$

can be written as $\sum_{k=1}^{5} \sigma_i \vec{\mathbf{u}}_i \vec{\mathbf{v}}_i^T$ where the singular triples $(\sigma_i, \vec{\mathbf{u}}_i, \vec{\mathbf{v}}_i)$ are

$$\left(402.64, \tfrac{1}{8}\begin{bmatrix} 2 \\ -4 \\ -2 \\ -2 \\ 6 \end{bmatrix}, \tfrac{1}{8}\begin{bmatrix} -5 \\ -1 \\ 3 \\ 4 \\ 2 \\ 3 \end{bmatrix}\right), \left(223.60, \tfrac{1}{8}\begin{bmatrix} -1 \\ -2 \\ 1 \\ -7 \\ -3 \end{bmatrix}, \tfrac{1}{8}\begin{bmatrix} 1 \\ -5 \\ 3 \\ 2 \\ -4 \\ -3 \end{bmatrix}\right), \left(0.24, \tfrac{1}{8}\begin{bmatrix} -1 \\ -2 \\ -7 \\ 1 \\ -3 \end{bmatrix}, \tfrac{1}{8}\begin{bmatrix} 5 \\ 2 \\ 0 \\ 5 \\ -1 \\ 3 \end{bmatrix}\right),$$

$$\left(0.16, \tfrac{1}{8}\begin{bmatrix} -7 \\ 2 \\ -1 \\ -1 \\ 3 \end{bmatrix}, \tfrac{1}{8}\begin{bmatrix} 3 \\ 0 \\ 6 \\ -3 \\ 3 \\ 1 \end{bmatrix}\right), \text{ and } \left(0.08, \tfrac{1}{8}\begin{bmatrix} -3 \\ -6 \\ 3 \\ 3 \\ -1 \end{bmatrix}, \tfrac{1}{8}\begin{bmatrix} -2 \\ 5 \\ 3 \\ -1 \\ -5 \\ 0 \end{bmatrix}\right).$$

Find $\vec{\mathbf{x}}$ so that $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$, if $\vec{\mathbf{b}} = \begin{bmatrix} 8 \\ -47 \\ -10 \\ -64 \\ 28 \end{bmatrix}$. Actually, that's easy to ask RREF: $\vec{\mathbf{x}} \approx \begin{bmatrix} 1.75 \\ 8.68 \\ 9.53 \\ -1.16 \\ -6.07 \\ 2.31 \end{bmatrix}$.

What I meant to ask was find me a smaller $\vec{\mathbf{x}_0}$ that still basically works. I'm thinking around 2% the size of the perfect $\vec{\mathbf{x}}$, but allowing around 2% relative error. Explicitly $\|\vec{\mathbf{x}_0}\| \leq 0.3 \approx 0.02\|\vec{\mathbf{x}}\|$ and $\|A\vec{\mathbf{x}_0} - \vec{\mathbf{b}}\| \leq 1.8 \approx 0.02\|\vec{\mathbf{b}}\|$.

Explain how to find small almost-solutions:

Write $\vec{b}$ in terms of $\vec{u}_i$: $\vec{b} \approx \sum_{i=1}^{5} \frac{(\vec{u}_i^T b)}{(\vec{u}_i^T \vec{u}_i)} \vec{u}_i$

$\vec{b} = 65\vec{u}_1 + 55\vec{u}_2 + 1\vec{u}_3 + 1\vec{u}_4 + 1\vec{u}_5$

Write $\vec{x}$ interms of $\vec{v}_i$: $\vec{x} = \sum x_i \vec{v}_i$

$\|x\|^2 = \sum x_i^2$ like in #2

Write $A\vec{x}$ in terms of $\vec{u}_i$: $A\vec{x} = \sum_{i=1}^{5} \frac{\vec{u}_i^T A\vec{x}}{\vec{u}_i^T \vec{u}_i} \vec{u}_i$

$\vec{u}_i^T A\vec{x} = \vec{u}_i^T \left(\sum_{j=1}^{5} \sigma_j \vec{u}_j \vec{v}_j^T\right)\left(\sum_{k=1}^{5} x_k \vec{v}_k\right)$

$= \sum_{j=1}^{5} \sum_{k=1}^{5} \sigma_j x_k \underset{\uparrow \text{ 0 unless } i=j}{(\vec{u}_i^T \vec{u}_j)} \underset{\nwarrow \text{ 0 unless } j=k}{(\vec{v}_j^T \vec{v}_k)}$

$= \sigma_i x_i$

$A\vec{x} = \sum_{i=1}^{5} \sigma_i x_i \vec{u}_i$

so $\|A\vec{x}-\vec{b}\|^2 = \sum_{i=1}^{5} (\sigma_i x_i - b_i)^2$ where $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix} = \begin{bmatrix} 65 \\ 55 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

b_i is $\dfrac{\vec{u}_i^T \vec{b}}{\vec{u}_i^T \vec{u}_i}$

So the smallest $\|A\vec{x}-\vec{b}\|$ is with

$$x_i = \frac{b_i}{\sigma_i}$$

but $\|x\|^2 = \sum x_i^2 = \left(\frac{65}{402.64}\right)^2 + \left(\frac{55}{223.60}\right)^2$ (small)

$+ \left(\frac{1}{0.24}\right)^2 + \left(\frac{1}{0.16}\right)^2 + \left(\frac{1}{0.08}\right)^2$ (big)

What if we set $x_3 = x_4 = x_5 = 0$ instead?

$$\frac{\|A\vec{x}-\vec{b}\|^2}{\|\vec{b}\|^2} = \frac{0^2+0^2+1^2+1^2+1^2}{65^2+55^2+1^2+1^2+1^2} = \frac{3}{7253}, \quad \sqrt{\sim} = 0.0203\ldots \approx 2\%$$

$$\frac{\|x\|^2}{\|\text{exact } x\|^2} = \frac{\left(\frac{65}{402.64}\right)^2 + \left(\frac{55}{223.60}\right)^2 + 0+0+0}{\left(\frac{65}{402.64}\right)^2 + \left(\frac{55}{223.60}\right)^2 + \left(\frac{1}{0.24}\right)^2 + \left(\frac{1}{0.16}\right)^2 + \left(\frac{1}{0.08}\right)^2}$$

$$= \frac{0.0866}{212.7602}, \quad \sqrt{\sim} = 0.0202 \approx 2\%$$

so our $\vec{x}$ is 98% smaller!

and is 98% accurate!