MA322-007 May 5 - Final Exam Name: ____________________

1. What are the eigenpairs of:

(a) $A = 25\vec{\mathbf{u}}\vec{\mathbf{u}}^T = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$ if $\vec{\mathbf{u}} = \frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$?

(b) $B = \vec{\mathbf{v}}\vec{\mathbf{v}}^T = \begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix}$ if $\vec{\mathbf{v}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$?

(c) $C = \frac{4}{7}\vec{\mathbf{w}}\vec{\mathbf{w}}^T - \frac{1}{7}\vec{\mathbf{p}}\vec{\mathbf{p}}^T = \begin{bmatrix} -3 & 4 & 1 \\ 4 & 0 & 4 \\ 1 & 4 & 5 \end{bmatrix}$ if $\vec{\mathbf{w}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\vec{\mathbf{p}} = \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}$?

2. $A = \begin{bmatrix} 4 & 12 \\ 12 & 4 \end{bmatrix}$ has eigenpairs $(\lambda_i, \vec{\mathbf{u}}_i)$: $(20, \frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix})$ and $(-5, \frac{1}{5}\begin{bmatrix} -4 \\ 3 \end{bmatrix})$.

(a) Find a simple expression for $D(i,j) = \vec{\mathbf{u}}_i^T \vec{\mathbf{u}}_j$

(b) Find a simple expression for $L(x,y) = \|\vec{\mathbf{v}}\|^2$ if $\vec{\mathbf{v}} = x\vec{\mathbf{u}}_1 + y\vec{\mathbf{u}}_2$

(c) Find a simple expression for $E(x,y) = \vec{\mathbf{v}}^T A \vec{\mathbf{v}}$ if $\vec{\mathbf{v}} = x\vec{\mathbf{u}}_1 + y\vec{\mathbf{u}}_2$

(d) What values of x, y maximize $E(x,y)$ while keeping $L(x,y) = 1$?

(e) What values of x, y minimize $E(x,y)$ while keeping $L(x,y) = 1$?

3. $B = \begin{bmatrix} 78 & 52 \\ 4 & 111 \end{bmatrix}$ can be written as $B = \sigma_1 \vec{\mathbf{u}}_1 \vec{\mathbf{v}}_1^T + \sigma_2 \vec{\mathbf{u}}_2 \vec{\mathbf{v}}_2^T$ where $(\sigma_i, \vec{\mathbf{u}}_i, \vec{\mathbf{v}}_i)$ are

$$\left(130, \frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix}, \frac{1}{13}\begin{bmatrix} 5 \\ 12 \end{bmatrix}\right) \quad \text{and} \quad \left(65, \frac{1}{5}\begin{bmatrix} -4 \\ 3 \end{bmatrix}, \frac{1}{13}\begin{bmatrix} 12 \\ -5 \end{bmatrix}\right).$$

You can use that $\vec{\mathbf{u}}_i^T \vec{\mathbf{u}}_j = \vec{\mathbf{v}}_i^T \vec{\mathbf{v}}_j$ is equal to 1 if $i = j$ and is equal to 0 if $i \neq j$.

(a) Find numbers r, s such that $\vec{\mathbf{b}} = r\vec{\mathbf{u}}_1 + s\vec{\mathbf{u}}_2$ if $\vec{\mathbf{b}} = \begin{bmatrix} 1352 \\ 3211 \end{bmatrix}$

(b) Find a simple expression for the numbers p, q such that $B\vec{\mathbf{w}} = p\vec{\mathbf{u}}_1 + q\vec{\mathbf{u}}_2$ if $\vec{\mathbf{w}} = x\vec{\mathbf{v}}_1 + y\vec{\mathbf{v}}_2$

(c) Find a simple expression for $L(x, y) = \|\vec{\mathbf{w}}\|^2$ if $\vec{\mathbf{w}} = x\vec{\mathbf{v}}_1 + y\vec{\mathbf{v}}_2$

3. (continued) $B = \begin{bmatrix} 78 & 52 \\ 4 & 111 \end{bmatrix} = \sigma_1 \vec{\mathbf{u}}_1 \vec{\mathbf{v}}_1^T + \sigma_2 \vec{\mathbf{u}}_2 \vec{\mathbf{v}}_2^T$ where $(\sigma_i, \vec{\mathbf{u}}_i, \vec{\mathbf{v}}_i)$ are

$$\left(130, \tfrac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix}, \tfrac{1}{13}\begin{bmatrix} 5 \\ 12 \end{bmatrix}\right) \qquad \text{and} \qquad \left(65, \tfrac{1}{5}\begin{bmatrix} -4 \\ 3 \end{bmatrix}, \tfrac{1}{13}\begin{bmatrix} 12 \\ -5 \end{bmatrix}\right).$$

(d) Find a simple expression for $E(x, y) = \|B\vec{\mathbf{w}} - \vec{\mathbf{b}}\|^2$ with x, y as in (a) and (b)

(e) Find x, y such that $E(x, y)$ is as small as possible

Bonus: As in #3, find $\vec{\mathbf{w}}$ so that $\|B\vec{\mathbf{w}} - \vec{\mathbf{b}}\| < 25\% \|\vec{\mathbf{b}}\|$ and $\|\vec{\mathbf{w}}\| \leq 26$. Equivalently, find x, y such that $E(x, y) < 0.25^2 \|\vec{\mathbf{b}}\|^2 \approx 750000$ but $L(x, y) \leq 26^2$.