MA322-007 May 4 - Practice Exam Name: ______________________

1. What are the eigenpairs of:

(a) $A = 26\vec{\mathbf{u}}\vec{\mathbf{u}}^T = \frac{1}{13}\begin{bmatrix} 50 & 120 \\ 120 & 288 \end{bmatrix}$ if $\vec{\mathbf{u}} = \frac{1}{13}\begin{bmatrix} 5 \\ 12 \end{bmatrix}$?

(b) $B = \vec{\mathbf{v}}\vec{\mathbf{v}}^T = \begin{bmatrix} 16 & 20 \\ 20 & 25 \end{bmatrix}$ if $\vec{\mathbf{v}} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$?

(c) $C = \vec{\mathbf{w}}\vec{\mathbf{w}}^T - \vec{\mathbf{p}}\vec{\mathbf{p}}^T = \begin{bmatrix} -12 & -16 & 30 \\ -16 & -21 & 36 \\ 30 & 36 & -27 \end{bmatrix}$ if $\vec{\mathbf{w}} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$ and $\vec{\mathbf{p}} = \begin{bmatrix} 4 \\ 5 \\ -6 \end{bmatrix}$?

2. The 4×4 matrix $A = \begin{bmatrix} 32 & -5 & 2 & 25 \\ -5 & 8 & -1 & -2 \\ 2 & -1 & 8 & 5 \\ 25 & -2 & 5 & 32 \end{bmatrix}$ has eigenpairs $(\lambda_i, \vec{\mathbf{v}}_i)$:

$$\left(58, \frac{1}{10}\begin{bmatrix} 7 \\ -1 \\ 1 \\ 7 \end{bmatrix}\right), \quad \left(10, \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right), \quad \left(8, \frac{1}{10}\begin{bmatrix} 1 \\ 7 \\ -7 \\ 1 \end{bmatrix}\right), \text{ and } \quad \left(4, \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}\right).$$

(a) Find a simple expression for $D(i, j) = \vec{\mathbf{v}}_i^T \vec{\mathbf{v}}_j$

(b) Find a simple expression for $E(a, b, c, d) = \vec{\mathbf{v}}^T A \vec{\mathbf{v}}$ if $\vec{\mathbf{v}} = a\vec{\mathbf{v}}_1 + b\vec{\mathbf{v}}_2 + c\vec{\mathbf{v}}_3 + d\vec{\mathbf{v}}_4$.

(c) Find a simple expression for $L(a, b, c, d) = \|\vec{\mathbf{v}}\|^2$ if $\vec{\mathbf{v}} = a\vec{\mathbf{v}}_1 + b\vec{\mathbf{v}}_2 + c\vec{\mathbf{v}}_3 + d\vec{\mathbf{v}}_4$.

(d) Find a vector $\vec{\mathbf{v}}$ with length $\|\vec{\mathbf{v}}\| = 1$ that maximizes the energy $E = \vec{\mathbf{v}}^T A \vec{\mathbf{v}}$. What is that maximum energy E?

(e) Find a vector $\vec{\mathbf{v}}$ with length $\|\vec{\mathbf{v}}\| = 1$ that minimizes the energy $E = \vec{\mathbf{v}}^T A \vec{\mathbf{v}}$. What is that minimum energy E?

3. Suppose $A = \sum_{i=1}^{r} \sigma_i \vec{\mathbf{u}}_i \vec{\mathbf{v}}_i^T$ with $\vec{\mathbf{u}}_i^T \vec{\mathbf{u}}_j = \vec{\mathbf{v}}_i^T \vec{\mathbf{v}}_j$ equal to 1 if $i = j$ and equal to 0 otherwise. Here r is a small positive integer, and σ_i are positive real numbers.

(a) Find a simple expression for the numbers b_i such that $\sum_{i=1}^{r} b_i \vec{\mathbf{u}}_i$ is as close as possible to $\vec{\mathbf{b}}$

(b) Find a simple expression for the numbers c_i such that $A\vec{\mathbf{x}} = \sum_{i=1}^{r} c_i \vec{\mathbf{u}}_i$ if $\vec{\mathbf{x}} = \sum_{i=1}^{r} x_i \vec{\mathbf{v}}_i$.

(c) Find a simple expression for $E(b_i, x_i) = \|A\vec{\mathbf{x}} - \vec{\mathbf{b}}\|^2$ if $\vec{\mathbf{b}} = \sum_{i=1}^{r} b_i \vec{\mathbf{u}}_i$ and $\vec{\mathbf{x}} = \sum_{i=1}^{r} x_i \vec{\mathbf{v}}_i$.

(d) Find a simple expression for $L(x_i) = \|\vec{\mathbf{x}}\|^2$ if $\vec{\mathbf{x}} = \sum_{i=1}^{r} x_i \vec{\mathbf{v}}_i$.

(e) Find a simple expression for the numbers x_i such that $A\vec{\mathbf{x}}$ is as close as possible to $\vec{\mathbf{b}}$ if $\vec{\mathbf{x}} = \sum_{i=1}^{r} x_i \vec{\mathbf{v}}_i$.

4. The 5×6 matrix $A = \begin{bmatrix} -66.47 & 4.86 & 27.15 & 43.38 & 39.11 & 48.20 \\ 118.83 & 60.05 & -96.45 & -114.68 & -22.32 & -54.55 \\ 66.26 & -4.92 & -27.27 & -43.47 & -39.14 & -48.31 \\ 38.46 & 134.89 & -111.12 & -99.22 & 72.63 & 35.63 \\ -199.25 & 14.63 & 81.84 & 129.95 & 117.46 & 144.66 \end{bmatrix}$

can be written as $\displaystyle\sum_{k=1}^{5} \sigma_i \vec{\mathbf{u}}_i \vec{\mathbf{v}}_i^T$ where the singular triples $(\sigma_i, \vec{\mathbf{u}}_i, \vec{\mathbf{v}}_i)$ are

$$\left(402.64, \tfrac{1}{8}\begin{bmatrix} 2 \\ -4 \\ -2 \\ -2 \\ 6 \end{bmatrix}, \tfrac{1}{8}\begin{bmatrix} -5 \\ -1 \\ 3 \\ 4 \\ 2 \\ 3 \end{bmatrix}\right), \left(223.60, \tfrac{1}{8}\begin{bmatrix} -1 \\ -2 \\ 1 \\ -7 \\ -3 \end{bmatrix}, \tfrac{1}{8}\begin{bmatrix} 1 \\ -5 \\ 3 \\ 2 \\ -4 \\ -3 \end{bmatrix}\right), \left(0.24, \tfrac{1}{8}\begin{bmatrix} -1 \\ -2 \\ -7 \\ 1 \\ -3 \end{bmatrix}, \tfrac{1}{8}\begin{bmatrix} 5 \\ 2 \\ 0 \\ 5 \\ -1 \\ 3 \end{bmatrix}\right),$$

$$\left(0.16, \tfrac{1}{8}\begin{bmatrix} -7 \\ 2 \\ -1 \\ -1 \\ 3 \end{bmatrix}, \tfrac{1}{8}\begin{bmatrix} 3 \\ 0 \\ 6 \\ -3 \\ 3 \\ 1 \end{bmatrix}\right), \text{ and } \left(0.08, \tfrac{1}{8}\begin{bmatrix} -3 \\ -6 \\ 3 \\ 3 \\ -1 \end{bmatrix}, \tfrac{1}{8}\begin{bmatrix} -2 \\ 5 \\ 3 \\ -1 \\ -5 \\ 0 \end{bmatrix}\right).$$

Find $\vec{\mathbf{x}}$ so that $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$, if $\vec{\mathbf{b}} = \begin{bmatrix} 8 \\ -47 \\ -10 \\ -64 \\ 28 \end{bmatrix}$. Actually, that's easy to ask RREF: $\vec{\mathbf{x}} \approx \begin{bmatrix} 1.75 \\ 8.68 \\ 9.53 \\ -1.16 \\ -6.07 \\ 2.31 \end{bmatrix}$.

What I meant to ask was find me a smaller $\vec{\mathbf{x_0}}$ that still basically works. I'm thinking around 2% the size of the perfect $\vec{\mathbf{x}}$, but allowing around 2% relative error. Explicitly $\|\vec{\mathbf{x_0}}\| \leq 0.3 \approx 0.02\|\vec{\mathbf{x}}\|$ and $\|A\vec{\mathbf{x_0}} - \vec{\mathbf{b}}\| \leq 1.8 \approx 0.02\|\vec{\mathbf{b}}\|$.

Explain how to find small almost-solutions:

2. $A = \begin{bmatrix} 2 & 36 \\ 36 & 23 \end{bmatrix}$ has eigenpairs $(\lambda_i, \vec{\mathbf{u}}_i)$: $(50, \frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix})$ and $(-25, \frac{1}{5}\begin{bmatrix} -4 \\ 3 \end{bmatrix})$.

(a) Find a simple expression for $D(i, j) = \vec{\mathbf{u}}_i^T \vec{\mathbf{u}}_j$

(b) Find a simple expression for $L(x, y) = \|\vec{\mathbf{v}}\|^2$ if $\vec{\mathbf{v}} = x\vec{\mathbf{u}}_1 + y\vec{\mathbf{u}}_2$

(c) Find a simple expression for $E(x, y) = \vec{\mathbf{v}}^T A \vec{\mathbf{v}}$ if $\vec{\mathbf{v}} = x\vec{\mathbf{u}}_1 + y\vec{\mathbf{u}}_2$

(d) What values of x, y maximize $E(x, y)$ while keeping $L(x, y) = 1$?

(e) What values of x, y minimize $E(x, y)$ while keeping $L(x, y) = 1$?

3. $B = \begin{bmatrix} 27 & 200 \\ 136 & 225 \end{bmatrix}$ can be written as $B = \sigma_1 \vec{\mathbf{u}}_1 \vec{\mathbf{v}}_1^T + \sigma_2 \vec{\mathbf{u}}_2 \vec{\mathbf{v}}_2^T$ where $(\sigma_i, \vec{\mathbf{u}}_i, \vec{\mathbf{v}}_i)$ are

$$\left(325, \frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix}, \frac{1}{13}\begin{bmatrix} 5 \\ 12 \end{bmatrix}\right) \quad \text{and} \quad \left(65, \frac{1}{5}\begin{bmatrix} -4 \\ 3 \end{bmatrix}, \frac{1}{13}\begin{bmatrix} 12 \\ -5 \end{bmatrix}\right).$$

(a) Find numbers r, s such that $\vec{\mathbf{b}} = r\vec{\mathbf{u}}_1 + s\vec{\mathbf{u}}_2$ if $\vec{\mathbf{b}} = \begin{bmatrix} -119 \\ 183 \end{bmatrix}$

(b) Find a simple expression for the numbers p, q such that $B\vec{\mathbf{w}} = p\vec{\mathbf{u}}_1 + q\vec{\mathbf{u}}_2$ if $\vec{\mathbf{w}} = x\vec{\mathbf{v}}_1 + y\vec{\mathbf{v}}_2$

(c) Find a simple expression for $L(x, y) = \|\vec{\mathbf{w}}\|^2$ if $\vec{\mathbf{w}} = x\vec{\mathbf{v}}_1 + y\vec{\mathbf{v}}_2$

(d) Find a simple expression for $E(x, y, r, s) = \|B\vec{\mathbf{w}} - \vec{\mathbf{b}}\|^2$ with x, y, r, s as in (a) and (b)

(e) Find x, y such that $E(x, y, r, s)$ is as small as possible

Bonus: As in #3, find $\vec{\mathbf{w}}$ so that $\|B\vec{\mathbf{w}} - \vec{\mathbf{b}}\| < 5\%\|\vec{\mathbf{b}}\|$ and $\|\vec{\mathbf{w}}\| \leq 3$. Equivalently, find x, y such that $E(x, y, r, s) < 0.05^2\|\vec{\mathbf{b}}\|^2 \approx 120$ but $L(x, y) \leq 3^2$. Answer on the back.