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On Some Almost Everywhere Symmetry Theorems

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1. Introduction

In this note we consider problems of the following type: Let D be a bounded domain in Euclidean n space (R^n) and suppose u is a positive solution to $Lu = 0$ in D with $u(x) \to 0$, $|\nabla u(x)| \to$ constant, as $x \to \partial D$, in an appropriate sense. Show that D is a ball and u is radially symmetric about the center of D. This problem was solved very elegantly by Serrin [25] under the assumption that ∂D is of class C^2, $u \in C^2(\bar{D})$, and

$$Lu = a(u, |\nabla u|)\Delta u + b(u, |\nabla u|)u_{x_i}u_{x_j}u_{x_ix_j} - c(u, |\nabla u|) = 0, \tag{1.1}$$

where a, b, c, are continuously differentiable in each variable. Also L is elliptic and repeated indices denote summation from 1 to n. Immediately following Serrin's article, Weinberger [30] gave another proof of Serrin's theorem when $Lu = \Delta u + 1 = 0$. From Weinberger's proof it is clear that the assumption, $\partial D \in C^2$ is unnecessary in this special case, provided the boundary conditions are interpreted as,

(A) Given $\epsilon > 0$ there exists a neighborhood N of ∂D such that $u(x) < \epsilon$, $||\nabla u(x)| - a| < \epsilon$, when $x \in N$.

In (A), a denotes a positive constant.

Garafalo and the first author [17] extended Weinberger's proof to weak solutions u in D of

$$\operatorname{div}(|\nabla u|^{-1} f'(|\nabla u|)\nabla u) = -1, \tag{1.2}$$

where $f \in C^2(0, \infty)$ and for some p, $1 < p < \infty$, $c_1, c_2 > 0$,

$$c_1(t^p - 1) \le t f'(t) \le c_2(t^p + 1), \ t > 0, \tag{1.3}$$

$$c_1 \le t f''(t)/f'(t) \le c_2, \ t > 0, \tag{1.4}$$

$$\int_D |\nabla u|^p dx < +\infty. \tag{1.5}$$

Thus for u satisfying (1.2), (1.5), with boundary values as in (A) it is still true that D is a ball and u is radially symmetric about the center of D.

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The second author [29] has recently shown that the same conclusion is valid when u satisfies (1.1) with boundary values as in (A). His results are also valid for equations of the form

$$\operatorname{div}(|\nabla u|^{-1} f'(|\nabla u|)\nabla u) = d(u, |\nabla u|), \tag{1.6}$$

where f, u, are as in (1.3)-(1.5), provided f has continuous third derivatives and d is continuously differentiable in each of its arguments. Details of this proof will appear elsewhere, however we give a brief outline here.

Most of the effort in the proof is devoted to formalizing the statement: If near the boundary of a general region, u is C^2 and satisfies an elliptic equation, as well as (A), then this is enough to prove some boundary regularity. More specifically,

Theorem A. *Let u satisfy (1.1) in D with boundary values as in (A). Then given $x_0 \in \partial D$ there exists $r > 0$ so that $B(x_0, r) \cap D$ consists of at most two components D', D'', satisfying*

(i) $x_0 \in \partial D' \cap \partial D''$

(ii) $\partial D' \cap B(x_0, r)$, $\partial D'' \cap B(x_0, r)$ *are $C^{2,\alpha}$ surfaces for some* α, $0 < \alpha < 1$

(iii) $\partial D \cap B(x_0, r) = (\partial D' \cap B(x_0, r)) \cup (\partial D'' \cap B(x_0, r))$.

In Theorem A, $B(x_0, r) = \{x : |x - x_0| < r\}$. Note that (i)-(iii) does not imply ∂D is $C^{2,\alpha}$ in the usual sense, but intuitively, ∂D is $C^{2,\alpha}$ from "each side".

To begin the proof of Theorem A, the second author first studies curves in D along which ∇u is tangent (the trajectories of u). If $t > 0$ is small enough it follows from (A) and positivity of u that for each x_0 in $\{x : u(x) = t\}$ there exists exactly one trajectory beginning at x_0 and ending at $\psi(x_0)$ in ∂D, along which u decreases. Moreover, it is easily seen that ψ is a continuous function from $\{x : u(x) = t\}$ onto ∂D. Note from (A) and the implicit function theorem that $\{x : u(x) = t\}$ has a finite number of components. Hence ∂D has a finite number of components, since ψ is continuous. Using these trajectories and estimates on u, it can also be shown for given $x_0 \in \partial D$ and $\sigma > 0$ small, that there exists $r > 0$, a direction ν, and a region D' with

(a) $x_0 \in \partial D'$

(b) $D' \cap B(x_0, r) \subseteq D \cap B(x_0, r) \cap H(\sigma r)$

(c) $D' \cap H(-\sigma r) \cap B(x, r) = H(-\sigma r) \cap B(x_0, r)$

where $H(t) = \{x : (x - x_0) \cdot \nu < t\}$. (a)-(c) are conditions similar to the flatness condition introduced by Alt and Cafferelli in [3] (see also [4]). The method used in these papers can be suitably modified to show that preliminary flatness in the above sense can be improved in a smaller ball about x_0 of radius $c_0 r$ $(0 < c_0 < 1)$.

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Iterating this result it follows that ∂D is $C^{1,\alpha}$ from each side at x. To improve the regularity a hodograph type transformation of Kinderlehrer and Nirenberg [21] is then used so that regularity results of Agmon-Douglis-Nirenberg [1] can be applied to obtain ∂D is $C^{2,\alpha}$ from each side. Finally, Serrin's argument can be adapted, to obtain that D is a ball.

2. Almost Everywhere Symmetry Theorems

Next we consider some symmetry problems where boundary condition (A) can be weakened at the expense of assuming more about ∂D. To this end we say that a bounded domain D is Lipschitz provided for each $y \in \partial D$ there is $r = r(y) > 0$ and a truncated right circular cylinder Z, with radius r, y on the axis at the center of the cylinder, and the property: If the axis of the cylinder is chosen parallel to the x_n axis, then there exists a Lipschitz function ψ on R^{n-1} $n \geq 2$, such that $\partial D \cap Z = Z \cap \{(x', \psi(x')) : x' \in R^{n-1}\}$ and $D \cap Z = Z \cap \{(x', x_n) : x_n > \psi(x')\}$. Also the bases of Z are at some positive distance from ∂D. From compactness we see that ∂D is contained in the union of a finite number of such cylinders. We say that u defined on $D \cap Z$ has a radial limit at $y = (y', \psi(y'))$ in $\partial D \cap Z$ provided $\lim_{t \to 0} u(y', \psi(y') + |t|)$ exists finitely. If $0 < r_0 < r$ and $(x', \psi(x') + 2r_0)$ is in D whenever $(x', \psi(x')) \in \partial D \cap Z$ put

$$u^*(y) = u^*(y', \psi(y')) = \sup_{|t| < r_0} |u(y', \psi(y') + |t|)|.$$

u^* is called the radial maximal function of u relative to r_0. Let H^k denote k dimensional Hausdorff measure. We note that on $\partial D \cap Z$ we have for $y \in \partial D$,

$$dH^{n-1}y = \sqrt{1 + |\nabla\psi|^2(y')}dy'.$$

For D a bounded Lipschitz domain replace boundary condition (A) by

(A*) $\lim_{x \to y} |\nabla u(x)| = a$, radially, for H^{n-1} a.e. $y \in \partial D$, while $u(x) \longrightarrow 0$ continuously as $x \to \partial D$ (as in (A)).

We remark that the assumption $u \to 0$ continuously, is true in a Lipschitz domain for a solution u to any of the above partial differential equations provided u can be approximated arbitrarily close in the norm of a certain Sobolev space (depending on the P.D.E.) by infinitely differentiable functions with compact support in D. In case u is Green's function for D we can prove almost everywhere symmetry theorems rather easily. More specifically,

Theorem 1. *Let $D \subseteq R^n$ be a bounded Lipschitz domain and suppose that u is Green's function for D with pole at $0 \in D$. If u satisfies boundary condition (A*), then D is a ball with center at 0 and u is radially symmetric about* 0.

Our proof of Theorem 1 will involve, among other things, a simple barrier argument.

Proof. Let $\omega_2 = (2\pi)^{-1}$, and $\omega_n = \{(n-2)H^{n-1}[\partial B(0,1)]\}^{-1}$ if $n > 2$. Then by definition, $u(x) - \omega_n|x|^{2-n}$ is harmonic and bounded in D for $n > 2$ (if $n = 2$, replace $|x|^{2-n}$ by $\log(|x|^{-1})$) while $u(x) \to 0$ as $x \to \partial D$ in the sense of Perron-Wiener-Brelot, so continuously for a Lipschitz domain (see [18, section 1.6.3]). The first step is to show that

$$aH^{n-1}(\partial D) = 1. \tag{2.1}$$

To prove (2.1) let Ω be a smooth domain with $\bar{\Omega}$ (the closure of Ω) contained in D. Then from the divergence theorem and the usual limiting argument it follows that

$$-\int_{\partial\Omega} \nabla u \cdot \nu dH^{n-1} = 1 \tag{2.2}$$

provided $0 \in \Omega$. Here ν is the outer unit normal to Ω. The idea now is to use (A*) and choose a sequence of smooth domains $(\Omega_j)_1^\infty$ such that $\bar{\Omega}_j \subseteq \Omega_{j+1}$, $\bigcup_{j=1}^{\infty}\Omega_j = D$, and with the property that (2.2) with $\Omega = \Omega_j$ approaches (2.1) as $j \to \infty$. To justify the limits we shall need to know that:

(+) The radial maximal function of ∇u for some $r_0 > 0$, taken componentwise and denoted $(\nabla u)^*$, is square integrable with respect to H^{n-1} measure on $\partial D((\nabla u)^* \in L_2(\partial D))$,

(++) $\nabla u(x) \to -an(y)$ radially for H^{n-1} almost every $y \in \partial D$ where $n(y) = \frac{(\nabla\psi(y'),-1)}{\sqrt{1+|\nabla\psi(y')|^2}}$, $y \in \partial D$, is the outer unit normal to D.

(+) and (++) were proved by Dahlberg in [9, Thm. 3], [10, Thm. 2]. The $L_2(\partial D)$ norm of $(\nabla u)^*$ depends only on the Lipschitz norm of the functions ψ defining ∂D. Another way to prove (+) and (++) is to note that $u(x) - \omega_n|x|^{2-n}$ has tangential derivatives on ∂D which are in $L_2(\partial D)$. It then follows from a theorem of Verchota (see [28, Cor. 3.5]) that u can be represented as a single layer potential for which (+) and (++) hold. To get $(\Omega_j)_1^\infty$, we first smoothly approximate ψ locally from above in such a way that the sequence of approximants has uniformly bounded Lipschitz norm. Second piece together the resulting graphs to obtain $(\Omega_j)_1^\infty$ (see [28, Thm. 1.12] for more details). From (+), (++), and Lebesgue dominated convergence we deduce that (2.1) is the limit of (2.2) as $j \to \infty$.

Next given $y \in \partial D$ we claim that

$$\limsup_{x \to y} |\nabla u(x)| \le a \tag{2.3}$$

To prove this claim, we observe that $|\nabla u|$ is subharmonic in $D \cap B(y,r)$ for $r > 0$ small enough. Let $(\delta_j)_1^\infty$ be a sequence of small positive numbers with $\lim_{j\to\infty} \delta_j = 0$, and put

$$D + (0,\delta_j) = \{x + (0,\delta_j) = (x', x_n + \delta_j) : x \in D\}.$$

Proof. Let $\omega_2 = (2\pi)^{-1}$, and $\omega_n = \{(n-2)H^{n-1}[\partial B(0,1)]\}^{-1}$ if $n > 2$. Then by definition, $u(x) - \omega_n|x|^{2-n}$ is harmonic and bounded in D for $n > 2$ (if $n = 2$, replace $|x|^{2-n}$ by $\log(|x|^{-1})$) while $u(x) \to 0$ as $x \to \partial D$ in the sense of Perron-Wiener-Brelot, so continuously for a Lipschitz domain (see [18, section 1.6.3]). The first step is to show that

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$$D + (0, \delta_j) = \{x + (0, \delta_j) = (x', x_n + \delta_j) : x \in D\}.$$

Choose a sequence of smooth domains $(\Omega_j)_1^\infty$ whose boundaries are locally the graphs of Lipschitz functions with uniformly bounded Lipschitz norm and for which

$$(D + (0, \delta_j)) \cap B(y, 2r) \subseteq \Omega_j \subseteq D \cap B(y, 8r), \quad j = 1, 2, ...$$

Let h_j be the least harmonic majorant of

$$(|\nabla u| - a)^\wedge = \max[|\nabla u| - a, 0]$$

in Ω_j and let $g_j(\cdot, w)$, $j = 1, 2, ...,$ be Green's function for Ω_j with pole at $w \in \Omega_j$. Then from the classical Poisson integral formula for smooth domains (see [18, section 1.5]) we have

$$(|\nabla u| - a)^\wedge(x) \leq h_j(x) = \int_{\partial\Omega_j} |\nabla g_j|(z, x)(|\nabla u| - a)^\wedge(z) dH^{n-1}z \quad (2.4)$$

whenever $x \in \Omega_j$. From (+) we see for given $x \in B(y, r) \cap D$ and $j \geq j_0$ large, that

$$\int_{\partial\Omega_j} |\nabla g_j|^2(z, x) dH^{n-1}z \leq k,$$

where k may depend on $x, r,$ and D but is independent of $j \geq j_0$. Using this inequality, (+), (++), (A*), and letting $j \to \infty$ in (2.4) we see from Hölder's inequality and Lebesgue dominated convergence that for properly chosen $(\Omega_j)_1^\infty$, we have

$$(|\nabla u| - a)^\wedge(x) \leq h(x), \quad x \in D \cap B(y, r),$$

where h is harmonic in $D \cap B(y, r)$ with boundary value 0 on $\partial D \cap B(y, r)$ in the sense of Perron-Wiener-Brelot. Because a Lipschitz domain is regular for the Dirichlet problem, it follows first that h has a continuous extension to $(D \cup \partial D) \cap B(y, r)$ with $h \equiv 0$ on $\partial D \cap B(y, r)$ and there upon that (2.3) is true.

Finally we are in a position to prove Theorem 1. Let d be the distance from 0 to ∂D and let G be Green's function for $B(0, d)$ with pole at 0. Then clearly from the minimum principle for harmonic functions either $u - G$ is a positive harmonic function in $B(0, d)$ or $u \equiv G$ (so $D = B(0, d)$). Let $x_0 \in \partial B(0, d) \cap \partial D$ and observe from the mean value theorem of calculus that

$$-\frac{\partial}{\partial t} u(tx_0) \geq -\frac{\partial}{\partial t} G(tx_0),$$

for some $t < 1$ and arbitrarily near 1. Using this inequality, (2.1), and (2.3) we get

$$\frac{1}{H^{n-1}(\partial D)} = a \geq -\limsup_{t \to 1} \frac{\partial}{\partial t} u(tx_0) \geq -\lim_{t \to 1} \frac{\partial G}{\partial t}(tx_0) = a^*. \quad (2.5)$$

Since G satisfies boundary condition (A*) we can repeat the argument leading to (2.1) to obtain

$$\frac{1}{H^{n-1}(\partial B(0,d))} = a^*.$$

This equality and (2.5) yield

$$H^{n-1}(\partial D) \le H^{n-1}(\partial B(0,d)). \tag{2.6}$$

From the classical isoperimetric inequality:

$$H^n(D)^{1-1/n} \le (\nu_n)^{-1/n} n^{(1/n-1)} H^{n-1}(\partial D),$$

where $\nu_n = H^{n-1}(\partial B(0,1))$, we see that (2.6) can hold only if $D = B(0,d)$. Another way to see that (2.6) implies $D = B(0,d)$ is to project ∂D radially onto $\partial B(0,d)$ and use the fact that surface area decreases under this projection, unless $\partial D = \partial B(0,d)$. Q.E.D.

We remark that a proof of Theorem 1 for smooth domains can be given using Serrin's original argument or as in [23].

Theorem 1 generalizes to certain domains in Hyperbolic and Spherical n space (denoted H_n, S_n, respectively). In the usual way we identify H_n with $B(0,1)$ under the Riemannian metric,

$$g_{ij}(x) = 4\delta_{ij}(1-|x|^2)^{-2}, \; x \in B(0,1),$$

and S_n with R^n under the Riemannian metric,

$$g_{ij}(x) = 4\delta_{ij}(1+|x|^2)^{-2}, \; x \in R^n, \; 1 \le i, \; j \le n.$$

Here, δ_{ij} denotes the Kronecker delta and we also use $(g^{ij}) = (g_{ij})^{-1}$, $g = \det(g_{ij})$. The definition of a bounded Lipschitz domain D is unchanged, provided bounded is interpreted with respect to the usual distance function for H_n, S_n. By definition, if u denotes Green's function for $D \subseteq H_n$ or $D \subseteq S_n$ with pole at $0 \in D$, then $0 = \tilde{\Delta} u(x)$ for $x \in D - \{0\}$, where

$$\tilde{\Delta} u = g^{-1/2}\frac{\partial}{\partial x_i}(g^{1/2} g^{ij} u_{x_j}) = \frac{1}{4}(1 \pm |x|^2)^n \frac{\partial}{\partial x_i}\left[(1 \pm |x|^2)^{2-n}\frac{\partial u}{\partial x_i}\right]. \tag{2.7}$$

Here the $+$ sign is taken if $D \subseteq S_n$ and the $-$ sign if $D \subseteq H_n$. Moreover, if θ has compact support in D, then

$$\int_D (\nabla\theta \cdot \nabla u)(1 \pm |x|^2)^{2-n} dH^n x = \theta(0). \tag{2.8}$$

Again the + sign is taken if $D \subseteq S_n$ and the − sign if $D \subseteq H_n$. Computing the Hyperbolic and Spherical gradients with respect to each (g_{ij}) we find that (A*) should be replaced with

(A**) $\lim_{x \to y} |\nabla u(x)| = a(1 \pm |y|^2)^{-1}$, for H^{n-1} a.e. $y \in \partial D$, while $u(x) \to 0$ continuously as $x \to \partial D$.

The sign convention is the same as above. With this notation we prove

Theorem 2. *Let u be Green's function for a bounded Lipschitz domain D with pole at $0 \in D$. If $D \subseteq H_n$, then D is a Hyperbolic ball while if $D \subseteq S_n$ and $H^n(D) \leq \frac{1}{2} H^n(S_n)$ (considered as sets on the unit sphere in R^{n+1}), then D is a Spherical ball.*

Proof. We argue as in Theorem 1. In place of (2.1) we show

$$a \int_{\partial D} (1 \pm |x|^2)^{1-n} dH^{n-1} x = 1, \tag{2.9}$$

where again the + sign is taken if $D \subseteq S_n$ and the − sign if $D \subseteq H_n$. The proof of (2.9) is essentially the same as the proof of (2.1) once we show (+) and (++) (with $\nabla u(x)$ replaced by $(1 \pm |x|^2)\nabla u(x)$) are valid in this situation. One way to prove (+) and (++) is to use the mapping, $(x', \psi(x') + \lambda) \to (x', \lambda)$, $x' \in R^{n-1}$, $\lambda > 0$, to map, $B(y, r) \cap D$, $y \in \partial D$, onto a portion of a half space. If $q(x', \lambda) = u(x', \psi(x') + \lambda)$, then q satisfies a divergence type equation for which the results of Fabes, Jerison, and Kenig [15] can be applied (see also [16, Thm. 3.5]). Doing this, we get (+), (++). Another proof of (+), (++), can be given, by using Rellich-Necas-Pohożaev type inequalities (see [17, 19, 24,28]) to show that $|\nabla u|$ has a certain weak limit in $L_2(\partial D)$. The rest of Dahlberg's proof can then essentially be repeated to get (+), (++). Using (+), (++), we obtain (2.1) in the same way as previously. To prove

$$\limsup_{x \to y} |\nabla u(x)| \leq a(1 \pm |y|^2)^{-1}, \tag{2.10}$$

for all $y \in \partial D$, let $v(x) = (\epsilon + |\nabla u(x)|^2)^{1/2}$, $x \in D$, for given $\epsilon > 0$. We differentiate (2.7) with respect to x_k, $1 \leq k \leq n$. From the resulting equalities and (2.7) we deduce for r, $\epsilon > 0$ small enough,

$$\Delta v(x) \geq -c(|\nabla u(x)| + 1), \quad x \in D \cap B(y, 4r). \tag{2.11}$$

Using (2.11), the Riesz representation formula for subharmonic functions (see [18, Thm. 3.14]) and arguing as in the proof of (2.3) we find that

$$v(x) \leq p(x) + b(x), \quad x \in B(y, r) \cap D,$$

where b is harmonic in $B(y, r) \cap D \subseteq R^n$ with

$$\lim_{x \to z} b(x) = [a^2(1 \pm |z|^2)^{-2} + \epsilon]^{1/2}, \quad z \in \partial D \cap B(y, r), \tag{2.12}$$

and

$$p(x) = c \int_{B(y,2r)\cap D} g(y,x)(|\nabla u|(y)+1)dy, \quad x \in D \cap B(y,2r).$$

Here, $g(\cdot, x)$ is Green's function for $D \subseteq R^n$ with pole at x. Now from (2.8) and (A**) it is easily seen for $r > 0$ small enough that $|\nabla u|$ is square integrable with respect to H^n measure on $D \cap B(y, 2r)$. Using this fact and Sobolev's Theorem (see [27, Ch. 5]) we obtain p is integrable to the m-th power in $B(y, 2r)$ where $m = 2n/(n-2)$ if $n > 2$ and $n < \infty$ if $m = 2$. Since b has continuous boundary values on $B(y, r) \cap \partial D$, we conclude that v is integrable to the m-th power in $B(y, r/2)$. Using Sobolev's Theorem again and repeating the above argument a finite number of times we see that $|\nabla u| \le k < \infty$ in $B(y, r/2)$. From this inequality, (A**), and Harnack's inequality (see (4.25)) it follows easily that $\lim_{x\to z} p(x) = 0$, $z \in \partial D \cap B(y, r)$. Using this equality, (2.12), and letting $\epsilon \to 0$, we find (2.10) is true.

To prove Theorem 2, let G be Green's function for $B(0, d)$ with pole at 0, where d is the distance from 0 to ∂D in each geometry. Then G satisfies (2.7), (2.8), with u replaced by G and D by $B(0, d)$. From uniqueness of G, we see that G is radially symmetric. From the maximum principle for elliptic P.D.E.'s we also have $G \le u$ in $B(0, d)$. Hence if $x_0 \in \partial B(0, d) \cap \partial D$, then from (2.10) we deduce as in (2.5)

$$a(1 \pm |x_0|^2)^{-1} \ge -\limsup_{t\to 1} \frac{\partial}{\partial t} u(tx_0) \ge -\lim_{t\to 1} \frac{\partial G}{\partial t}(tx_0) = a^*(1 \pm |x_0|^2)^{-1}, \tag{2.13}$$

so $a \ge a^*$. We note that (2.9) also holds with a replaced by a^* and D by $B(0, d)$ because G satisfies the same hypotheses as u. Thus

$$1 = a^* \int_{\partial B(0,d)} (1 \pm |x|^2)^{1-n} dH^{n-1}x = a \int_{\partial D} (1 \pm |x|^2)^{1-n} dH^{n-1}x. \tag{2.14}$$

Recall that the + sign is taken if $D \subseteq S_n$ and the − sign if $D \subseteq H_n$. Projecting ∂D onto $\partial B(0, d)$ and using (2.13) we see for $D \subseteq H_n$ that (2.14) can only hold when $D = B(0, d)$. If $D \subseteq S_n$, we identify D with its spherical image by way of stereographic projection. Then from the classical spherical isoperimetric inequality we have

$$H^{n-1}(\partial D) \ge H^{n-1}(\partial P), \tag{2.15}$$

where P is a spherical ball (cap) with the same H^n measure as D. Also from (2.13), (2.14), we see that

$$H^{n-1}(\partial D) \le H^{n-1}(\partial B(0, d)), \tag{2.16}$$

for D, $B(0, d)$, contained in the unit sphere of R^{n+1}. Finally observe that if $P_1 \subseteq P_2 \subseteq Q$, where P_1, P_2 are spherical balls (caps), and Q is a hemisphere, then $H^{n-1}(\partial P_1) \le H^{n-1}(\partial P_2)$. In view of this fact, (2.15), and

(2.16) we conclude $D = B(0,d)$, whenever $H^n(D) \le \frac{1}{2}H^n(S_n)$. Q.E.D.

3. Parabolic Symmetry Theorems

Let $D \subseteq R^n$ be a bounded Lipschitz domain, as in section 2. Let u be a function defined on $D \times (0,T)$, $0 < T < \infty$. If $(y,t) \in \partial D \times (0,T)$, define the radial limit of u as in section 2 relative to $D \times \{t\}$. Replace (A*) by

(A⁺) $\lim_{x \to y} |\nabla u|(x,t) = a(t)$, radially for H^n almost every $(y,t) \in \partial D \times (0,T)$, $0 < t < T$, while $u(x,t) \to 0$ continuously as $(x,t) \to \partial D \times [0,T)$.

We prove

Theorem 3. *Let D be a Lipschitz domain, $0 \in D$, and suppose that u is Green's function for the heat equation in $D \times [0,T)$ with pole at $(0,0)$. Then D is a ball with center at $(0,0)$ and for fixed t, $0 < t < T$, $u(\cdot,t)$ is radially symmetric about the center of D.*

We remark that by definition

$$p(x,t) = u(x,t) - (4\pi t)^{-n/2} e^{-(|x|^2/4t)}, \quad (x,t) \in D \times [0,T),$$

is a bounded solution to the heat equation in $D \times [0,T)(\Delta_x p = p_t)$ and u has boundary value zero in the Perron-Wiener-Brelot sense. Here Δ_x and ∇_x denote the Laplacian and gradient with respect to $x \in R^n$, only. Since every point in $\partial D \times [0,T)$ is regular for the heat equation, it follows that the assumption $u(x) \to 0$ continuously as $x \to \partial D \times [0,T)$ is automatic in this case. As motivation for Theorem 3, we also remark that Alessandrini and Garofalo generalized Serrin's theorem to smooth cylinders in [2], so Theorem 1 should have a similar generalization.

Proof. The proof of Theorem 3 is similar to the proof of Theorem 1. In place of (2.1) we show

$$H^{n-1}(\partial D)\left(\int_0^{T_1} a(t)dt\right) + \int_D u(x,T_1)dH^n x = 1, \tag{3.1}$$

whenever $0 < T_1 < T$. For this purpose let Ω be a smooth domain with $\bar{\Omega} \subset D$. Applying the divergence theorem in $[\Omega \times (0,T)] - E$, to $\nabla_x u$ where

$$E = \{(x,t) : |t|^{1/3}, |x| \le \epsilon\},$$

and letting $\epsilon \to 0$ we get

$$-\int_0^{T_1}\left[\int_{\partial\Omega} \nabla_x u \cdot \nu dH^{n-1}x dt + \int_\Omega u(x,T_1)dH^n x = 1,\right. \tag{3.2}$$

where ν is the outer unit normal to Ω. As in section 2 the idea now is to approximate D by a sequence of smooth domains $(\Omega_j)_1^\infty$ in such a way that (3.2) with $\Omega = \Omega_j$ approaches (3.1) as $j \to \infty$. To do this we need to show as in section 2 that

$(\cdot)$ $(\nabla_x u)^* \in L_2[\partial D \times (0,T)]$

$(\cdot\cdot)$ $\nabla_x u(x,t) \to -a(t)n(y)$ for H^n a.e. $(y,t) \in \partial D \times (0,T)$, where $n(y)$ is as in $(++)$.

$(\cdot)$ and $(\cdot\cdot)$ follow from the work of Fabes and Salsa [14, Thms. 1.3, 1.4, and 3.2]. They generalized Dahlberg's Theorem to the heat equation in cylinders of the form $D \times (0,T)$. $(\cdot)$ and $(\cdot\cdot)$ also are a consequence of a theorem of Brown (see [5, Thm. 6.1], [6]) who among other results obtained analogues of Verchota's work for the heat equation in cylinders. From $(\cdot)$ and $(\cdot\cdot)$ we conclude that (3.1) is the limit of (3.2) with $\Omega = \Omega_j$ as $j \to \infty$. Next we claim for H^1 almost every $t \in (0,T)$ that

$$\limsup_{(x,t)\to(y,t)} |\nabla u(x,t)| \le a(t), \quad \text{radially, for all } y \in \partial D. \tag{3.3}$$

The proof of (3.3) is essentially the same as the proof of (2.3) if we assume for example that $a(t)$ is continuous on $(0,T)$. Otherwise, we must use slightly deeper results of the above authors: Let h be the unique solution to the heat equation in $D \times (0,T)$, with $h^* \in L_2[\partial D \times (0,T)]$, $h(x,0) = 0$ continuously, $x \in D$, and

$$\lim_{(x,t)\to(y,t)} h(x,t) = a(t) \quad \text{radially, for } H^n \text{ a.e. } (y,t) \text{ in } \partial D \times (0,T).$$

The existence of h follows from $(\cdot)$ and either [14, Thm. 3.2] or [5, Thm. 8.1]. Since $|\nabla u|$ is a subsolution to the heat equation it follows as in section 2 that for given $\epsilon > 0$ and $(y,t) \in \partial D \times (0,T)$, there exists $r_1 > 0$ with

$$|\nabla u|(x,s)| \le h(x,s) + \epsilon, \quad (x,s) \in [B(y,r_1) \cap D] \times (0,T).$$

From this inequality we see it suffices to prove

$$\lim_{x\to y} h(x,t) = a(t), \quad \text{radially, for a.e. } t \in (0,T), \text{ and all } y \in \partial D, \tag{3.4}$$

in order to get (3.3). From $(\cdot), (\cdot\cdot)$, we deduce that parabolic measure (see [31, section 2]) with respect to (0,T) is equal to

$$a(T-t)dH^{n-1}y dt \quad \text{for } H^n \text{ a.e. } (y,t) \in \partial D \times (0,T).$$

From this deduction and a theorem of Kemper ([20, Thm. 2.6]) it follows that (3.4) actually holds whenever

$$\lim_{\delta\to 0}\left[\int_0^\delta |a(s)-a(t)|a(T-s)ds \Big/ \left(\int_0^\delta a(T-s)ds\right)\right] = 0, \tag{3.5}$$

$0 < t < T$. Finally, (3.5) follows from the usual Lebesgue differentiation Theorem, $(\cdot)$, and $(\cdot\cdot)$, once it is shown $a(t) \neq 0$ for a.e. $t \in (0, T)$. This last inequality, again by the above deduction, is equivalent to the assertion that H^n measure on $\partial D \times (0, T)$ is absolutely continuous with respect to parabolic measure at (0,T) which likewise is true by [14, Thm. 3.1]. We conclude from (3.5), (3.4), that (3.3) is true. To complete the proof of Theorem 3, let G be Green's function with pole at (0,0) for the heat equation in $B(0, d) \times (0, T)$. Again d is the distance from (0,0) to ∂D. Then clearly, $G \leq u$, so from (3.3) we see as in section 2 that

$$a^*(t) = \lim_{x \to y} |\nabla G(x, t)| \leq a(t), \text{ radially for a.e. } t \in (0, T), \tag{3.6}$$

whenever $y \in \partial B(0, d)$. Thus from (3.1),

$$\begin{aligned} H^{n-1}(\partial B(0,d)) \left(\int_0^{T_1} a^*(t) dt \right) + \int_D G(x, T_1) dH^n x \qquad (3.7) \\ \leq H^{n-1}(\partial D) \left(\int_0^{T_1} a(t) dt \right) + \int_D u(x, T_1) dH^n x = 1, \end{aligned}$$

with equality only if $D = B(0, d)$. Now equality must hold in (3.7) because (3.1) is also true with u replaced by G, D by $B(0, d)$, and $a(t)$ by $a^*(t)$. Hence $D = B(0, d)$. Since the boundary values of u are invariant under rotations in x, we conclude from uniqueness of u, that $u(\cdot, t)$ is radial, $t \in (0, T)$. Q.E.D.

Next we note that if

$$k(x,t) = \begin{cases} (4\pi t)^{-n/2} \exp\left[-\frac{|x|^2}{4t}\right], & \text{if } x \in R^n, \ t > 0; \\ 0, & \text{if } x \in R^n, \ t \leq 0. \end{cases}$$

denotes the Green's function for the heat equation in $R^n \times R$, then for given $\lambda > 0$,

$$|\nabla_y k|(y, t) = \lambda |y|/(2t), \text{ on } \{(x, t) : k(x, t) = \lambda\}. \tag{3.8}$$

Using (3.8) we shall obtain a different generalization of Theorem 1 to domains whose boundaries can be rough in the time variable. To this end suppose now $D \subseteq R^n \times R$ is bounded and for given $(y, t) \in \partial D \cap [R^n \times (0, T)]$ there exists $r > 0$ such that after a possible rotation in the x variable:

$$Z \cap \partial D = \{(x', x_n, s) : x_n = \psi(x', s), x' \in R^{n-1}, s \in R\} \cap Z$$

$$Z \cap D = \{(x', x_n, s) : x_n > \psi(x', s), x' \in R^{n-1}, s \in R\} \cap Z$$

where $Z \subset R^n \times R$ is a truncated circular cylinder of radius r with $(y,t) = (y', \psi(y',t), t)$ at the center of the cylinder and axis parallel to the x_n axis. Also the bases of Z have a positive distance to ∂D. In case $n = 1$ delete x' from the above equation. Here ψ is a function on $R^{n-1} \times R$ with compact support and the following properties: For each fixed t, $\psi(\cdot, t)$ is Lipschitz on R^{n-1} with

$$\|\psi(\cdot,t)\|^* \le a_1 < \infty, \tag{3.9}$$

while for each fixed $x' \in R^{n-1}$,

$$\psi(x',t) = \int_R |s-t|^{-1/2} b(x',s)\,ds, \quad t \in R,$$

where $b(x',\cdot)$ is of bounded mean oscillation on R with

$$\|b(x',\cdot)\|^\wedge \le a_2 < \infty \tag{3.10}$$

Again if $n = 2$, remove equation (3.9) and delete x' from (3.10). Also, $\|\ \|^*$, $\|\ \|^\wedge$, denote the Hölder and BMO norms, respectively. Let

$$D^* = D \cap [R^n \times (0,T)] \text{ and } \partial' D^* = \partial D \cap [R^n \times (0,T)]$$

and put

$$d\sigma(y',t) = \sqrt{1 + |\nabla_{y'}\psi|^2(y',t)}\ dy'dt, \ (y,t) \in \partial' D^* \cap Z.$$

It is easily seen that σ is well defined on $\partial' D^*$ independently of y. In fact if $D(t) = D \cap (R^n \times \{t\})$ and f is integrable with respect to σ, then $\int_{\partial' D^*} f d\sigma = \int_0^T \left(\int_{\partial D(t)} f dH^{n-1}\right) dt$. The radial limit of a function u at $(y,t) \in \partial' D^*$ is defined to be $\lim_{\alpha \to 0} u(y', \psi(y',t) + |\alpha|, t)$ provided this limit exists, and if $(x', \psi(x',s) + 2r_0, s) \subseteq D^* \cap Z$, whenever $(x', \psi(x',s), s)$ is in $\partial' D^* \cap Z$, then the nontangential maximal function, u^*, relative to $r_0 > 0$ at (y,t) is,

$$u^*(y,t) = \sup_{|\alpha| \le r_0} |u(y', \psi(y',t) + |\alpha|, t)|.$$

Replace (A*) by

(A~) $\lim_{(x,s)\to(y,t)} |\nabla u|(x,s) = a|y|/t$, radially, for σ almost every (y,t) in $\partial' D^*$ while u has continuous boundary values 0 on $\partial' D^*$

Since the only case of interest in the theorem to follow is when $(0,0) \in \partial D$, we must extend the definition of the Green's function with pole at (0,0). For this purpose suppose $(0,0) \in \bar{D}$ and for some $\tau > 0$ that

$$\{(x,t) : k(x,t) > \tau\} \cap [R^n \times (0,T)] \subseteq D^*. \tag{3.11}$$

Let $u > 0$ be the positive solution to the heat equation in D for which

$$u(x,t) = k(x,t) + q(x,t), \quad (x,t) \in D,$$

where q is the bounded solution to the heat equation in D with boundary values: $q = -k$ on $\partial D - \{0\}$, in the sense of Perron-Wiener-Brelot. Existence of q, $-\tau \le q \le 0$, follows from the usual Perron family argument, thanks to (3.11). Note that u is just the Green's function for D with pole at (0,0) when $(0,0) \in D$. Finally we point out that for D satisfying the above conditions every point in $\partial' D^*$ is regular (see [31, section 1]). Hence the assumption, $u(x) \to 0$ as $x \to \partial' D^*$, continuously, is unnecessary for the Green's function of D with pole at (0,0). With this notation we prove

Theorem 4. *Let D be as above and let u be Green's function for the heat equation in D with pole at $(0,0) \in \bar{D}$. There exists $a_0 > 0$ such that if $a_1, a_2 \le a_0$ in (3.9), (3.10), and u has boundary values as in $(A^\sim)$, then for some $\lambda > 0$, $\partial' D^* \subseteq \{(x,t) : k(x,t) = \lambda\}$ and $u \equiv k - \lambda$ in D^*.*

Note that Theorem 1 could be restated, as above.

Proof. The proof of Theorem 4 is similar to the proof of Theorem 3. In place of (3.1) we want to show for arbitrary T_1, T_2, $0 < T_1 < T_2 < T$,

$$a \int_{T_1}^{T_2} \left(\int_{\partial D(t)} |x| dH^{n-1}x \right) \frac{dt}{t} + \int_{D(T_2)} u(x,T_2) dH^n x = \int_{D(T_1)} u(x,T_1) dH^n x \tag{3.12}$$

Let Ω be a smooth domain with $\bar{\Omega} \subseteq D$. Then from the divergence theorem we deduce for $\Omega^* = \Omega \cap (R^n \times [T_1, T_2])$, $0 < T_1 < T_2$, and $\Omega(t) = \Omega \cap (R^n \times \{t\})$, $0 < t < T$,

$$- \int_{\partial\Omega^*} \nabla_x u \cdot \nu_x dH^n - \int_{\partial\Omega^*} u \nu_t dH^n = 0 = \int_{\Omega(T_2)} u(x,T_2) dH^n x - \int_{\Omega(T_1)} u(x,T_1) dH^n x$$
$$- \int_{T_1}^{T_2} \left(\int_{\partial\Omega(t)} \nabla_x u \cdot \nu_x' dH^{n-1}x \right) dt - \int_{T_1}^{T_2} \left(\int_{\partial\Omega(t)} u \nu_t' dH^{n-1}x \right) dt, \tag{3.13}$$

where $\nu = (\nu_x, \nu_t)$ is the outer unit normal to Ω, $\nu_x' = |\nu_x|^{-1}(\nu_x, 0)$, $\nu_t' = (0, \nu_t)|\nu_x|^{-1}$. Again we shall obtain (3.12) as the limit of (3.13) when $\Omega \in (\Omega_j)_1^\infty$. To justify the limit we need a stronger version of $(\cdot)$, $(\cdot\cdot)$, namely for a_0 small enough,

$(-)$ $\nabla_x u$ taken componentwise is locally square integrable with respect to σ on $\partial' D^*$

$(--)$ $\nabla_x u \to -a(|y|/t) n(y,t)$, radially, for σ almost every $(y,t) \in \partial' D^*$ where $n(y,t) = \frac{(\nabla_{y'} \psi(y',t), -1, 0)}{\sqrt{1 + |\nabla_{y'} \psi(y',t)|^2}}$

$(-), (--)$ follow from the work of Murray and the first author in [22]. The sequence $(\Omega_j)_1^\infty$ can be obtained by piecing together smooth approximates to ψ from above locally; since by compactness, $\partial' D^* \cap (R^n \times [T_1, T_2])$ is contained in a finite union of cylinders. It should be noted that convolution of ψ with an approximant identity in both the x and t variables separately, gives a smooth function with Lipschitz and BMO norms still bounded by a_1, a_2, respectively. Using this fact, $(-), (--)$, dominated convergence, and taking a limit in (3.13) with $\Omega = \Omega_j$ as $j \to \infty$, we get (3.12).

Next, we let $T_1 \to 0$ in (3.12) and use the fact that

$$\int_{D(T_1)} u(x, T_1) dH^n x \leq \int_{R^n \times \{T_1\}} k(x, T_1) dH^n x = 1,$$

to deduce

$$a \int_0^{T_2} \left(\int_{\partial D(t)} |x| dH^{n-1} x \right) t^{-1} dt + \int_{D(T_2)} u(x, T_2) dH^n x \leq 1. \quad (3.14)$$

Clearly, (3.14) implies that $0 \in \partial D$. We shall also need

$$\limsup_{(x,s) \to (y,t)} |\nabla u|(x, s) \leq a|y|/t, \quad \text{whenever } (y, t) \in \partial' D^*. \quad (3.15)$$

This inequality follows from the work in [22] in a way similar to the proof of (2.3). We omit the details. To continue the proof of Theorem 4, let $\lambda(\epsilon)$, $0 < \epsilon < \frac{1}{4}$, be the smallest of the numbers γ such that

$$q(x, t) \geq -\gamma + \epsilon \ln|x|, \quad \text{for all } (x, t) \in D^* \cap [R^n \times (0, T_2)],$$

where q is as in the definition of u. Using (3.11), and the fact that $u < k$ in D^* we see there exists $\epsilon_0 > 0$ (small) and $0 < b_1 < b_2 < \infty$, such that

$$b_1 < \lambda(\epsilon) < b_2, \quad 0 \leq \epsilon \leq \epsilon_0. \quad (3.16)$$

Moreover, since $q(x, t) + \lambda(\epsilon) - \epsilon \ln|x|$, is a supersolution to the heat equation in D^*, and q is continuous and bounded on $D - \{0\}$, we see from the minimum principle for supersolutions to the heat equation that

$$q(x, t) + \lambda(\epsilon) - \epsilon \ln|x| = 0,$$

for some $x = x(\epsilon)$, $t = t(\epsilon)$, with $(x, t) \neq (0, 0)$ and $(x, t) \in \partial D \cap \{R^n \times [0, T_2]\}$. Moreover, since $q \equiv 0$ on $(R^n \times \{0\}) \cap [\partial D - \{(0, 0)\}]$ we see from (3.16) that for $\epsilon_0 > 0$ small enough we have $t(\epsilon) > 0$, $0 < \epsilon \leq \epsilon_0$. Using this fact, (3.15), and radial symmetry of $k(\cdot, t)$ as in (2.5) we obtain at $(x, t) = (x(\epsilon), t(\epsilon))$,

$$a \frac{|x|}{t} \geq \frac{|x|}{2t} k(x, t) - \frac{\epsilon}{|x|} \quad (3.17)$$

We now let $\epsilon \to 0$ and consider two cases: either (a) $\lim_{\epsilon\to 0} t(\epsilon) = 0$, which by the above reasoning implies $\lim_{\epsilon\to 0} x(\epsilon) = 0$ or (b) $\lim_{\epsilon\to 0} t(\epsilon) = t_0 > 0$, $\lim_{\epsilon\to 0} x(\epsilon) = x_0 \neq 0$, and $\lim_{\epsilon\to 0} \lambda(\epsilon) = \lambda_0$, for $\epsilon \in (\epsilon_i)_1^\infty$. In case (b) we see that $\lambda_0 = \lambda(0)$ and $k(x_0, t_0) = \lambda_0$. From (3.17) we conclude that in case (b),

$$\{(y,s) : k(y,s) > \lambda_0\} \cap [R^n \times (0, T_2)] \subseteq D^* \tag{3.18}$$

and

$$a \geq \frac{\lambda_0}{2}. \tag{3.19}$$

If case (a) occurs observe from (3.11) and (3.16) that for ϵ_1 small enough, $0 < \epsilon \leq \epsilon_1 < \epsilon_0$, $x = x(\epsilon)$, $t = t(\epsilon)$, we have

$$-\tau \leq -k(x,t) = q(x,t) = -\lambda(\epsilon) + \epsilon \ln|x| \leq -\frac{1}{2}b_1.$$

From this inequality we see for ϵ_1 small enough that there exists b_3, $0 < b_3 < \infty$, with

$$||x|^2 + 2nt \ln t| \leq b_3 t, \quad 0 < \epsilon \leq \epsilon_1.$$

Hence, $\lim_{\epsilon\to 0}(t/|x|^2) = 0$. Using this inequality in (3.17) and the fact that $k(x,t) \geq \lambda(\epsilon)$, ϵ small, we obtain for $\lambda_0 = \limsup_{\epsilon\to 0} \lambda(\epsilon)$ that (3.19) is still true. Also (3.18) remains valid, as is easily seen. Let

$$W = \{(y,s) : k(y,s) > \lambda_0\},$$

$$W(t) = W \cap (R^n \times \{t\}).$$

From (3.18) and the maximum principle for the heat equation we observe first that $u \geq k - \lambda_0$ in $D \cap [R^n \times (0, T_2)]$ and second that

$$\min_{x\in\partial D(t)} |x| \geq \max_{x\in\partial W(t)} |x|, \quad 0 < t < T_2.$$

Since $W(t)$ is a ball in $R^n \times \{t\}$ it follows from (3.19), the above observations, the isoperimetric inequality, and (3.14) that

$$\frac{\lambda_0}{2}\int_0^{T_2}\left(\int_{\partial W(t)} |x| dH^{n-1}x\right) t^{-1}dt + \int_{W(T_2)} (k(x,T_2) - \lambda_0) dH^n x$$

$$\leq a\int_0^{T_2}\left(\int_{\partial D(t)} |x| dH^{n-1}x\right) t^{-1}dt + \int_{D(T_2)} u(x,T_2) dH^n x \leq 1,$$

with equality only if $u \equiv k - \lambda_0$ in $D \cap [R^n \times (0, T_2)]$. Moreover equality must hold in this inequality, as it follows from (3.18) and the same argument used in proving (3.14) that

$$\frac{\lambda_0}{2}\int_0^{T_2}\left(\int_{\partial W(t)} |x| dH^{n-1}x\right) t^{-1}dt + \int_{W(T_2)} (k(x,T_2) - \lambda_0) dH^n x = 1.$$

Thus, $u \equiv k - \lambda_0$ in $D \cap [R^n \times (0, T_2)]$ and because T_2 is arbitrary, $0 < T_2 < T$, the proof is complete. Q.E.D.

4. Sets of Finite Perimeter

In this section we suppose that D is a bounded domain of finite perimeter: By definition D is of finite perimeter if whenever ϕ is a smooth vector field defined on R^n, and $|\phi(x)| \leq 1$, $x \in R^n$, then

$$\int_D \nabla \cdot \phi dH^n \leq M < \infty,$$

If D is of finite perimeter it follows (see [13, section 5.8]) that

$$\int_D \nabla \cdot \phi dx = \int_{\partial^* D} \phi \cdot n(x) dH^{n-1} x, \tag{4.1}$$

where $\partial^* D$ (the reduced boundary of D), is the set of points where a certain measure has a derivative with respect to another. For our purposes it is enough to know that $\partial^* D$ is H^{n-1} a.e. equivalent to

$$\partial_* D = \{x \in \partial D : \limsup_{r \to 0} r^{-n} \min[H^n(B(x,r) \cap D),\ H^n(B(x,r) - D)] > 0\},$$

the so called measure theoretic boundary (see [13, section 5.8]) of D. We note that $n(y)$, $y \in \partial_* D$, is a measure theoretic outer normal in the sense that for H^{n-1} a.e. $y \in \partial_* D$,

$$\lim_{r \to 0} \{r^{-n} H^n [B(y,r) \cap D \cap K^+(y)]\} = 0 \tag{4.2}$$

$$\lim_{r \to 0} \{r^{-n} H^n [(B(y,r) - D) \cap K^-(y)]\} = 0,$$

where

$$K^+(y) = \{x \in R^n : n(y) \cdot (x - y) > 0\},$$
$$K^-(y) = \{x \in R^n : n(y) \cdot (x - y) < 0\}.$$

Approximating $n(y)$, $y \in \partial_* D$ by smooth functions on R^n we see from (4.1) that, $H^{n-1}(\partial_* D) < +\infty$. This inequality is also sufficient for D to be of finite perimeter [13, section 5.8]. We shall discuss possible extensions of Theorem 1 in domains D of finite perimeter with

$$H^{n-1}(\partial D - \partial_* D) = 0. \tag{4.3}$$

Observe that a Lipschitz domain clearly satisfies these conditions. One way to state boundary condition (A) which avoids the definition of radial limits and is equivalent for Lipschitz domains due to Dahlberg's theorem; is to require that for each Borel subset $E \subseteq \partial D$,

(A^) $\mu(E) = aH^{n-1}(E) = aH^{n-1}(E \cap \partial_* D)$, where μ is harmonic measure of ∂D relative to 0, while $u(x) \to 0$ as $x \to \partial D$, continuously.

More specifically, let f be a continuous function on ∂D and suppose H_f is the harmonic solution to the Dirichlet problem obtained by way of the usual Perron family argument. Then

$$|H_f(x)| \leq \max_{x \in \partial D} |f(x)|, \quad x \in \partial D.$$

From the Riesz representation theorem it follows that there exists a regular Borel measure μ on ∂D so that the functional, $f \to H_f(0)$, can be represented as

$$H_f(0) = \int f d\mu. \tag{4.4}$$

We would like to be able to extend Theorem 1 with (A) replaced by (A^) to domains of the above type. This it turns out is impossible. In fact there exists simply connected domains $D \subset R^2$ (other than disks), which are bounded by a rectifiable Jordan curve (so D is of finite perimeter and (4.3) holds) for which harmonic measure with respect to 0 is a constant multiple of H^1 on ∂D (condition (A^)). For construction of these domains, classically called non Smirnov domains see [11, section 10.4] for references. An examination of the proof of Theorem 1 reveals that (2.3) must fail since from (4.4) and (A^) we deduce

$$1 = \mu(\partial D) = aH^{n-1}(\partial D),$$

which is (2.1). Indeed, for a non Smirnov domain it is true that $|\nabla u(x)| \to +\infty$ as $x \to y$ for some $y \in \partial D$. Therefore we assume for some $\lambda > 0$ that $B(0, 2\lambda) \subseteq D$ and

$$|\nabla u(x)| \leq k < \infty, \quad x \in D - B(0, \lambda). \tag{4.5}$$

We prove

Theorem 5. *Let $D \subseteq R^n$ be a bounded domain of finite perimeter for which (4.3) is valid. Let u be Green's function for D with pole at $0 \in D$ and suppose that u satisfies (A^), (4.5). Then D is a ball with center at zero and u is radially symmetric about 0.*

As mentioned above the same argument as in Theorem 1 can be used to prove Theorem 5, once we show (2.3) holds in this situation. Thus we only prove (2.3).

Proof. The proof is essentially due to Alt and Caffarelli [3, Thm. 6.3]. Let $y \in \partial D, r < |y|/2$, and put

$$\sigma(r, u) = \sigma(r, u, y) = [H^{n-1}(\partial B(y, r))]^{-1} \left(\int_{\partial B(y,r)} u dH^{n-1} \right). \tag{4.6}$$

Observe from (4.6) and (4.5) that

$$\sigma(r,u) \le ckr. \tag{4.7}$$

From the Riesz representation formula for subharmonic functions (see [18, (3.9.1), (3.9.4)]) we have for $\nu_n = H^n(\partial B(0,1))$ as in section 1,

$$0 = u(y) = \nu_n \sigma(r,u) - \int_0^r \mu(B(y,t) \cap \partial D) t^{1-n} dt. \tag{4.8}$$

Using (A^), (4.6), (4.7), and (4.8) it follows that

$$H^{n-1}(B(y,r/2) \cap \partial D) r^{1-n} \le cr^{-1} \int_{r/2}^r H^{n-1}(B(y,t) \cap \partial D) t^{1-n} dt \tag{4.9}$$

$$\le a^{-1} c\sigma(r,u) r^{-1} \le c.$$

In (4.7), (4.9), as in the rest of this section, c denotes a positive constant depending only on n, k, a, not necessarily the same at each occurrence. We note from (4.7)-(4.9) and (A^) that for $\delta > 0$ small and $y \in \partial D$ that

$$crH^{n-1}[B(y,r) \cap \partial D](\delta r)^{1-n} \ge \int_{\delta r}^r t^{1-n} \mu[B(y,t) \cap \partial D] dt$$

$$= \sigma(r,u) - \sigma(\delta r, u) \ge \sigma(r,u) - c\delta r.$$

From this inequality we see that if $k_1 \sigma(r,u) \ge r$, then there exists $\delta = \delta(k_1) > 0$ such that

$$H^{n-1}[B(y,r) \cap \partial D] \ge c(k_1) r^{n-1}, \tag{4.10}$$

where $c(k_1)$ is a positive constant depending on k_1, n, k, a.

Next let $(r_m)_1^\infty$ be a decreasing sequence of positive numbers with $\lim_{m\to\infty} r_m = 0$. Extend u to a continuous subharmonic function on $R^n - \{0\}$ by defining $u \equiv 0$ on $R^n - D$. For fixed $y \in \partial D$ let

$$v_m(x) = r_m^{-1} u[y + r_m x], \quad x \in R^n - \{-y/r_m\}, \; m = 1, 2, \ldots$$

Then from (4.5) we see that $(v_m)_1^\infty$ is a sequence of uniformly bounded Lipschitz functions in $R^n - \{-y/r_m : m = 1, 2, \ldots\}$. Thus a subsequence converges uniformly on compact subsets to a Lipschitz function v on R^n. In fact we claim for H^{n-1} a.e. $y \in \partial D$ that $v(x) = 0$ when $n(y) \cdot x > 0$, and

$$v(x) = -a(n(y) \cdot x), \quad \text{when } n(y) \cdot x \le 0. \tag{4.11}$$

To prove this claim we need the fact that for H^{n-1} a.e. $y \in \partial_* D$ ([13, section 5.7, Cor. 1]),

$$\lim_{r\to 0} [r^{1-n} H^{n-1}(\partial_* D \cap B(y,r))] = \alpha_n \tag{4.12}$$

where α_n is the volume of the unit ball in $R^{n-1}(\alpha_2 = 2)$. In view of (4.3) we can replace $\partial_* D$ by ∂D in this inequality. Now suppose that $y \in \partial D$ is a point where (4.2) and (4.12) hold. From (4.2) we see that $v \geq 0$ is subharmonic on R^n with $v \equiv 0$ on $\{x : n(y) \cdot x \geq 0\}$. Also from (4.8), (4.12), and (A^), we find

$$\sigma(\rho, v) = \sigma(\rho, v, 0) = \alpha_n a \rho \nu_n^{-1}, \quad 0 < \rho < \infty, \tag{4.13}$$

while from (4.5) it follows that for H^n a.e. x,

$$|v(x)| \leq c|x|, \quad x \in R^n. \tag{4.14}$$

From (4.14) and the Riesz representation formula for subharmonic functions in a halfspace we see that

$$v(x) = -\beta(n(y) \cdot x) - q(x), \quad n(y) \cdot x < 0, \tag{4.15}$$

where q is a Green's potential and $\beta \geq 0$. Now it follows from essentially the Phragmén-Lindelöf theorem (see [12]) that $\lim_{\rho \to \infty} \rho^{-1}\sigma(\rho, -q) = 0$. This equality and (4.13) imply $q \equiv 0$. Putting (4.15) with $q \equiv 0$ into (4.13) and using the divergence theorem we get $\beta = a$, so (4.11) is true.

Since each subsequence of $(v_m)_1^\infty$ converges to v and $(r_m)_1^\infty$ is arbitrary we conclude from (4.5) that

$$|u(x+y) + a(n(y) \cdot x)|\, |x|^{-1} \to 0, \tag{4.16}$$

uniformly as $|x| \to 0$. We note that if h is harmonic in $B(x_0, s)$ then from the Poisson integral formula it is easily shown that

$$|\nabla h(x)| \leq c s^{-(n+1)} \left(\int_{B(x_0,s)} |h| dH^n \right), \quad x \in B(x_0, s/2). \tag{4.17}$$

Let $d(x, \partial D)$ denote the distance from x to ∂D. Using (4.17) in (4.16) we conclude that if $k_2 d(x+y, \partial D) > |x|$, k_2 large, and $\eta > 0$ is given, then there exists $r_0 = r_0(\eta, k_2, y) > 0$ such that

$$||\nabla u(x+y)| - a| \leq \eta, \quad |x| \leq r_0. \tag{4.18}$$

Since (4.18) holds for H^{n-1} a.e. $y \in \partial D$, we see for fixed η, k_2, and

$$E(\epsilon) = \{y \in \partial D : r_0(\eta, k_2, y) \geq \epsilon\},$$

that

$$\lim_{\epsilon \to 0} H^{n-1}[\partial D - E(\epsilon)] = 0. \tag{4.19}$$

Next put $w(x) = \max[|\nabla u(x)| - a, 0]$, $x \in D - \{0\}$ and observe that w is subharmonic in $D - \{0\}$. Let $g(\cdot, y)$ be Green's function for D with pole at $y \in D$ and recall that $u = g(\cdot, 0)$. For fixed $x_0 \neq 0$ in $D - B(0, 3/2\lambda)$ let $r > 0$ be such that

$$B(0,\lambda) \cap \{x : g(x, x_0) \leq r\} = \{\phi\}.$$

If $D_1 = \{x : g(x, x_0) > r\} - B(0, \lambda)$, we also choose r so that $|\nabla g(\cdot, x_0)| \neq 0$ on ∂D_1, and $x_0 \in D_1 - B(0, \frac{3}{2}\lambda)$. Then from Green's second identity and subharmonicity of w we deduce

$$\begin{aligned} w(x_0) &\leq -\int_{\partial D_1} w(y)\frac{\partial g}{\partial \nu}(y, x_0)\,dH^{n-1}y + \int_{\partial D_1} (g(y, x_0) - r)|\nabla w(y)|\,dH^{n-1}y \qquad (4.20)\\ &\leq -\int_{\{y: g(y,x_0)=r\}} w(y)\frac{\partial g}{\partial \nu}(y, x_0)\,dH^{n-1}y + \int_{\partial B(0,\lambda)} (w(y)|\nabla g(y, x_0)| \\ &\qquad + g(y, x_0)|\nabla w(y)|)\,dH^{n-1}y = I_1(x_0) + I_2(x_0), \end{aligned}$$

where ν is the outer unit normal to D_1. From Harnack's inequality we have $g(y, x_0) \leq cu(x_0)$ for $y \in B(0, \lambda)$ and $x_0 \in D - B(0, \frac{3}{2}\lambda)$. From this inequality, (4.17), and (A^) we get

$$I_2(x_0) \to 0 \text{ continuously, as } x_0 \to \partial D. \tag{4.21}$$

From(4.20) and (4.21) we see that in order to prove (2.3) it suffices to show for fixed $x_0 \in D - B(0, 3/2\lambda)$ that

$$I_1(x_0) \to 0 \quad \text{as } r \to 0. \tag{4.22}$$

As for (4.22) suppose $g(y, x_0) = r$, $d(y, \partial D) \geq k_3 r$. Then for r sufficiently small, say $0 < r \leq r_1$, we see from Harnack's inequality that there exists $c_1 = c_1(x_0, r_1, n) > 0$ such that

$$(c_1)^{-1} r = (c_1)^{-1} g(y, x_0) \leq u(y) \leq c_1 g(y, x_0) = c_1 r, \tag{4.23}$$

so from (4.17) we have for $r_1 > 0$ small, $0 \leq r \leq r_1$,

$$|\nabla u(y)| \leq \frac{c}{k_3 r}\sigma\left(\frac{1}{2}k_3 r, u, y\right) \leq \frac{c}{k_3 r}u(y) \leq \frac{c\, c_1}{k_3}.$$

Thus if $k_3 = k_3(a, x_0, r_1, n)$ is large enough, then $w(y) = 0$ and it follows that

$$I_1(x_0) = -\int_F w(y)\frac{\partial g}{\partial \nu}(y, x_0)\,dH^{n-1}y \tag{4.24}$$

where

$$F = \{y : g(y, x_0) = r,\ d(y, \partial D) < k_3 r\}.$$

Fix $k_3 > 0$ to be the smallest number such that (4.24) holds. If $y \in F$ observe from (4.23) and (4.5) that

$$kd(y, \partial D) \geq u(y) \geq (c_1)^{-1} g(y, x_0) = (c_1)^{-1} r. \qquad (4.25)$$

Given $\epsilon > 0$, let $k_2 = 8kk_3c_1$, $r < \epsilon/k_2$, and let F_1 be the set of all $y \in F$ such that there exists $z \in E(\epsilon)$ with $y \in B(z, 8k_3r)$. Then from (4.25), (4.18), we find that $w(y) \leq \eta$. Hence

$$-\int_{F_1} w(y) \frac{\partial g}{\partial \nu}(y, x_0) dH^{n-1} y \leq -\eta \int_{\{y: g(y,x_0)=r\}} \frac{\partial g}{\partial \nu}(y, x_0) dH^{n-1} y = c\eta. \qquad (4.26)$$

To handle the integral over $F - F_1$ we use a well known covering lemma (see [13, 1.5.2]) to get (z_m), $z_m \in F - F_1$, such that $F - F_1 \subseteq \cup B(z_m, 4k_3r)$ and each point in the union is contained in at most $N = N(n)$ balls. Now from (4.17), (4.23), and (4.5) we deduce for $y \in D$ and $s = d(y, \partial D) \leq 8k_3r$,

$$|\nabla g(y, x_0)| \leq cs^{-1} \sigma\left(\frac{s}{2}, g(\cdot, x_0), y\right) \leq c\, c_1 s^{-1} \sigma\left(\frac{s}{2}, u, y\right) \leq c_2, \qquad (4.27)$$

for some $c_2 = c_2(x_0, r_1, n, k) > 0$. Let $L_m = B(z_m, 4k_3r) \cap F$. Then from (4.27), (4.5), and the divergence theorem we obtain

$$-\int_{L_m} w(y) \frac{\partial g}{\partial \nu}(y, x_0) dH^{n-1} y \leq -k \int_{L_m} \frac{\partial g}{\partial \nu}(y, x_0) dH^{n-1} y \qquad (4.28)$$

$$\leq k \int_{\partial B(z_m, 4k_3r)} |\nabla g(y, x_0)| dH^{n-1} y \leq kc\, c_2 (k_3 r)^{n-1} = c_3 r^{n-1},$$

where c_3 depends on a, x_0, r_1, n, k, and k_3. Again by well known estimates for subharmonic functions and (4.23) we see there exists z_m^* in ∂D with $|z_m^* - z_m| < k_3 r$ and

$$\sigma(2k_3r, u, z_m^*) \geq cu(z_m) \geq c(c_1)^{-1} g(z_m, x_0) = c_4 r.$$

Hence if r is replaced by $2k_3r$ in (4.10) and $k_1 = 2k_3/c_4$, then from (4.10) we obtain

$$r^{n-1} \leq c_5 H^{n-1}[B(z_m^*, 2k_3r) \cap \partial D] \leq c_5 H^{n-1}[B(z_m, 4k_3r) \cap \partial D],$$

where $c_5 = c_5(a, x_0, r_1, n, k, k_1, k_3) > 0$. Using this inequality in (4.28) we conclude

$$-\int_{L_m} w(y) \frac{\partial g}{\partial \nu}(y, x_0) dH^{n-1} y \leq c_6 H^{n-1}[B(z_m, 4k_3r) \cap \partial D],$$

where c_6 has the same dependence as c_5. Summing this inequality it follows that

$$-\int_{F-F_1} w(y)\frac{\partial g}{\partial \nu}(y,x_0)dH^{n-1}y \leq -\sum \int_{L_m} w(y)\frac{\partial g}{\partial \nu}(y,x_0)dH^{n-1}y \tag{4.29}$$

$$\leq c_6\left(\sum_m H^{n-1}[B(z_m,4k_3r)\cap\partial D]\right)$$

$$\leq c\, c_6 H^{n-1}\left\{[\bigcup_m B(z_m,4k_3r)]\cap\partial D\right\}$$

$$\leq c\, c_6 H^{n-1}(\partial D - E(\epsilon)),$$

because

$$\partial D \cap \bigcup_m B(z_m,4k_3r) \subseteq \partial D - E(\epsilon).$$

Combining (4.29), (4.26), and (4.24), we conclude

$$I_1(x_0) \leq c\eta + c\, c_6 H^{n-1}(\partial D - E(\epsilon)).$$

Since the right-hand side is independent of r, $0 < r < r_1$, we have

$$\xi = \limsup_{r\to 0} I_1(x_0) \leq c[\eta + c_6 H^{n-1}(\partial D - E(\epsilon))].$$

Next we let $\epsilon \to 0$ and use (4.19) to obtain $\xi \leq c\,\eta$. Finally letting $\eta \to 0$ we get (4.22). Thus (2.3) is valid and Theorem 5 follows from our earlier work. Q.E.D.

5. Remarks and Problems

(1) As mentioned in section 1, the second author in his thesis, requires f to have continuous third partials and d to have continuous first partials in each variable; in order to conclude that a solution u to (1.6) under boundary condition (A) is radially symmetric. Can the same conclusion be made under weaker regularity assumptions on f, d? If for example, d is only bounded while f is C^∞ and uniformly convex, then classical Schauder type estimates give $u \in C^{1,\alpha}(D)$ for $0 < \alpha < 1$ and it can be shown as outlined in section 1 that ∂D is $C^{1,\alpha}$ from each side. However to use Serrin's argument we need ∂D to be C^2. Also if $f'(0) = 0$ or ∞, and f is only C^2, then it is not known for some functions d whether a Hopf boundary maximum principle holds for solutions to (1.6). This maximum principle is needed in Serrin's argument.

(2) When can the Green's function in Theorems 1,2,5, be replaced by a solution u to either (1.1) or (1.6)? For a general L as in (1.1) this question could be difficult, since an answer appears linked with determining the sets of L elliptic measure zero. If

$$Lu = \Delta u + f(u, |\nabla u|) = 0,$$

where $f > 0$ is C^1 in both u and $|\nabla u|$, then in R^2, it can be shown (using the fact that a certain function of $|\nabla u|$ is a super solution to a uniformly elliptic P.D.E.) that boundary condition (A*) forces ∂D to be C^2 when D is Lipschitz. Serrin's method can then be applied. In R^n, $n > 2$, super solution estimates are no longer available and the procedure for showing ∂D smooth is much more involved. However it appears likely that a new method of Caffarelli [7,8] can be used in the Lipschitz case (Theorem 1) to show that boundary condition (A*) forces ∂D to be C^2. Serrin's argument can then be applied to get D is a ball. If Caffarelli's method works, then parabolic analogues of Theorem 3 in Lipschitz cylinders for

$$u_t = \Delta u + f(u, \nabla u)$$

should also hold. Still, though, a more direct approach to these problems which requires only subsolution estimates, would be preferable.

Also, for more general domains D, in Theorem 5, it is probably not possible to first show that a boundary condition similar to (A^) forces ∂D to be smooth. In fact we do not know how to show this, even in R^2. One essential difference between this case and the Lipschitz case is that $|\nabla u|$, a priori, need not be bounded away from zero in a neighborhood of a boundary point, so super solution estimates appear difficult. Moreover in R^3, Alt and Caffarelli [3, section 2.7] point out that there exists a positive Lipschitz harmonic function u in the exterior of a cone K with $u = 0$, $|\nabla u| = a$, continuously on K, except at the vertex of the cone, and

$$k^{-1} \le |\nabla u| \le k \quad \text{for some } k, \quad 0 < k < +\infty,$$

in a neighborhood of the vertex. Clearly K is not smooth in any neighborhood of its vertex. The above authors also show for a similar problem that ∂D is locally smooth for H^{n-1} a.e. $y \in \partial D$. Thus can Serrin's argument be extended to domains that are locally smooth outside of a small exceptional set. We have had no luck in trying this approach. If

$$Lu = \Delta u + 1 = 0, \tag{5.1}$$

analogues of Theorems 1,4, can be obtained using Weinberger's original method, and arguments similar to those for the Green's function. We briefly sketch the proof of Theorem 1 for a solution u to (5.1) satisfying (A*).

In place of (2.2) it can be shown that

$$\int_D (n|\nabla u|^2 + 2u)dH^n = na^2 H^n(D). \tag{5.2}$$

For (5.2) we use the Rellich-Necas-Pohożaev formula [17,19,24,28]

$$-\int_{\partial\Omega} [(x\cdot\nu)|\nabla u|^2 - 2(\nabla u\cdot\nu)(x\cdot\nabla u) - 2(x\cdot\nu)u + (2n-2)u(\nabla u\cdot\nu)]dH^{n-1} \\ = \int_\Omega (n|\nabla u|^2 + 2u)dH^n, \tag{5.3}$$

where ν is the outer unit normal to the smooth domain Ω with $\Omega \subseteq D$. Choosing a sequence of smooth domains $(\Omega_j)_1^\infty$ as in section 1 and using the radial limit theorems mentioned there, we obtain (5.2) as the limit of (5.3) with $\Omega = \Omega_j$ as $j \to \infty$. (2.3) remains true in this case and its proof is essentially unchanged, since $|\nabla u|$ is subharmonic and its radial maximal function is in $L_2(\partial D)$. Now

$$\Delta(n|\nabla u|^2 + 2u) = 2n \sum_{i,j} (u_{x_i x_j})^2 - 2 \geq 2(\Delta u)^2 - 2 \geq 0, \tag{5.4}$$

where we have used Schwarz's inequality. Thus $n|\nabla u|^2 + 2u$ is subharmonic in D and so from (A*), (2.3), and the maximum principle for subharmonic functions we have

$$n|\nabla u|^2 + 2u \leq na^2 \tag{5.5}$$

in D with equality at any point of D only if $n|\nabla u|^2 + 2u \equiv na^2$. In view of (5.2), (5.5), it follows that $n|\nabla u|^2 + 2u \equiv na^2$. From the case of equality in Schwarz's inequality, we conclude from (5.4) that $(u + \frac{1}{2n}|x|^2)_{x_i x_j} \equiv 0$ in D, $1 \leq i, j \leq n$, which clearly implies D is a ball and u is radially symmetric about the center of D.

To obtain a version of Theorem 3 for solutions u to $u_t - \Delta u - 1 = 0$, where u satisfies boundary condition (A*) and $u(x,0) \equiv 0$, $x \in D$, continuously, we argue as in Garafalo and Alessandrini [2] to get

$$u(x,t) = \psi(x) + \sum_{m=1}^{\infty} b_m e^{-\lambda_m t}\phi_m(x), \quad x \in D, \quad 0 < t < T, \tag{5.6}$$

where $\Delta\psi = -1$ in D with $\psi = 0$ continuously on ∂D. Here $(\phi_m)_1^\infty$ is a complete set of orthonormal eigenfunctions for the Laplacian with Dirichlet boundary conditions and $\lambda_m \leq \lambda_{m+1}$, $m = 1, 2, ...$, are the eigenvalues for the Laplacian. Also

$$b_m = -\int_D \phi_m \psi dH^n, \quad m = 1, 2, ...$$

For fixed m we write

$$\phi_m = h + p$$

where h is harmonic in D and p is a solution to $\Delta p = -\lambda_m \phi_m$ in some ball B with $\bar{D} \subset B$ and $p = 0$ continuously on ∂B (Define $\phi_m \equiv 0$ outside D). The $L_2(\partial D)$ norm of the tangential derivatives of h can be estimated in terms of those of p, which in turn follow from well known estimates on λ_m, ϕ_m. Doing this and using Verchota's theorem again (see section 2) we get

(i) $(\nabla\phi_m)^* \in L_2(\partial D)$ with norm $\le cm^l$, for some $l = l(n) > 0$
(ii) $\nabla\phi_m(y) = \lim_{x\to y} \nabla\phi_m(x) = \pm|\nabla\phi_m(y)|n(y)$ radially for H^{n-1} a.e. y in ∂D, $m = 1, 2, \ldots$

Similar statements are true for ψ. Using (i), (ii), (A*), and (5.6) we deduce

$$-a(t) = \nabla\psi(y)\cdot n(y) + \sum_{m=1}^{\infty} b_m e^{-\lambda_m t}\nabla\phi_m(y)\cdot n(y), \tag{5.7}$$

for H^{n-1} a.e. $y \in \partial D$. If y_1, y_2, satisfy this equality, then

$$\nabla\psi(y_1)\cdot n(y_1) - \nabla\psi(y_2)\cdot n(y_2)$$

$$= \sum_{m=1}^{\infty} b_m e^{-\lambda_m t}[\nabla\phi_m(y_1)\cdot n(y_1) - \nabla\phi_m(y_2)\cdot n(y_1)] \tag{5.8}$$

(5.8) holds for $t > 0$ since the right hand side is real analytic in t for $t > 0$. Letting $t \to \infty$ we get, $|\nabla\psi(y)| = a$ for H^{n-1} a.e. $y \in \partial D$. From our previous proof of Theorem 1 for ψ we now conclude that D is a ball and by uniqueness of u that $u(\cdot, t)$ for $0 < t < T$ is symmetric about the center of D.

(3) Is the assumption, $H^n(D) \le \frac{1}{2}H^n(S_n)$, necessary in Theorem 2? Although we know of no counterexamples, it should be pointed out here that Serrin's Theorem is false when D is not contained in a hemisphere. The authors would like to thank Robert Molzon for pointing out this fact to us, by way of the following example, which is apparently due to Carlos Berenstein. Let

$$x_1 = \rho\cos\theta,\ x_2 = \rho\sin\theta\ \sin\phi,\ x_3 = \rho\sin\theta\ \cos\theta,$$

$$0 < \theta \le \pi, \quad 0 \le \phi < 2\pi, \quad \text{and}$$

$$\rho = (x_1^2 + x_2^2 + x_3^2)^{1/2}$$

be spherical coordinates in three space and define α, $0 < \alpha < \pi/2$, by $\cos\alpha = \frac{1}{\sqrt{3}}$. Let

$$u(\theta, \phi) = \frac{1}{2} - \frac{3}{2}\cos^2\theta, \quad \alpha < \theta < \pi - \alpha, \quad 0 < \phi \le 2\pi.$$

If $D = \{(\theta, \phi) : \alpha < \theta < \pi - \alpha, \quad 0 \le \phi < 2\pi\}$, then $u > 0$ in D and

$$\tilde{\Delta} u = -6u \quad \text{in } D \subseteq S_3,$$

where $\tilde{\Delta}$ denotes the spherical Laplacian, while $u = 0, \quad |\nabla u| = 3 \sin\alpha \cos\alpha$ on ∂D. Clearly D is not a spherical ball.

(4) Can a_0 be replaced in Theorem 4 by ∞?

(5) Does Theorem 5 remain valid if we do not assume (4.3) but still (a) D is of finite perimeter, (b) μ is a constant multiple of H^{n-1} measure on ∂D, and (c) (4.5) holds? Shapiro asks in [26] whether there exists "a pseudosphere in 3 space, that is, a surface homeomorphic (but not congruent) to a sphere with respect to which the average of each harmonic function equals the value of the function at some fixed point." In a future paper we shall give an affirmative answer to Shapiro's question.

The reader is invited to state and prove parabolic analogues of Theorem 5 for the Green's function and solutions u to $u_t = \Delta_x u + 1$. For this latter equation it appears difficult to use the argument of Alessandrini and Garofalo, since it is hard to see how (4.5) (where ∇ is replaced by ∇_x)can be used to estimate the eigenfunctions of the Laplacian in (5.7).

REFERENCES

[1] S. Agmon, A. Douglis, and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions*, Comm Pure Appl. Math. **12**, 623-727 (1959).

[2] G. Alessandrini and N. Garafalo, *Symmetry for degenerate parabolic equations*, to appear.

[3] H. Alt and L. Caffarelli, *Existence and regularity for a minimum problem with free boundary*, J. Reine Angew. Math. **325**, 105-144 (1981).

[4] H. Alt, L. Caffarelli, and A. Friedman, *A free boundary problem for quasi-linear elliptic equations*, Ann. Scoula Norm. Sup. Pisa (4) **11**, 1-44 (1984).

[5] R. Brown, *The method of layer potentials for the heat equation in Lipschitz cylinders*, Amer. J. Math. **111**, 339-379 (1989).

[6] R. Brown, *The initial-Neumann problem for the heat equation in Lipschitz cylinders*, to appear.

[7] L. Caffarelli, *A Harnack inequality approach to the regularity of free boundaries. PartI: Lipschitz free boundaries are $C^{1,\alpha}$*, Revista Mathematica Iberoamericana **3**, 139-162 (1987).

[8] L. Caffarelli, *A Harnack inequality approach to the regularity of free boundaries. Part II: Flat free boundaries are Lipschitz*, Comm. on Pure and Appl. Math. **42**, 55-78 (1989).

[9] B. Dahlberg, *Estimates on harmonic measure*, Arch. Rational Mech. Anal. **65**, 275-288 (1977).

[10] B. Dahlberg, *On the Poisson integral for Lipschitz and C^1 domains*, Studia Math. **86**, 7-24 (1979).

[11] P. Duren, *Theory of H^p spaces*, Academic Press, 1970.

[12] M. Essén and J. Lewis, *The generalized Ahlfors-Heins theorem in certain d-dimensional cones*, Math. Scand. **33**, 113-129 (1973).

[13] L. Evans and R. Gariepy, *Lecture notes on measure theory and fine properties of functions*, EPSCoR preprint series, University of Kentucky.

[14] E. Fabes and S. Salsa, *Estimates of caloric measure and the initial-Dirichlet problem for the heat equation in Lipschitz cylinders*, Trans. Amer. Math. Soc. **279**, 635-650 (1983).

[15] E. Fabes, D. Jerison, and C. Kenig, *Necessary and sufficient conditions for absolute continuity of elliptic harmonic measure*, Annals of Math. **119**, 121-141 (1984).

[16] R. Fefferman, C. Kenig, and J. Pipher, *The theory of weights and the Dirichlet problem for elliptic equations*, to appear.

[17] N. Garafalo and J. Lewis, *A symmetry result related to some overdetermined boundary value problems*, American Journal Math. **111**, 9-33 (1989).

[18] Hayman and Kennedy, *Subharmonic functions 1*, Academic Press, (1976).

[19] D. Jerison and C. Kenig, *The Neumann problem on Lipschitz domains*, Bull. Amer. Math. Soc. **4**, 203-207 (1981).

[20] J. Kemper, *Temperatures in several variables: Kernel functions, representations and parabolic boundary values*, Trans. Amer. Math. Soc. **167**, 243-262 (1972).

[21] D. Kinderlehrer and L. Nirenberg, *Regularity in free boundary problems*, Ann. Scoula Norm. Sup. Pisa (4) **4**, 372-391 (1977).

[22] J. Lewis and M. Murray, *The Dirichlet and Neumann problems for the heat equation in domains with time dependent boundaries*, in preparation.

[23] Payne and Schaeffer, *Duality theorems in some overdetermined boundary value problems*, to appear in Math. Methods Appl. Sci.

[24] P. Pucci and J. Serrin, *A general variational identity*, Indiana Univ. Math. J. **35**, (1986), no. 3.

[25] J. Serrin, *A symmetry problem in potential theory*, Arch. Rational Mech. Anal. **43**, 304-318 (1971).

[26] H. Shapiro, *Remarks concerning domains of Smirnov type*, Michigan Math. J. **13**, 341-348 (1966).

[27] E. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, 1970.

[28] G. Verchota, *Layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domains*, Jour. of Functional Analysis **59**, 572-611 (1984).

[29] A. Vogel, *Regularity and symmetry for general regions having a solution to certain overdetermined boundary values problems*, thesis, University of Kentucky, 1989.

[30] H. Weinberger, *Remark on the preceeding paper of Serrin*, Arch. Rat. Mech. Anal. **43**, 319-320 (1971).

[31] J. Wu, *On parabolic measures and subparabolic functions*, Trans. Amer. Math. Soc. **251**, 171-185 (1979).