3. $B = \begin{bmatrix} 27 & 200 \\ 136 & 225 \end{bmatrix}$ can be written as $B = \sigma_1 \vec{\mathbf{u}}_1 \vec{\mathbf{v}}_1^T + \sigma_2 \vec{\mathbf{u}}_2 \vec{\mathbf{v}}_2^T$ where $(\sigma_i, \vec{\mathbf{u}}_i, \vec{\mathbf{v}}_i)$ are

$$\left(325, \tfrac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix}, \tfrac{1}{13}\begin{bmatrix} 5 \\ 12 \end{bmatrix}\right) \quad \text{and} \quad \left(65, \tfrac{1}{5}\begin{bmatrix} -4 \\ 3 \end{bmatrix}, \tfrac{1}{13}\begin{bmatrix} 12 \\ -5 \end{bmatrix}\right).$$

(a) Find numbers r, s such that $\vec{\mathbf{b}} = r\vec{\mathbf{u}}_1 + s\vec{\mathbf{u}}_2$ if $\vec{\mathbf{b}} = \begin{bmatrix} -119 \\ 183 \end{bmatrix}$

$\vec{u}_1^T \cdot \vec{b} = \vec{u}_1^T (r\vec{u}_1 + s\vec{u}_2) = r\vec{u}_1^T\vec{u}_1 + s\vec{u}_1^T\vec{u}_2 = r(1) + s(0) = r$

$\vec{u}_2^T\vec{b} = u_2^T(r\vec{u}_1 + s\vec{u}_2) = r\vec{u}_2^T\vec{u}_1 + s\vec{u}_2^T\vec{u}_2 = r(0) + s(1) = s$

so $r = \vec{u}_1^T\vec{b} = \frac{1}{5}(3(-119) + 4(183)) = \frac{375}{5} = 75$

and $s = \vec{u}_2^T\vec{b} = \frac{1}{5}((-4)(-119) + 3(183)) = \frac{1025}{5} = 205$

(b) Find a simple expression for the numbers p, q such that $B\vec{\mathbf{w}} = p\vec{\mathbf{u}}_1 + q\vec{\mathbf{u}}_2$ if $\vec{\mathbf{w}} = x\vec{\mathbf{v}}_1 + y\vec{\mathbf{v}}_2$

$B\vec{w} = (325\,\vec{u}_1\vec{v}_1^T + 65\,\vec{u}_2\vec{v}_2^T)(x\vec{v}_1 + y\vec{v}_2)$

$= 325x\,\vec{u}_1\vec{v}_1^T\vec{v}_1 + 325y\,\vec{u}_1\vec{v}_1^T\vec{v}_2 + 65x\,\vec{u}_2\vec{v}_2^T\vec{v}_1 + 65y\,\vec{u}_2\vec{v}_2^T\vec{v}_2$

$= 325x\,\vec{u}_1(1) + 325y\,\vec{u}_1(0) + 65x\,\vec{u}_2(0) + 65y\,\vec{u}_2(0)$

So $p = 325x$ $\quad q = 65y$

(c) Find a simple expression for $L(x, y) = \|\vec{\mathbf{w}}\|^2$ if $\vec{\mathbf{w}} = x\vec{\mathbf{v}}_1 + y\vec{\mathbf{v}}_2$

$\|\vec{w}\|^2 = (x\vec{v}_1 + y\vec{v}_2)^T(x\vec{v}_1 + y\vec{v}_2) = x^2\vec{v}_1^T\vec{v}_1 + xy\vec{v}_1^T\vec{v}_2 + yx\vec{v}_2^T\vec{v}_1 + y^2\vec{v}_2^T\vec{v}_2$

$= x^2(1) + xy(0) + yx(0) + y^2(1)$

$= x^2 + y^2$

(d) Find a simple expression for $E(x, y, r, s) = \|B\vec{\mathbf{w}} - \vec{\mathbf{b}}\|^2$ with x, y, r, s as in (a) and (b)

$\|B\vec{w} - \vec{b}\|^2 = \|(325x - 75)\vec{u}_1 + (65y - 205)\vec{u}_2\|^2$

$= (325x - 75)^2 + (65y - 205)^2$

(e) Find x, y such that $E(x, y, r, s)$ is as small as possible

$x = \frac{75}{325} \qquad y = \frac{205}{65} \qquad \vec{w} = \frac{75}{325}\frac{1}{13}\begin{bmatrix} 5 \\ 12 \end{bmatrix} + \frac{205}{65}\frac{1}{13}\begin{bmatrix} 12 \\ -5 \end{bmatrix}$

$\|\vec{w}\|^2 = \left(\frac{75}{325}\right)^2 + \left(\frac{205}{65}\right)^2 = 10 > 9$

Bonus: As in #3, find $\vec{\mathbf{w}}$ so that $\|B\vec{\mathbf{w}} - \vec{\mathbf{b}}\| < 5\%\|\vec{\mathbf{b}}\|$ and $\|\vec{\mathbf{w}}\| \leq 3$. Equivalently, find x, y such that $E(x, y, r, s) < 0.05^2\|\vec{\mathbf{b}}\|^2 \approx 120$ but $L(x, y) \leq 3^2$. Answer on the back.

One way is to say x is more important than y,

so let $x = \frac{75}{325}$ and give y the "rest"

$$y = \sqrt{9 - \left(\frac{75}{325}\right)^2} = \frac{\sqrt{1512}}{13} \quad \text{so } \|w\|^2 = 9$$

Then $E(x,y) = \left(325 \frac{75}{325} - 75\right)^2 + \left(65 \frac{\sqrt{1512}}{13} - 205\right)^2$

$$\approx 0^2 + 111.889 < 120$$