Good Ways

2. $A = \begin{bmatrix} 2 & 36 \\ 36 & 23 \end{bmatrix}$ has eigenpairs $(\lambda_i, \vec{\mathbf{u}}_i)$: $(50, \frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix})$ and $(-25, \frac{1}{5}\begin{bmatrix} -4 \\ 3 \end{bmatrix})$.

$\vec{u}_1$ $\vec{u}_2$

(a) Find a simple expression for $D(i,j) = \vec{\mathbf{u}}_i^T \vec{\mathbf{u}}_j$

$\vec{u}_1^T\vec{u}_1 = \frac{1}{5}\frac{1}{5}(3^2+4^2) = \frac{25}{25} = 1$

$\vec{u}_1^T\vec{u}_2 = \frac{1}{5}\frac{1}{5}(3(-4)+4(3)) = \frac{0}{25} = 0 = \vec{u}_2^T\vec{u}_1$

$\vec{u}_2^T\vec{u}_2 = \frac{1}{5}\frac{1}{5}((-4)^2+3^2) = \frac{25}{25} = 1$

So $\vec{u}_i^T\vec{u}_j = \begin{cases} 1 \text{ if } i=j \\ 0 \text{ if } i\neq j \end{cases}$

(b) Find a simple expression for $L(x,y) = \|\vec{\mathbf{v}}\|^2$ if $\vec{\mathbf{v}} = x\vec{\mathbf{u}}_1 + y\vec{\mathbf{u}}_2$

$\|\vec{v}\|^2 = \vec{v}^T\vec{v} = (x\vec{u}_1 + y\vec{u}_2)^T(x\vec{u}_1 + y\vec{u}_2)$

$= x^2\vec{u}_1^T\vec{u}_1 + xy\vec{u}_1^T\vec{u}_2 + yx\vec{u}_2^T\vec{u}_1 + y^2\vec{u}_2^T\vec{u}_2$

$= x^2(1) + xy(0) + yx(0) + y^2(1)$

$= x^2 + y^2$

(c) Find a simple expression for $E(x,y) = \vec{\mathbf{v}}^T A\vec{\mathbf{v}}$ if $\vec{\mathbf{v}} = x\vec{\mathbf{u}}_1 + y\vec{\mathbf{u}}_2$

$A\vec{v} = xA\vec{u}_1 + yA\vec{u}_2 = x(50\vec{u}_1) + y(-25\vec{u}_2)$

$= 50x\,\vec{u}_1 - 25y\,\vec{u}_2$

$\vec{v}^TA\vec{v} = (x\vec{u}_1 + y\vec{u}_2)^T(50x\,\vec{u}_1 - 25y\,\vec{u}_2)$

$= 50x^2\vec{u}_1^T\vec{u}_1 - 25xy\,\vec{u}_1^T\vec{u}_2 + 50xy\,\vec{u}_2^T\vec{u}_1 - 25y^2\vec{u}_2^T\vec{u}_2 = 50x^2(1) - 25xy(0) + 50xy(0) - 25y^2(1) = 50x^2 - 25y^2$

(d) What values of x, y maximize $E(x,y)$ while keeping $L(x,y) = 1$?

If $L(x,y)=1$, then $x^2+y^2=1$, $y^2 = 1-x^2$ and $-1 \le x \le 1$

$E(x,y) = 50x^2 - 25(1-x^2) = 75x^2 - 25$

crit values: $50x = 0 \Rightarrow x = 0$

x	$E(x,y)$	
-1	50	
0	-25	← min
1	50	

$x=0, y=\pm 1, E=-25$

(e) What values of x, y minimize $E(x,y)$ while keeping $L(x,y) = 1$?

Ditto but $x = \pm 1, y = 0, E = 50$

2. $A = \begin{bmatrix} 2 & 36 \\ 36 & 23 \end{bmatrix}$ has eigenpairs $(\lambda_i, \vec{\mathbf{u}}_i)$: $(50, \frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix})$ and $(-25, \frac{1}{5}\begin{bmatrix} -4 \\ 3 \end{bmatrix})$.

(a) Find a simple expression for $D(i,j) = \vec{\mathbf{u}}_i^T \vec{\mathbf{u}}_j$

Here are bad/hard ways to do (b) and (c)

(b) Find a simple expression for $L(x,y) = \|\vec{\mathbf{v}}\|^2$ if $\vec{\mathbf{v}} = x\vec{\mathbf{u}}_1 + y\vec{\mathbf{u}}_2 = \begin{bmatrix} \frac{3}{5}x - \frac{4}{5}y \\ \frac{4}{5}x + \frac{3}{5}y \end{bmatrix}$

$$\|v\|^2 = \left(\tfrac{3}{5}x - \tfrac{4}{5}y\right)^2 + \left(\tfrac{4}{5}x + \tfrac{3}{5}y\right)^2 = \tfrac{9}{25}x^2 - \tfrac{24}{25}xy + \tfrac{16}{25}y^2 + \tfrac{16}{25}x^2 + \tfrac{24}{25}xy + \tfrac{9}{25}x^2$$

$$= \tfrac{25}{25}x^2 + 0xy + \tfrac{25}{25}y^2$$

$$= x^2 + y^2$$

(c) Find a simple expression for $E(x,y) = \vec{\mathbf{v}}^T A \vec{\mathbf{v}}$ if $\vec{\mathbf{v}} = x\vec{\mathbf{u}}_1 + y\vec{\mathbf{u}}_2$, $A\vec{v} = \begin{bmatrix} 2\left(\frac{3}{5}x - \frac{4}{5}y\right) + 36\left(\frac{4}{5}x + \frac{3}{5}y\right) \\ 36\left(\frac{3}{5}x - \frac{4}{5}y\right) + 23\left(\frac{4}{5}x + \frac{3}{5}y\right) \end{bmatrix}$

$$A\vec{v} = \begin{bmatrix} \left(\frac{6}{5} + \frac{144}{5}\right)x + \left(\frac{-8}{5} + \frac{108}{5}\right)y \\ \left(\frac{108}{5} + \frac{92}{5}\right)x + \left(\frac{-144}{5} + \frac{69}{5}\right)y \end{bmatrix} = \begin{bmatrix} 30x - 20y \\ 40x - 15y \end{bmatrix}$$

$$\vec{v}^T A\vec{v} = \left(\tfrac{3}{5}x - \tfrac{4}{5}y\right)(30x - 20y) + \left(\tfrac{4}{5}x + \tfrac{3}{5}y\right)(40x - 15y) = \left(\tfrac{90}{5} + \tfrac{160}{5}\right)x^2 + \left(\tfrac{-60}{5} - \tfrac{120}{5} + \tfrac{60}{5} + \tfrac{120}{5}\right)xy + \left(\tfrac{-80}{5} - \tfrac{45}{5}\right)y^2$$

$$= 50x^2 - 25y^2$$

(d) What values of x, y maximize $E(x,y)$ while keeping $L(x,y) = 1$?

(e) What values of x, y minimize $E(x,y)$ while keeping $L(x,y) = 1$?

Harder / Longer 2

2. The 4×4 matrix $A = \begin{bmatrix} 32 & -5 & 2 & 25 \\ -5 & 8 & -1 & -2 \\ 2 & -1 & 8 & 5 \\ 25 & -2 & 5 & 32 \end{bmatrix}$ has eigenpairs $(\lambda_i, \vec{\mathbf{v}}_i)$:

$$\left(58, \tfrac{1}{10}\begin{bmatrix} 7 \\ -1 \\ 1 \\ 7 \end{bmatrix}\right), \quad \left(10, \tfrac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right), \quad \left(8, \tfrac{1}{10}\begin{bmatrix} 1 \\ 7 \\ -7 \\ 1 \end{bmatrix}\right), \text{ and } \quad \left(4, \tfrac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}\right).$$

Do Rank 2 instead?

(a) Find a simple expression for $D(i,j) = \vec{\mathbf{v}}_i^T \vec{\mathbf{v}}_j$

$\vec{v}_1^T \vec{v}_1 = \frac{1}{10}\begin{bmatrix} 7 & -1 & 1 & 7 \end{bmatrix}\frac{1}{10}\begin{bmatrix} 7 \\ -1 \\ 1 \\ 7 \end{bmatrix} = \frac{1}{100}(7^2+1+1+7^2) = \frac{100}{100} = 1 = \vec{v}_3^T \vec{v}_3$

$\vec{v}_1^T \vec{v}_2 = \vec{v}_2^T \vec{v}_1 = \frac{1}{20}(-7-1+1+7) = 0$, etc

$\vec{v}_i^T \vec{v}_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

(b) Find a simple expression for $E(a,b,c,d) = \vec{\mathbf{v}}^T A \vec{\mathbf{v}}$ if $\vec{\mathbf{v}} = a\vec{\mathbf{v}}_1 + b\vec{\mathbf{v}}_2 + c\vec{\mathbf{v}}_3 + d\vec{\mathbf{v}}_4$.

$(a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3 + d\vec{v}_4)^T A(a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3 + d\vec{v}_4)$

$= (a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3 + d\vec{v}_4)^T (a A\vec{v}_1 + bA\vec{v}_2 + cA\vec{v}_3 + dA\vec{v}_4)$

$= (a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3 + d\vec{v}_4)^T (a\,58\vec{v}_1 + b\,10\vec{v}_2 + c\,8\vec{v}_3 + d\,4\vec{v}_4)$

$= 58a^2 + 10b^2 + 8c^2 + 4d^2$

(c) Find a simple expression for $L(a,b,c,d) = \|\vec{\mathbf{v}}\|^2$ if $\vec{\mathbf{v}} = a\vec{\mathbf{v}}_1 + b\vec{\mathbf{v}}_2 + c\vec{\mathbf{v}}_3 + d\vec{\mathbf{v}}_4$.

$\|v\|^2 = a^2 + b^2 + c^2 + d^2$

(d) Find a vector $\vec{\mathbf{v}}$ with length $\|\vec{\mathbf{v}}\| = 1$ that maximizes the energy $E = \vec{\mathbf{v}}^T A \vec{\mathbf{v}}$. What is that maximum energy E?

Maximize $58a^2 + 10b^2 + 8c^2 + 4d^2$

Given $a^2 + b^2 + c^2 + d^2 = 1$

$a = \pm 1$, $b = c = d = 0$, $\vec{v} = \pm\vec{v}_1 = \pm\frac{1}{10}\begin{bmatrix} 7 \\ -1 \\ 1 \\ 7 \end{bmatrix}$

(e) Find a vector $\vec{\mathbf{v}}$ with length $\|\vec{\mathbf{v}}\| = 1$ that minimizes the energy $E = \vec{\mathbf{v}}^T A \vec{\mathbf{v}}$. What is that minimum energy E?

Minimize $58a^2 + 10b^2 + 8c^2 + 4d^2$

Given $a^2 + b^2 + c^2 + d^2 = 1$

$d = \pm 1$, $a = b = c = 0$, $\vec{v} = \pm\vec{v}_4 = \pm\frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$