MA322-007 May 5 - Final Exam

Name: KEY

1. What are the eigenpairs of:

(a) $A = 25\vec{\mathbf{u}}\vec{\mathbf{u}}^T = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$ if $\vec{\mathbf{u}} = \frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$?

$\|\vec{u}\|^2 = \frac{1}{5^2}(3^2+4^2) = \frac{25}{25} = 1$

$(25, \vec{u})$

$\left(0, \begin{bmatrix} -4 \\ 3 \end{bmatrix}\right)$

$(\lambda \vec{u}\vec{u}^T)(\vec{u}) = \lambda \vec{u} \|\vec{u}\|^2$

$(\lambda \|\vec{u}\|^2, \vec{u})$

if $\vec{u} \perp \vec{v}$ then $(\lambda \vec{u}\vec{u}^T)(\vec{v})$

$= \lambda \vec{u}(\vec{u}^T\vec{v})$

$= \lambda \vec{u}(0)$

$= \vec{0}$

$(0, \vec{v})$

(b) $B = \vec{\mathbf{v}}\vec{\mathbf{v}}^T = \begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix}$ if $\vec{\mathbf{v}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$?

$\|\vec{v}\|^2 = 2^2 + 3^2 = 4 + 9 = 13$

$(13, \vec{v})$

$\left(0, \begin{bmatrix} -3 \\ 2 \end{bmatrix}\right)$

(c) $C = \frac{4}{7}\vec{\mathbf{w}}\vec{\mathbf{w}}^T - \frac{1}{7}\vec{\mathbf{p}}\vec{\mathbf{p}}^T = \begin{bmatrix} -3 & 4 & 1 \\ 4 & 0 & 4 \\ 1 & 4 & 5 \end{bmatrix}$ if $\vec{\mathbf{w}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\vec{\mathbf{p}} = \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}$?

$C\vec{w} = \frac{4}{7} w w^T w - \frac{1}{7} p p^T w = \frac{4}{7} w (1^2+2^2+3^2) - \frac{1}{7} p (5-8+3) = \frac{4(14)}{7}\vec{w} - \vec{0}$

$\left(8, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)$

$C\vec{p} = \frac{4}{7} w (5-8+3) - \frac{1}{7} p (5^2+(-4)^2+1^2) = -\frac{1}{7}(42) p$

$\left(-6, \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}\right)$

$\vec{q} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1+2+3}{1^2+2^2+3^2}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \frac{5-4+1}{5^2+(4)^2+1^2}\begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{6}{14}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \frac{2}{42}\begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 - \frac{3}{7} - \frac{5}{21} \\ 1 - \frac{6}{7} + \frac{4}{21} \\ 1 - \frac{9}{7} - \frac{1}{21} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \end{bmatrix}$

$\left(0, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right)$

$\lambda_1 \quad \vec{u}_1 \qquad \lambda_2 \quad \vec{u}_2$

2. $A = \begin{bmatrix} 4 & 12 \\ 12 & 4 \end{bmatrix}$ has eigenpairs $(\lambda_i, \vec{\mathbf{u}}_i)$: $(20, \frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix})$ and $(-5, \frac{1}{5}\begin{bmatrix} -4 \\ 3 \end{bmatrix})$.

(a) Find a simple expression for $D(i,j) = \vec{\mathbf{u}}_i^T \vec{\mathbf{u}}_j$

$$\vec{u}_1^T \vec{u}_1 = \frac{1}{5}[3\ 4] \cdot \frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \frac{1}{5}\frac{1}{5}(3^2+4^2) = \frac{25}{25} = 1$$

$$u_2^T u_1 = u_1^T u_2 = \frac{1}{5}[3\ 4] \cdot \frac{1}{5}\begin{bmatrix} -4 \\ 3 \end{bmatrix} = \frac{1}{5}\frac{1}{5}(3(-4)+4(3)) = \frac{0}{25} = 0$$

$$u_2^T u_2 = \frac{1}{5}[-4\ 3] \cdot \frac{1}{5}\begin{bmatrix} -4 \\ 3 \end{bmatrix} = \frac{1}{5}\frac{1}{5}((-4)(-4)+(3)(3)) = \frac{25}{25} = 1$$

$$D(i,j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

(b) Find a simple expression for $L(x,y) = \|\vec{\mathbf{v}}\|^2$ if $\vec{\mathbf{v}} = x\vec{\mathbf{u}}_1 + y\vec{\mathbf{u}}_2$

$$\|\vec{v}\|^2 = v^T v = (x\vec{u}_1 + y\vec{u}_2)^T (x\vec{u}_1 + y\vec{u}_2) = x^2 D(1,1) + xy\, D(1,2) + yx D(2,1) + y^2 D(2,2)$$
$$= x^2(1) + xy(0) + yx(0) + y^2(1)$$
$$= x^2 + y^2$$

(c) Find a simple expression for $E(x,y) = \vec{\mathbf{v}}^T A \vec{\mathbf{v}}$ if $\vec{\mathbf{v}} = x\vec{\mathbf{u}}_1 + y\vec{\mathbf{u}}_2$

$$v^T A \vec{v} = (x\vec{u}_1 + y\vec{u}_2)^T A (x\vec{u}_1 + y\vec{u}_2) = (x\vec{u}_1 + y\vec{u}_2)^T (xA\vec{u}_1 + yA\vec{u}_2)$$
$$= (xu_1 + yu_2)^T (20x\vec{u}_1 - 5y\vec{u}_2) = 20x^2 D(1,1) - 5xy D(1,2) + 20xy D(2,1) - 5y^2 D(2,2)$$
$$= 20x^2 - 5y^2$$

(d) What values of x, y maximize $E(x,y)$ while keeping $L(x,y) = 1$?

if $x^2 + y^2 = 1$ then $y^2 = 1 - x^2$ ~~and~~ $-1 \le x \le 1$

$E = 20x^2 - 5(1-x^2) = 20x^2 - 5 + 5x^2 = 25x^2 - 5$

$E' = 50x$, which is 0 iff $x = 0$

x	y	E	
-1	0	20	max
0	±1	-5	min
1	0	20	

$x = \pm 1$
$y = 0$
$E = 20$
max

(e) What values of x, y minimize $E(x,y)$ while keeping $L(x,y) = 1$?

$x = 0,\ y = \pm 1,\ E = -5$ min

3. $B = \begin{bmatrix} 78 & 52 \\ 4 & 111 \end{bmatrix}$ can be written as $B = \sigma_1 \vec{\mathbf{u}}_1 \vec{\mathbf{v}}_1^T + \sigma_2 \vec{\mathbf{u}}_2 \vec{\mathbf{v}}_2^T$ where $(\sigma_i, \vec{\mathbf{u}}_i, \vec{\mathbf{v}}_i)$ are

$\left(130, \frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix}, \frac{1}{13}\begin{bmatrix} 5 \\ 12 \end{bmatrix}\right)$ and $\left(65, \frac{1}{5}\begin{bmatrix} -4 \\ 3 \end{bmatrix}, \frac{1}{13}\begin{bmatrix} 12 \\ -5 \end{bmatrix}\right)$.

You can use that $\vec{\mathbf{u}}_i^T \vec{\mathbf{u}}_j = \vec{\mathbf{v}}_i^T \vec{\mathbf{v}}_j$ is equal to 1 if $i = j$ and is equal to 0 if $i \neq j$.

(a) Find numbers r, s such that $\vec{\mathbf{b}} = r\vec{\mathbf{u}}_1 + s\vec{\mathbf{u}}_2$ if $\vec{\mathbf{b}} = \begin{bmatrix} 1352 \\ 3211 \end{bmatrix}$

$\vec{u}_i \cdot \vec{b} = r\vec{u}_1^T\vec{u}_i + s\vec{u}_2^T\vec{u}_i = r\,D(1,i) + s\,D(2,i) = \begin{cases} r & \text{if } i=1 \\ s & \text{if } i=2 \end{cases}$

$r = \vec{b}^T\vec{u}_1 = [1352 \quad 3211]\frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \frac{1}{5}(4056 + 12844) = \frac{1}{5}(16900) = 3380$

$s = \vec{b}^T\vec{u}_2 = [1352 \quad 3211]\frac{1}{5}\begin{bmatrix} -4 \\ 3 \end{bmatrix} = \frac{1}{5}(-5408 + 9633) = \frac{1}{5}(4225) = 845$

(b) Find a simple expression for the numbers p, q such that $B\vec{\mathbf{w}} = p\vec{\mathbf{u}}_1 + q\vec{\mathbf{u}}_2$ if $\vec{\mathbf{w}} = x\vec{\mathbf{v}}_1 + y\vec{\mathbf{v}}_2$

$B\vec{w} = (\sigma_1\vec{u}_1\vec{v}_1^T + \sigma_2\vec{u}_2\vec{v}_2^T)(x\vec{v}_1 + y\vec{v}_2) = x\sigma_1\vec{u}_1\vec{v}_1^T\vec{v}_1 + y\sigma_1\vec{u}_1\vec{v}_1^T\vec{v}_2 + x\sigma_2 u_2 v_2^T v_1 + y\sigma_2 v_2^T v_2$

$= x\sigma_1 u_1 (1) + y\sigma_1 u_1 (0) + x\sigma_2 u_2 (0) + y\sigma_2 (1)$

$= \underbrace{x\sigma_1}_{p}\vec{u}_1 + \underbrace{y\sigma_2}_{q}\vec{u}_2$

$p = x\sigma_1 = 130x$

$q = y\sigma_2 = 65y$

(c) Find a simple expression for $L(x, y) = \|\vec{\mathbf{w}}\|^2$ if $\vec{\mathbf{w}} = x\vec{\mathbf{v}}_1 + y\vec{\mathbf{v}}_2$

$\|w\|^2 = (x\vec{v}_1 + y\vec{v}_2)^T(x\vec{v}_1 + y\vec{v}_2) = x^2 D(1,1) + xy\,D(1,2) + yx\,D(2,1) + y^2 D(2,2)$

$= x^2 + y^2$

3. (continued) $B = \begin{bmatrix} 78 & 52 \\ 4 & 111 \end{bmatrix} = \sigma_1 \vec{\mathbf{u}}_1 \vec{\mathbf{v}}_1^T + \sigma_2 \vec{\mathbf{u}}_2 \vec{\mathbf{v}}_2^T$ where $(\sigma_i, \vec{\mathbf{u}}_i, \vec{\mathbf{v}}_i)$ are

$$\left(130, \tfrac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix}, \tfrac{1}{13}\begin{bmatrix} 5 \\ 12 \end{bmatrix}\right) \quad \text{and} \quad \left(65, \tfrac{1}{5}\begin{bmatrix} -4 \\ 3 \end{bmatrix}, \tfrac{1}{13}\begin{bmatrix} 12 \\ -5 \end{bmatrix}\right).$$

(d) Find a simple expression for $E(x, y) = \|B\vec{\mathbf{w}} - \vec{\mathbf{b}}\|^2$ with x, y as in (a) and (b)

$Bw = 130x\,\vec{u}_1 + 65y\,\vec{u}_2$

$-\ b = -3380\,\vec{u}_1 - 845\,\vec{u}_2$

$\|B\vec{\omega} - \vec{b}\|^2 = (130x - 3380)^2 D(1,1) + \sim D(1,2) + \sim D(2,1) + (65y - 845)^2 D(2,2)$

$= (130x - 3380)^2 + (65y - 845)^2$

Simplifying doesn't help

(e) Find x, y such that $E(x, y)$ is as small as possible

$0^2 + 0^2$ is min, $x = \frac{338\cancel{0}}{13\cancel{0}}$ $\quad y = \frac{845}{65}$

$13\overline{)338} = 26$, 26, 78 $\quad 65\overline{)845} = 13$, 65, 195, 195, 0

$x = 26 \quad y = 13$

$\left(\|w\|^2 = 26^2 + 13^2 > 26^2 \text{ by kind of a lot} \right)$

Bonus: As in #3, find $\vec{\mathbf{w}}$ so that $\|B\vec{\mathbf{w}} - \vec{\mathbf{b}}\| < 25\%\|\vec{\mathbf{b}}\|$ and $\|\vec{\mathbf{w}}\| \leq 26$. Equivalently, find x, y such that $E(x, y) < 0.25^2\|\vec{\mathbf{b}}\|^2 \approx 750000$ but $L(x, y) \leq 26^2$.

$x = 26 \quad y = 0$ has $E = (845)^2 < 750000$

even better is

$x = 24.01\ldots \quad y = \ldots$

Calc1 is ok, but leads to quartic