VERY WEAK SOLUTIONS OF PARABOLIC SYSTEMS OF p-LAPLACIAN TYPE

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Abstract. We show that the standard assumptions on weak solutions to certain parabolic systems can be weakened and still the usual regularity properties of solutions can be obtained. In order to do this, we derive estimates for the solutions below the natural exponent and then apply reverse Hölder inequalities.

1. Introduction

Our work is motivated by classical Weyl's lemma: If a locally integrable function satisfies Laplace's equation in the sense of distributions, then it is real analytic. In other words, only a very modest requirement on the regularity of a solution is needed for a partial differential equation to make sense and then the equation gives extra regularity. We are interested in nonlinear parabolic systems of partial differential equations so that a counterpart of Weyl's lemma is too much to hope for, but the question of relaxing the standard Sobolev type assumptions on weak solutions and still obtaining regularity theory is the objective of our work.

We consider solutions to second order parabolic systems

$$\frac{\partial u_i}{\partial t} = \operatorname{div} A_i(x,t,\nabla u) + B_i(x,t,\nabla u), \qquad i = 1,\dots,N. \tag{1.1}$$

In particular, we are interested in systems of p-Laplacian type. The principal prototype is the p-parabolic system

$$\frac{\partial u_i}{\partial t} = \operatorname{div}(|\nabla u|^{p-2}\nabla u_i), \qquad i = 1,\dots,N,$$

with $1 < p < \infty$. Equations of type (1.1) have recently received new interest, see for example [D] and [KLi]. Solutions to (1.1) are usually taken in a weak sense and they are assumed to belong to a parabolic Sobolev space of order p. However, the weak formulation of (1.1) makes sense under a weaker assumption that the solution belongs to a Sobolev space of order r for some $r < p$. Thus we define very weak solutions of (1.1) to be those functions which satisy the usual integral identity

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associated with the weak formulation of (1.1) and which belong apriori to a weaker Sobolev space than the usual one. We show that very weak solutions are actually weak solutions when $p > 2n/(n+2)$, so possess the usual regularity properties of such solutions as boundedness, Hölder continuity and higher integrability, see [D] and [KLe]. In short, we are able to pass from an exponent below to an exponent which is above the natural Sobolev exponent for such a partial differential equation. We conclude this paper by making some brief remarks concerning the singular case $1 < p \leq 2n/(n+2)$. We remind the reader that for this range of p, weak solutions do not have to be even locally bounded.

In the elliptic case when the system is

$$\operatorname{div} A_i(x,t,\nabla u) + B_i(x,t,\nabla u) = 0, \qquad i = 1,\dots,N, \tag{1.2}$$

it is known that very weak solutions are weak solutions. When $p = 2$ and the system is linear this is due to Meyers [M]. Later Elcrat and Meyers [ME] extended the result to cover the case $1 < p < \infty$. They used a duality argument which is not available in the nonlinear situation. Later Iwaniec and coauthors (see [I] and [IS]) developed methods which proved the result for equations of p-Laplacian type and an alternative approach which also worked for higher order systems was given in [L]. Even though none of these methods apply directly to the parabolic case, our result is based on [L].

The major difficulty in dealing with a very weak solution u is that u times a cutoff function cannot be used as a test function in the weak formulation of the equation. This is a consequence of the assumption that u belongs to a Sobolev space below the natural exponent p. In [L] suitable test functions are constructed by using the Whitney extension theorem to extend u off the set where a certain maximal function is bounded. This approach appears to have first been used in [AF]. In the present case we encounter major difficulties with this approach. For example there is no natural maximal function of $|\nabla u|$. We use the so called strong maximal function. Extension of u off the set where this maximal function is bounded has to be done relative to weighted parabolic rectangles whose side length in either space or time depends on the given bound. Showing that such an extension can be used to get the usual Caccioppoli type inequality for the parabolic p-Laplacian involves some very delicate estimates especially as regards this inequality on time slices. Finally we obtain reverse Hölder inequalities similar to those obtained for weak solutions in [KLe].

Another problem is that in [L] an important part of the argument uses the fact that the Hardy-Littlewood maximal function raised to a sufficiently small positive power is an A_p weight in the sense of Muckenhoupt, thanks to a result of Coifman and Rochberg. In the parabolic case the strong maximal function need not have this property. We give an alternative argument which turns out to be somewhat simpler than the one in [L] even in the elliptic case.

As outlined above our argument is rather delicate and somewhat technical. In fact in an early preprint this paper was combined with [KLe] but in order to keep the reader from being swamped with technicalities we decided to divide it into two papers. Thus the reader is advised to have [KLe] at hand as we simply refer to the relevant parts in [KLe] instead of repeating all details here.

As far as we know there are no earlier results which deal with such fundamental questions as integrability below the natural exponent for the gradients of solutions to systems of nonlinear parabolic partial differential equations. Our results appear to be new even when $p = 2$.

2. Main result for very weak solutions

Let $\Omega \subset \mathbf{R}^n$ be an open set and let $W^{1,r}(\Omega)$ denote the Sobolev space of real valued functions g such that $g \in L^r(\Omega)$ and the distributional first partial derivatives $\partial g/\partial x_i$, $i = 1, 2, \dots, n$, exist in Ω and belong to $L^r(\Omega)$. The space $W^{1,r}(\Omega)$ is equipped with the norm

$$\|g\|_{1,r,\Omega} = \|g\|_{r,\Omega} + \sum_{i=1}^{n} \left\|\partial g/\partial x_i\right\|_{r,\Omega}.$$

Given $O \subset \mathbf{R}^n$ open, N a positive integer, $-\infty \le S < T \le \infty$, let

$$u = (u_1, \dots, u_N)\colon O \times (S, T) \to \mathbf{R}^N$$

and suppose that whenever $p > \max\{2n/(n+2), 1\}$, $-\infty \le S < S_1 < T_1 < T \le \infty$ and $\overline{\Omega} \subset O$ we have

$$u \in L^2\big(\Omega \times [S_1, T_1]\big) \cap L^r\big([S_1, T_1]; W^{1,r}(\Omega)\big), \tag{2.1}$$

where $p - 1/2 < r < p$. Here the notation $L^r\big([S_1, T_1]; W^{1,r}(\Omega)\big)$ means that for almost every t, $S_1 < t < T_1$, with respect to one dimensional Lebesgue measure, the function $x \mapsto u(x, t)$ is in $W^{1,r}(\Omega)$ componentwise and

$$|||u|||^r_{r,\Omega} = \|u\|^r_{r,\Omega\times(S_1,T_1)} + \int_{S_1}^{T_1} \sum_{i=1}^{N} \left\|u_i(\cdot, t)\right\|^r_{1,r,\Omega} dt < \infty. \tag{2.2}$$

Let ∇u denote the distributional gradient of u (taken componentwise) in the x variable only.

We suppose that $A = (A_1, \dots, A_N)$, where

$$A_i = A_i(x, t, \nabla u)\colon O \times (S, T) \times \mathbf{R}^{nN} \to \mathbf{R}^n,$$

and $B = (B_1, \dots, B_N)$, where

$$B_i = B_i(x, t, \nabla u)\colon O \times (S, T) \times \mathbf{R}^{nN} \to \mathbf{R},$$

are Lebesgue $n+1$ measurable functions on $O\times(S, T)$. This is the case, for example, if A_i and B_i, $i = 1, 2, \dots, N$, satisfy the well known Carathéodory type conditions. We assume that there exist positive constants c_i, $i = 1, 2, 3$, such that

$$|A_i| \le c_1|\nabla u|^{p-1} + h_1 \tag{2.3}$$

$$|B_i| \le c_2\,|\nabla u|^{p-1} + h_2, \tag{2.4}$$

and

$$\sum_{i=1}^{N}\langle A_i, \nabla u_i\rangle \ge c_3|\nabla u|^p - h_3, \tag{2.5}$$

for $i = 1, 2, \ldots, N$, and almost every $(x, t) \in O \times (S, T)$. Here $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbf{R}^n$ and h_i, $i = 1, 2, 3$, are measurable functions in $O \times (S, T)$ so that

$$c_4 = \|h^p\|\|_{\widehat{q}, O\times(S,T)} < \infty, \tag{2.6}$$

where $\widehat{q} > 1$ and

$$h^p = (|h_1| + |h_2|)^{p/(p-1)} + |h_3|.$$

Finally u satisfying (2.1) is said to be a very weak solution in $O \times (S, T)$ to the nonlinear parabolic system

$$\frac{\partial u_i}{\partial t} = \operatorname{div} A_i(x, t, \nabla u) + B_i(x, t, \nabla u), \qquad i = 1, \ldots, N,$$

if the structural conditions (2.3)-(2.6) hold and

$$\int_S^T \int_O \sum_{i=1}^N \Big(- u_i \frac{\partial \phi_i}{\partial t} + \langle A_i, \nabla \phi_i \rangle - B_i \phi_i \Big) \, dx \, dt = 0 \tag{2.7}$$

for every test function $\phi = (\phi_1, \ldots, \phi_N) \in C_0^\infty(O \times (S, T))$. Observe that if r is replaced by p in (2.1), then u is said to be a weak solution.

The following theorem is our main result.

Theorem 2.8. *Let $p > 2n/(n+2)$. Then there exists $\beta > 0$ such that if u is a very weak solution to* (1.1) *with $r = p - \beta$, then*

$$u \in L^2\big(\Omega \times [S_1, T_1]\big) \cap L^{p+\beta}\big([S_1, T_1]; W^{1,p+\beta}(\Omega)\big),$$

where $\beta > 0$ depends only on n, p, $\widehat{q}$, and c_i, for $i = 1, 2, 3$, while $|||u|||_{p+\beta,\Omega}$ depends on these quantities as well as N, Ω, S_1, T_1 and c_4.

We present the proof of our main result in Section 4.

Remark 2.9. Theorem 2.8 implies that u is a weak solution to (1.1), so as in [D] it can be shown for $N = 1$ and $h_i \equiv 0$, $i = 1, 2, 3$, that u has a representative which is Hölder continuous on compact subsets of $O \times (S, T)$.

3. Preliminary reductions

Given r, $s > 0$, $(x, t) \in \mathbf{R}^{n+1}$, let

$$D_r(x) = \{y \in \mathbf{R}^n : |y_i - x_i| < r, \quad i = 1, 2, \ldots, n\}$$

and

$$Q_{r,s}(x, t) = D_r(x) \times (t - s, t + s)$$

a rectangle in $\mathbf{R}^{n+1}$. Let $|E|$ denote the Lebesgue $(n+1)$-measure of the measurable set E and if f is integrable on E with $0 < |E| < \infty$, then the integral average of f over E is

$$\fint_E f \, dx \, dt = \frac{1}{|E|} \int_E f \, dx \, dt.$$

If $Q_{\rho,s}(z,\tau)\subset O\times(S,T)$, then

$$I_\rho(t)=I_\rho(t,u,z,\tau)=m(D_\rho(z))^{-1}\int_{D_\rho(z)}u(x,t)\,dx,$$

whenever $\tau-s<t<\tau+s$. Here m denotes Lebesgue measure in $\mathbf{R}^n$ and the integral is taken componentwise.

Let Q be a rectangle in $\mathbf{R}^{n+1}$. We write

$$a=a(Q)=(a_1(Q),\dots,a_N(Q)),\quad\text{where}\quad a_i(Q)=\fint_Q u_i\,dx\,dt,$$

for $i=1,2,\dots,N$.

We begin with a useful lemma, which was proved for weak solutions in [KLe] (see Lemma 3.1). However, the same proof gives the result for very weak solutions as well.

Lemma 3.1. *Suppose that u is a very weak solution to* (1.1) *with $r>\min\{p-1/2,1\}$. If $Q_{4\rho,s}(z,\tau)\subset O\times(S,T)$, then there exists $\widehat\rho$, $\rho<\widehat\rho<2\rho$, and a constant c depending on p, n, c_1 and c_2, such that*

$$|I_{\widehat\rho}(t_2)-I_{\widehat\rho}(t_1)|\le cs\rho^{-1}\fint_{Q_{2\rho,s}(z,\tau)}\big(|\nabla u|^{p-1}+|h_1|+|h_2|\big)\,dx\,dt$$

for almost all t_i with $|t_i-\tau|<s$, $i=1,2$.

We assume that u is a very weak solution to (1.1) and

$$\max\big\{p-1/4,(1+2p)/3\big\}<r=p-\beta<p.$$

To prove Theorem 2.8, we essentially prove analogues of Propositions 4.2 and 4.14 of [KLe] with p replaced by $p-\beta$.

We assume, as we may, that $r=1$ and $(\widehat x,\widehat t)=(0,0)$, since otherwise we consider

$$v(x,t)=u(\widehat x+rx,\widehat t+r^pt)$$

for $(x,t)\in Q_{10,10^p}(0,0)$. It is easily seen that v is a weak solution to a partial differential equation similar to (1.1) and with the same structure. Proving claims for v with $r=1$ relative to (0,0) and then transforming back we get the result for the original u.

Let $\theta\in C_0^\infty(-1,1)$, such that θ equals a constant which is greater than 1/100 on $(-3/4,3/4)$, θ is even, positive, and

$$\int_{\mathbf{R}}\theta(\tau)\,d\tau=1.$$

If $f\colon Q_{10,10^p}(0,0)\to\mathbf{R}$ is locally Lebesgue integrable we put

$$f_\varepsilon(x,t)=c\,\gamma^{-1}\varepsilon^{-n-2}\int f(y,\tau)\,\theta\Big(\frac{t-\tau}{\gamma\varepsilon^2}\Big)\,\theta\Big(\frac{|x-y|}{\varepsilon}\Big)\,dy\,d\tau$$

whenever $0 < \varepsilon < 1/10$ and $(x,t) \in Q_{8,8^p}(0,0)$, where c is chosen so that

$$c \int_{\mathbf{R}^n} \int_{\mathbf{R}} \theta(\tau)\, \theta(|y|)\, d\tau\, dy = 1$$

and $\gamma > 0$ will be chosen later. Next let $\widehat{Q} = Q_{10^3\widehat{\rho},10^6\widehat{s}}(z,\tau) \subset Q_{2,2^p}(0,0)$, with $\widehat{\rho} \le \rho \le 10^2\widehat{\rho}$, and $\widehat{s} \le s \le 10^4\widehat{s}$. Set

$$\widetilde{\phi}(x,t) = \theta\Big(\frac{|x-z|}{4\rho}\Big)\, \theta\Big(\frac{|t-\tau|}{16s}\Big)$$

whenever $(x,t) \in \mathbf{R}^{n+1}$. Note that for fixed $x \in \mathbf{R}^n$ the function $\widetilde{\phi}(x,\cdot)$ is constant on $(\tau - 12s, \tau + 12s)$, $\widetilde{\phi}$ is constant on $Q_{3\rho,12s}(z,\tau)$, $\widetilde{\phi} \in C_0^\infty\big(Q_{4\rho,16s}(z,\tau)\big)$, and

$$\rho^{-1} \big\| |\nabla \widetilde{\phi}| \big\|_\infty + s^{-1} \Big\| \frac{\partial \widetilde{\phi}}{\partial t} \Big\|_\infty \le c(n) < \infty.$$

Let $Q_{\rho,s}(z,\tau) = Q$, fix ε with $0 < \varepsilon < 10^{-6} \min\{\rho, s\}$ and put

$$\widetilde{u}(x,t) = \begin{cases} (u(x,t) - a(Q))_\varepsilon \widetilde{\phi}(x,t), & \text{when} \quad (x,t) \in Q_{8,8^p}(0,0), \quad \text{and} \\ 0 & \text{otherwise.} \end{cases}$$

Let $\xi = \max\big\{p - 1/2, (1+p)/2\big\}$ and

$$\lambda_1^\xi = \fint_{\widehat{Q}} \big(|\nabla u| + |h|\big)^\xi \, dx\, dt. \tag{3.2}$$

Next for a locally integrable function $g\colon \mathbf{R}^n \to [-\infty, \infty]$, let Mg be the strong maximal function defined by

$$Mg(x,t) = \sup_{\widetilde{Q}} \fint_{\widetilde{Q}} |g| \, dz\, d\tau,$$

where the supremum is taken over all rectangles $\widetilde{Q}$ with sides parallel to the coordinate axes and $(x,t) \in \widetilde{Q}$. An iteration of the one-dimensional Hardy-Littlewood maximal theorem implies that

$$\|Mg\|_\sigma \le c(n,\sigma)\, \|g\|_\sigma$$

for $\sigma > 1$.

Let $\lambda \ge c_5(n,p)\lambda_1 = \lambda_2$, $Q^+ = Q_{6\rho,36s}(z,\tau)$, $\widehat{\xi} = \max\{p-1, 1\}$, and set

$$\begin{aligned} &E(\lambda) = \big\{(x,t) \in \mathbf{R}^{n+1} \colon \big(M((|\nabla u| + |h|)^{\widehat{\xi}} \chi_{Q^+})(x,t)\big)^{1/\widehat{\xi}} \le \lambda\big\}, \\ &S = \big\{(x,t) \in \mathbf{R}^{n+1} \colon |t - \tau| \le 16s\big\}, \\ &S' = \big\{(x,t) \in \mathbf{R}^{n+1} \colon |t - \tau| \le 6s\big\}, \\ &S'' = \big\{(x,t) \in \mathbf{R}^{n+1} \colon |t - \tau| \le 12s\big\}, \\ &\widehat{E}(\lambda) = E(\lambda) \cap S. \end{aligned} \tag{3.3}$$

Here χ_{Q^+} is the characteristic function of Q^+. From our definition of a weak solution and the Hardy-Littlewood maximal theorem it follows that there is $c_5(n,p) \ge 1$ so that

$$\widehat{E}(\lambda) \cap Q \ne \emptyset \quad \text{when} \quad \lambda \ge \lambda_2. \tag{3.4}$$

We shall need an analogue of Lemma 3.1 for $\widetilde{u}$.

Lemma 3.5. *Let $0 < \varepsilon < 10^{-6}\min\{\rho, s\}$, $\gamma = \lambda^{2-p}$ and $\widehat{\lambda} = \max\{\lambda_1^{p-1}s\rho^{-2}, \lambda\}$. Suppose that*

$$Q_{r,\gamma r^2}(x,t) \cap \widehat{E}(c_5\lambda) \neq \emptyset$$

for some $\lambda \geq \lambda_1$ and $Q_{r,\gamma r^2}(x,t) \subset S''$. Then there exists r^*, $r < r^* < 2r$, such that for $I_{r^*}(t_i) = I_{r^*}(t_i, \widetilde{u}, x, t)$, $i = 1, 2$, we have

$$|I_{r^*}(t_2) - I_{r^*}(t_1)| \leq c\widehat{\lambda}\min\{r, \rho\},$$

whenever $|t_i - t| \leq \gamma r^2$, for $i = 1, 2$. Here c depends on n, N, p, and c_i, $i = 1, 2, 3, 4$.

Proof. To prove Lemma 3.5, let $\delta, \eta > 0$ be small, $\psi_1 \in C_0^\infty(t_1 - \eta, t_2 + \eta)$ with $\psi_1 \equiv 1$ on (t_1, t_2) and $\psi_2 \in C_0^\infty(D_{r^*+\delta}(x))$ with $\psi_2 \equiv 1$ on $D_{r^*}(x)$. Let

$$\phi_j = (\widetilde{\phi}\psi_1\psi_2)_\varepsilon, \qquad j = 1, 2, \ldots, N,$$

We use $(0, \ldots, \phi_j, \ldots, 0)$ as a test function in (2.7). If we denote

$$\begin{aligned}
&\phi = (\phi_1, \ldots, \phi_N),\\
&\langle A(\cdot, \nabla u), \nabla\phi\rangle = (\langle A_1(\cdot, \nabla u), \nabla\phi_1\rangle, \ldots, \langle A_N(\cdot, \nabla u), \nabla\phi_N\rangle), \quad \text{and}\\
&B\phi = (B_1\phi_1, \ldots, B_N\phi_N),
\end{aligned}$$

we get using simple properties of convolutions that

$$\begin{aligned}
K_1 &= \int \widetilde{u}\frac{\partial}{\partial\tau}(\psi_1\psi_2)\,dz\,d\tau\\
&= -\int (u - a(Q))_\varepsilon\psi_1\psi_2\frac{\partial\widetilde{\phi}}{\partial\tau}\,dz\,d\tau + \int \big(\langle A(\cdot, \nabla u), \nabla\phi\rangle - B\phi\big)\,dz\,d\tau \qquad (3.6)\\
&= K_2 + K_3.
\end{aligned}$$

Letting first $\eta \to 0$ and then $\delta \to 0$ we find that

$$m(D_{r^*}(x))^{-1}K_1 \to -I_{r^*}(t_2) + I_{r^*}(t_1). \tag{3.7}$$

We have $K_2 \equiv 0$, since $\phi(x, \cdot)$ is constant on $[\tau - 12s, \tau + 12s]$. Also as $\eta, \delta \to 0$, we see that K_3 converges to the N vector whose ith component is

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_{D_{r^*}(x)} \big(\langle A_i(\cdot, \nabla u)_\varepsilon, \nabla\widetilde{\phi}\rangle - (B_i)_\varepsilon\widetilde{\phi}\big)\,dz\,d\tau\\
&\qquad - \int_{\partial D_{r^*}(x)\times(t_1,t_2)} \langle A_i(\cdot, \nabla u)_\varepsilon, \nu\rangle\widetilde{\phi}\,d\sigma\,d\tau,
\end{aligned} \tag{3.8}$$

where ν is the outer unit normal to $D_{r^*}(x)$ considered as a subset of $\mathbf{R}^n$, and σ is $n - 1$ dimensional surface area on the boundary of this set. The integrands in (3.8) are understood to be zero outside the support of $\widetilde{\phi}$.

Now we consider two cases. First suppose that $r \geq \rho/10^2$. Since $\widetilde{\phi}$ vanishes outside of $Q_{4\rho,16s}(z, \tau)$, it is easily seen that r^*, ρ^* and $\widetilde{x}$, can be chosen so that $r < r^* < 2r$, $\rho/10^2 \leq \rho^* \leq 20\rho$, and

$$\int_{D_{\rho^*}(\widetilde{x})} \widetilde{u}(z, t)\,dz = \int_{D_{r^*}(x)} \widetilde{u}(z, t)\,dz,$$

whenever $t \in \mathbf{R}$. Thus we can replace r^* by ρ^* in (3.6)-(3.8). Using (3.8), (2.3), (2.4), we find for properly chosen ρ^* and $t_1, t_2 \in \mathbf{R}$ that

$$\begin{aligned}\lim_{\eta,\delta\to 0} & m(D_{r^*}(x))^{-1}|K_3| \\ & \le cs\rho^{-1} \fint_{Q^+} \big(|\nabla u|^{p-1} + |h_1| + |h_2|\big)\,dz\,d\tau \le c\widehat{\lambda}\rho.\end{aligned} \tag{3.9}$$

Next if $r \le \rho/10^2$, we see from the definition of $\widehat{E}(\lambda)$ in (3.3) that (3.9) is still valid. To be more precise, if $r' = 2r + 2\varepsilon$ and $s' = \gamma(2r+2\varepsilon)^2$, then there is r^*, $r < r^* < 2r$, such that we have

$$\begin{aligned}\lim_{\eta,\delta\to 0} & m(D_{r^*}(x))^{-1}|K_3| \\ & \le c\gamma r \fint_{Q_{r',s'}(x,t)} \big(|\nabla u|^{p-1} + |h_1| + |h_2|\big)\chi_{Q^+}\,dz\,d\tau \le c\widehat{\lambda} r.\end{aligned} \tag{3.10}$$

From (3.6)-(3.10) we conclude that Lemma 3.5 is valid. $\square$

Since $\widehat{E}(\lambda)$ is closed and λ is as in (3.4) we can use a Whitney type argument see ([S, Chapter VI]), to divide $\mathbf{R}^{n+1} \setminus \widehat{E}(\lambda)$ into rectangles $Q_i = Q_{r_i,\gamma r_i^2}(x_i,t_i)$, $i = 1, 2, \dots$, with

$$10^{-10} n^{-1} d_\lambda\big(Q_i, \widehat{E}(\lambda)\big) \le r_i \le 10^{-5} n^{-1} d_\lambda\big(Q_i, \widehat{E}(\lambda)\big), \tag{3.11}$$

where

$$\begin{aligned}d_\lambda(G,H) = \inf\big\{ & |z_2 - z_1| + \lambda^{(p-2)/2}|\tau_2 - \tau_1|^{1/2} : \\ & (z_1,\tau_1) \in G,\ (z_2,\tau_2) \in H\big\}.\end{aligned} \tag{3.12}$$

With ε and λ fixed as in Lemma 3.5, we define $v = v(\cdot,\varepsilon,\lambda)$ on $\mathbf{R}^{n+1}$ by

$$v(x,t) = \begin{cases} \widetilde{u}(x,t), & \text{when} \quad (x,t) \in \widehat{E}(\lambda), \quad \text{and} \\ \sum_i a(Q_i,\widetilde{u}) w_i(x,t), & \text{when} \quad (x,t) \in \mathbf{R}^{n+1} \setminus \widehat{E}(\lambda). \end{cases}$$

Here $\{w_i\}$ is a partition of unity of $\mathbf{R}^{n+1} \setminus \widehat{E}(\lambda)$ adapted to the covering $\{Q_i\}$. By this we mean that for $i = 1, 2, \dots$ we have $w_i \in C_0^\infty(Q_{2r_i,4\gamma r_i^2}(x_i,t_i))$, $0 \le w_i \le 1$, $w_i \ge c(n)^{-1}$ on Q_i,

$$r_i^{-1}\|\nabla w_i\|_\infty + \gamma^{-1} r_i^{-2}\left\|\frac{\partial w_i}{\partial t}\right\|_\infty \le c$$

and

$$\sum_i w_i(x,t) = 1$$

for all $(x,t) \in \mathbf{R}^{n+1} \setminus \widehat{E}(\lambda)$.

We collect the basic properties of the function v into the following lemma.

Lemma 3.13. *There exists $c \geq 1$ with the same dependence as in Lemma 3.5 such that for every $\lambda \geq \lambda_2$ the following claims are true.*

(a) $\big\| |v\chi_{\mathbf{R}^{n+1}\setminus \widehat{E}(\lambda)}| \big\|_\infty \leq c\widehat{\lambda}\rho$.

(b) *The function $v(\cdot,t)$ is locally Lipschitz in the variable x on S' with Lipschitz constant independent of ε and t.*

(c) *The function v is locally Lipschitz on $S' \setminus \widehat{E}(\lambda)$ with Lipschitz constant independent of ε.*

(d)

$$\begin{aligned}\int_{S'\setminus \widehat{E}(\lambda)} &\left| \frac{\partial v}{\partial t}(\widetilde{u}-v) \right| dz\, d\tau \\ &\leq c\widehat{\lambda}^2\lambda^{p-2}|\mathbf{R}^{n+1}\setminus E(\lambda)| + cs^{-1}\int_{Q^+} |u-a(Q)|^2\, dz\, d\tau.\end{aligned}$$

(e)

$$\begin{aligned}\int_{S'\setminus \widehat{E}(\lambda)} &(|\nabla u| + h)^{p-1}(\rho^{-1}|v| + |\nabla v|)\, dz\, d\tau \\ &\leq c\widehat{\lambda}^2\lambda^{p-2}|\mathbf{R}^{n+1}\setminus E(\lambda)| + cs^{-1}\int_{Q^+} |u-a(Q)|^2\, dz\, d\tau.\end{aligned}$$

(f) *The function $(\widetilde{u}-v)^2$ has distributional partial derivatives in t on S'.*

Proof. Suppose that

$$Q_{10^{10}nr,\gamma 10^{20}n^2r^2}(x,t) \cap \widehat{E}(\lambda) \neq \emptyset$$

and let $r' = 2r + 2\varepsilon$ and $s' = \gamma(2r+2\varepsilon)^2$. Choose Q^- so that Lemma 3.1 holds with $Q_{\widehat{\rho},s}(z,\tau)$ replaced by Q^- and $Q_{5\rho,25s}(z,\tau) \subset Q^- \subset Q^+$. We claim that

$$T = \fint_{Q_{r',s'}(x,t)} |u-a(Q)|\chi_{Q^-}\, dz\, d\tau \leq c\rho\widehat{\lambda}. \tag{3.14}$$

Now

$$|a(Q) - a(Q^-)| \leq c\fint_{Q^-} |u - a(Q^-)|\chi_{Q^-}\, dz\, d\tau \leq c\rho\widehat{\lambda}$$

as we find from using Poincáre's inequality and Lemma 3.1. Thus it suffices to prove (3.14) with Q replaced by Q^-.

If $r' \geq \rho$, this inequality follows once again from Lemma 3.1 and Poincáre's inequality. Otherwise, let l be the least positive integer such that $2^l > \rho$. Choose $Q_i' = Q_{r_i',s'}(x,t)$ such that $2^i r' \leq r_i' \leq 2^{i+1}r'$ for $i = 1,2,\dots,l$ and Lemma 3.1 holds with $Q_{\widehat{\rho},s}(z,\tau)$ replaced by Q_i'. Using the triangle inequality, Lemma 3.1, Poincáre's inequality, (3.2) and (3.3) we get

$$\begin{aligned}T &\leq c\sum_{i=1}^{l} \fint_{Q_i'} |u - a(Q_i')|\chi_{Q^-}\, dz\, d\tau + \fint_{Q_l'} |u-a(Q^-)|\chi_{Q^-}\, dz\, d\tau \\ &\leq cr'\lambda\sum_{i=1}^{l} 2^i + c\rho\widehat{\lambda} \leq c\rho\widehat{\lambda}.\end{aligned}$$

Thus claim (3.14) is true.

Now suppose that $(x',t') \in \overline{Q}_i \subset \mathbf{R}^{n+1} \setminus \widehat{E}(\lambda)$. Let

$$\mu_i = \{j : w_j \neq 0 \text{ on } \operatorname{supp} w_i\}, \qquad i = 1, 2, \dots,$$

and observe from (3.11) and (3.12) that (3.14) holds with $r = r_j$. Hence

$$|v(x',t')| \leq c \sum_{j \in \mu_i} a(Q_j, |\widetilde{u}|) \leq c \sum_{j \in \mu_i} \fint_{Q_j} \big|(u - a(Q))_\varepsilon\big| \, dz \, d\tau \leq c\rho\widehat{\lambda}.$$

From this inequality we conclude that Lemma 3.13 (a) is valid.

Next let Q_i be a Whitney rectangle as above and suppose that $Q_i \cap S' \neq \emptyset$. Choose r and $(x,t) \in \widehat{E}(\lambda)$, so that

$$\bigcup_{j \in \mu_i} \operatorname{supp} w_j \subset Q_{r,\gamma r^2}(x,t) \tag{3.15}$$

and

$$c(p,n)^{-1} r \leq r_j \leq c(p,n) r \quad \text{for} \quad j \in \mu_i. \tag{3.16}$$

Again the existence of r follows from (3.11) and (3.12).

We consider two cases. If $Q_{r,\gamma r^2}(x,t) \subset S''$, let r^* be as in Lemma 3.5 and set $Q^* = Q_{r^*,\gamma r^2}(x,t)$. From Lemma 3.5, (3.14), and Poincáre's inequality we deduce for $j \in \mu_i$ that

$$\begin{aligned}
\big|a(Q_j, \widetilde{u}) - a(Q^*, \widetilde{u})\big| &\leq \fint_{Q^*} |\widetilde{u} - a(Q^*, \widetilde{u})| \, dz \, d\tau \\
&\leq c \fint_{Q^*} |\widetilde{u} - I_{r^*}(\cdot)| \, dz \, d\tau + c \sup_{|t'-t| \leq \gamma r^2} |I_{r^*}(t') - I_{r^*}(t)| \\
&\leq c \min\{r, \rho\} \fint_{Q^*} |\nabla \widetilde{u}| \, dz + c \min\{r, \rho\} \widehat{\lambda} \\
&\leq c \min\{r, \rho\} \rho^{-1} \fint_{Q^*} \big|(u - a(Q))_\varepsilon\big| \chi_{Q_{5\rho,25s}(z,\tau)} \, dz \, d\tau + c \min\{r, \rho\} \widehat{\lambda} \\
&\leq c \min\{r, \rho\} \widehat{\lambda}.
\end{aligned}$$

We conclude that

$$|\nabla v|(x',t') \leq c r_i^{-1} \sum_{j \in \mu_i} |a(Q_j, \widetilde{u}) - a(Q^*, \widetilde{u})| \leq c\widehat{\lambda}, \tag{3.17}$$

and

$$\left|\frac{\partial v}{\partial t}\right|(x',t') \leq c r_i^{-2} \gamma^{-1} \sum_{j \in \mu_i} |a(Q_j, \widetilde{u}) - a(Q^*, \widetilde{u})| \leq c\widehat{\lambda} \lambda^{p-2} r_i^{-1} \tag{3.18}$$

for almost every $(x',t') \in \overline{Q}_i$. If $Q_{r,\gamma r^2}(x,t) \cap (\mathbf{R}^{n+1} \setminus S'') \neq \emptyset$ then $c(n)\gamma r_i^2 \geq s$ and so using (3.14) we find as in (3.17) and (3.18) that

$$|\nabla v|(x',t') \leq c\rho\widehat{\lambda}/r_i \leq c\widehat{\lambda}\lambda^{(2-p)/2} \rho s^{-1/2} \tag{3.19}$$

and

$$\left|\frac{\partial v}{\partial t}\right|(x',t') \le cs^{-1}\widehat{\lambda}\rho \tag{3.20}$$

for almost every $(x',t') \in \overline{Q}_i$. Thus (c) of Lemma 3.13 is valid.

To complete the proof of (b) let $(x',t') \in S' \cap \widehat{E}(\lambda)$ and $(x,t') \in Q_i$. If (x'',t') is a point in $\widehat{E}(\lambda)$ on the line segment connecting these two points and nearest (x,t'), then for $|x-x'|$ small enough we deduce from (3.17), (3.18) and continuity of v that

$$\begin{aligned}|v(x,t')-v(x',t')| &\le |v(x,t')-v(x'',t')| + |v(x',t')-v(x'',t')|\\ &\le c\widehat{\lambda}\,|x-x'| + |v(x',t')-v(x'',t')|.\end{aligned}$$

We observe that

$$|v(x',t')-v(x'',t')| = |\widetilde{u}(x',t')-\widetilde{u}(x'',t')|.$$

Suppose that $|x-x'|$ is so small that $Q_{2r,4\gamma r^2}(x',t') \subset S''$ when $r = 2|x'-x''|$. Let r^* be as in Lemma 3.5 and set $Q_0^* = Q_{r^*,\gamma r^2}(x',t')$. Then clearly

$$|\widetilde{u}(x',t')-\widetilde{u}(x'',t')| \le |\widetilde{u}(x',t')-a(Q_0^*,\widetilde{u})| + |\widetilde{u}(x'',t')-a(Q_0^*,\widetilde{u})|.$$

To estimate the first term on the righthand side of this equation, let

$$Q_j^* = Q_{2^{-j}r,\gamma 4^{-j}r^2}(x',t'), \qquad j = 1,2,\dots.$$

Using Lemma 3.5 and arguing as in the proof of (3.17) and (3.18) we find that

$$\left|a(Q_{j+1}^*,\widetilde{u}) - a(Q_j^*,\widetilde{u})\right| \le c2^{-j}\widehat{\lambda}\,|x'-x''|,$$

for $j = 1,2,\dots$ From this inequality and continuity of $\widetilde{u}$ it follows that

$$|\widetilde{u}(x',t')-a(Q_0^*,\widetilde{u})| \le \sum_{i=1}^{\infty}\left|a(Q_{j+1}^*,\widetilde{u}) - a(Q_j^*,\widetilde{u})\right| \le c\widehat{\lambda}\,|x'-x''|.$$

The term $|u(x'',t')-a(Q_0^*,\widetilde{u})|$ can be estimated similarly. Hence

$$|v(x',t')-v(x,t')| \le c\widehat{\lambda}\,|x-x'|.$$

If $(x,t') \in \widehat{E}(\lambda)$ we can repeat the above argument with (x'',t) replaced by (x,t) to see that the above inequality is true. In view of this inequality, (3.17) and (3.19) we conclude that (b) of Lemma 3.13 holds.

To prove (d) and (e) we let Θ_1 be the set of those indices i for which there exists $Q_{r,\gamma r^2}(x,t)$ satisfying (3.15) and (3.16) with $\operatorname{supp} w_i \cap S' \ne \emptyset$ and $Q_{2r,4\gamma r^2}(x,t) \subset S''$. Put

$$\Theta_2 = \big\{i\colon \operatorname{supp} w_i \cap S' \ne \emptyset \quad \text{and} \quad i \notin \Theta_1\big\}.$$

From Lemma 3.13 (a), (3.17), (3.18) and the same argument as in proving these inequalities, we obtain

$$
\begin{aligned}
&\sum_{i\in\Theta_1}\int_{Q_i\cap S'}\left(\left|\frac{\partial v}{\partial t}\right||\widetilde{u}-v|+(|\nabla u|+h)^{p-1}(\rho^{-1}|v|+|\nabla v|)\right)dz\,d\tau\\
&\qquad\le c\widehat{\lambda}\lambda^{p-2}\sum_{i\in\Theta_1}\int_{Q_i\cap S'}r_i^{-1}|\widetilde{u}-a(\overline{Q}_i,\widetilde{u})|\,dz\,d\tau\\
&\qquad\quad+c\widehat{\lambda}\sum_{i\in\Theta_1}\sum_{j\in\mu_i}\int_{Q_i\cap S'}(|\nabla u|+h)^{p-1}\,dz\,d\tau\\
&\qquad\quad+c\widehat{\lambda}\lambda^{p-2}\sum_{i\in\Theta_1}r_i^{-1}|Q_i\cap S'|\sum_{j\in\mu_i}|a(Q_j,\widetilde{u})-a(Q_i,\widetilde{u})|\\
&\qquad\le c\widehat{\lambda}^2\lambda^{p-2}\sum_{i\in\Theta_1}|Q_i|\\
&\qquad\le c\widehat{\lambda}^2\lambda^{p-2}|\mathbf{R}^{n+1}\setminus E(\lambda)|.
\end{aligned}
\tag{3.21}
$$

Also using Hölder's inequality, (3.3), and the fact that $c(n)\gamma r_i^2\ge s$ if $i\in\Theta_2$, we obtain

$$
\begin{aligned}
&\sum_{i\in\Theta_2}\int_{Q_i\cap S'}\left(\left|\frac{\partial v}{\partial t}\right||\widetilde{u}-v|+(|\nabla u|+h)^{p-1}(\rho^{-1}|v|+|\nabla v|)\right)dz\,d\tau\\
&\qquad\le cs^{-1}\sum_{i\in\Theta_2}\sum_{j\in\mu_i}|Q_i\cap S|a(Q_j,|\widetilde{u}|)^2+c\widehat{\lambda}\sum_{i\in\Theta_2}\int_{Q_i\cap S'}(|\nabla u|+h)^{p-1}\,dz\,d\tau\\
&\qquad\quad+c\lambda^{(2-p)/2}s^{-1/2}\sum_{i\in\Theta_2}\int_{Q_i\cap S'}(|\nabla u|+h)^{p-1}\,dz\,d\tau\sum_{j\in\mu_i}a(Q_j,|\widetilde{u}|)\\
&\qquad\le cs^{-1}\sum_{i\in\Theta_2}\sum_{j\in\mu_i}|Q_i\cap S|a(Q_j,|\widetilde{u}|)^2+c\widehat{\lambda}\lambda^{p-1}\sum_{i\in\Theta_2}|Q_i\cap S|\\
&\qquad\le cs^{-1}\int_{Q^+}|u-a(Q)|^2\,dz\,d\tau+c\widehat{\lambda}\lambda^{p-1}|\mathbf{R}^{n+1}\setminus E(\lambda)|.
\end{aligned}
\tag{3.22}
$$

Here we have used the fact that $\widehat{E}(\lambda)=E(\lambda)\cap S$. Clearly, (3.21), (3.22) imply the claims (d) and (e) of Lemma 3.13.

To prove (f) observe from the usual Whitney type argument that $\widetilde{u}-v$ is continuous in $\mathbf{R}^{n+1}$ and vanishes on $\widehat{E}(\lambda)$. This fact, the claim (d), and a standard argument give (f). The proof of Lemma 3.13 is now complete. □

With λ still fixed we let $\varepsilon\to 0$ and note from simple properties of convolutions that $v(\cdot,\varepsilon,\lambda)\to w(\cdot,\lambda)$ pointwise for almost every (x,t). In fact if

$$
u'(x,t)=\begin{cases}(u(x,t)-a(Q,u))\widetilde{\phi}(x,t), & \text{when}\quad (x,t)\in Q_{4\rho,16s}(z,\tau),\quad\text{and}\\ 0 & \text{otherwise,}\end{cases}
$$

then

$$
w(x,t)=\begin{cases}u'(x,t), & \text{when}\quad (x,t)\in\widehat{E}(\lambda),\quad\text{and}\\ \sum_i a(Q_i,u')w_i(x,t), & \text{when}\quad (x,t)\in\mathbf{R}^{n+1}\setminus\widehat{E}(\lambda).\end{cases}
$$

Clearly (a)-(e) of Lemma 3.13 and (3.17), (3.18), (3.19), (3.20) are valid with v replaced by w. Moreover, Lemma 3.5 holds with $\widetilde{u}$ replaced by u'.

Lemma 3.23. *For almost every t with $(\mathbf{R}^n \times \{t\}) \cap S' \neq \emptyset$ the following is true: If $i \in \Theta_1$ and $\operatorname{supp} w_i \cap (\mathbf{R}^n \times \{t\}) \neq \emptyset$, then for $\lambda \geq \lambda_2$ we have*

$$\left| \int_{\mathbf{R}^n} (u' - a(Q_i, u'))\, w_i \, dx \right| \leq c r_i^{-1} \widehat{\lambda} \lambda^{p-2} |Q_i| \tag{3.24}$$

and

$$\left| \int_{\mathbf{R}^n} \langle (u' - a(Q_i, u')), (w - a(Q_i, u')) \rangle w_i(\cdot, t) \, dx \right| \leq c \widehat{\lambda}^2 \lambda^{p-2} |Q_i|, \tag{3.25}$$

where $c \geq 1$ has the same dependence as in Lemma 3.5.

Proof. We prove only (3.25) as the proof of (3.24) is similar. To begin we note from a now well known argument (see (3.14), (3.17), (3.18)) that

$$\int_{\mathbf{R}^{n+1}} |u^* - a(Q_i, u^*)| w_i \, dx \, dt \leq c \widehat{\lambda} \min\{r_i, \rho\} |Q_i|, \tag{3.26}$$

whenever $u^* = u'$ or $\overline{Q}_i \subset Q^+$ and $u^* = u$. This inequality and the same proof as in (3.17) and (3.18) imply that

$$\big\| (w - a(Q_i, u')) w_i \big\|_\infty \leq c \widehat{\lambda} \min\{r_i, \rho\}. \tag{3.27}$$

We consider two cases. If $r_i \geq \rho/20$, we can argue as in (3.14) to get

$$|a(Q_i, u')| \leq c \widehat{\lambda} \min\{r_i, \rho\}.$$

From this inequality and (3.27) we find in this case that

$$\begin{aligned} &\left| \int_{\mathbf{R}^n} \langle (u' - a(Q_i, u')), (w - a(Q_i, u')) \rangle w_i(\cdot, t) \, dx \right| \\ &\quad \leq c \widehat{\lambda}^2 \lambda^{p-2} |Q_i| + \left| \int_{\mathbf{R}^n} \langle u', w - a(Q_i, u') \rangle w_i(\cdot, t) \, dx \right| \\ &\quad = c \widehat{\lambda}^2 \lambda^{p-2} |Q_i| + I. \end{aligned} \tag{3.28}$$

To estimate the integral I let $\psi_1 \in C_0^\infty(t_1 - \eta, t_2 + \eta)$ as earlier. We define

$$\phi_j = \widetilde{\phi}(\widetilde{w}_j - a(Q_i, u_j')) w_i \psi_1, \qquad j = 1, 2, \ldots, N,$$

where $\widetilde{w}_j$ denotes the jth component of w and use $(0, \ldots, \phi_j, \ldots, 0)$ as a test function in (2.7) for $j = 1, 2, \ldots, N$. Setting $\widehat{u} = (\widehat{u}_1, \ldots, \widehat{u}_N)$, where

$$\widehat{u}_j = u_j' (\widetilde{w}_j - a(Q_i, u_j')) w_i, \qquad j = 1, 2, \ldots N,$$

and letting $\eta \to 0$ we get for almost every $t_1 < t_2$ with $|t - t_k| \leq \gamma r_i^2$, $k = 1, 2$, that

$$\begin{aligned} \left| \int_{\mathbf{R}^n} \big(\widehat{u}(\cdot, t_2) - \widehat{u}(\cdot, t_1)\big) \, dz \right| &\leq c \int_{\mathbf{R}^{n+1}} |u'| \left| \frac{\partial}{\partial t} \big((w - a(Q_i, u')) w_i \big) \right| dz \, d\tau \\ &\quad + c \int_{\mathbf{R}^{n+1}} \big(|\nabla u|^{p-1} + |h_1| + |h_2| \big) \big(|\nabla \phi| + |\phi| \big) \, dz \, d\tau \leq c \widehat{\lambda}^2 \lambda^{p-2} |Q_i|. \end{aligned}$$

The last inequality is obtained using (3.17) and (3.18) for w, (3.26), and (3.27). We deduce from the above inequality that

$$|I| \leq c(\gamma r_i^2)^{-1} \int_{t-\gamma r_i^2}^{t+\gamma r_i^2} \int_{\mathbf{R}^n} |\widehat{u}| \, dz \, d\tau + c\widehat{\lambda}^2 \lambda^{p-2} |Q_i| \leq c\widehat{\lambda}^2 \lambda^{p-2} |Q_i|.$$

Combining this inequality with (3.28), we get (3.25) when $r \geq \rho/20$.

Otherwise we note that either $u' \equiv 0$ on the support of $w_i \supset \overline{Q}_i$ in which case the integral in (3.25) is trivially zero or $\overline{Q}_i \subset Q_{5\rho,12s}(z,\tau)$. In this situation we once again use the fact that $\widetilde{\phi}(x,\cdot)$ is constant on $(\tau - 12s, \tau + 12s)$ to get for $(x,t) \in \operatorname{supp} w_i$ that

$$\begin{aligned}
\big|a(Q_i, u') - \widetilde{\phi}(x,t) a(Q_i, u - a(Q))\big| &\leq \fint_{Q_i} |u - a(Q,u)| |\widetilde{\phi}(x,t) - \widetilde{\phi}| \, dx \, dt \\
&\leq c r_i \rho^{-1} \fint_{Q_i} |u - a(Q,u)| \, dx \, dt \leq c\widehat{\lambda} r_i,
\end{aligned}$$

where the last inequality follows from (3.14). From this inequality and (3.27) we obtain

$$\begin{aligned}
&\left| \int_{\mathbf{R}^n} \langle (u' - a(Q_i, u')), (w - a(Q_i, u')) \rangle w_i(\cdot, t) \, dx \right| \\
&\qquad \leq c\widehat{\lambda}^2 \lambda^{p-2} |Q_i| + \left| \int_{\mathbf{R}^n} \langle (u - a(Q_i, u)) \widetilde{\phi}, (w - a(Q_i, u')) \rangle w_i(\cdot, t) \, dx \right|.
\end{aligned}$$

We can now define

$$\phi_j = \widetilde{\phi}(\widetilde{w}_j - a(Q_i, u_j')) w_i \psi_1, \qquad j = 1, 2, \ldots, N,$$

and proceed as in the previous case to estimate the last integral. Doing this we get (3.25). In view of our earlier remark we see that Lemma 3.23 is true. □

The next lemma is rather delicate and crucial for Theorem 2.8 to hold.

Lemma 3.29. *For almost every t with $(\mathbf{R}^n \times \{t\}) \cap S' \neq \emptyset$, we have for $\lambda \geq \lambda_2$ that*

$$\begin{aligned}
&\int_{\mathbf{R}^n \setminus \{x\colon (x,t) \in \widehat{E}(\lambda)\}} \big(|u'|^2 - |u' - w|^2\big)(\cdot, t) \, dz \\
&\qquad \geq -c\widehat{\lambda}^2 \lambda^{p-2} |\mathbf{R}^{n+1} \setminus E(\lambda)| - c \int_{Q^+} |u - a(Q)|^2 \, dz \, d\tau,
\end{aligned}$$

where $c \geq 1$ has the same dependence as in Lemma 3.5.

Proof. Let

$$\Lambda_1 = \big\{ i\colon |u'| + |w| \not\equiv 0 \quad \text{on} \quad \operatorname{supp} w_i \cap (\mathbf{R}^n \times \{t\}) \neq \emptyset \big\}$$

and put $\Lambda_2 = \Lambda_1 \cap \Theta_2$ and $\Lambda = \Lambda_1 \setminus \Lambda_2$. To prove Lemma 3.29 we write

$$\begin{aligned}\int_{\mathbf{R}^n \setminus \{x:\ (x,t)\in \widehat{E}(\lambda)\}} & \big(|u'|^2 - |u'-w|^2\big)(\cdot,t)\,dz \\ &= \sum_{i\in\Lambda}\int_{\mathbf{R}^n} w_i\big(|u'|^2 - |u'-w|^2\big)(\cdot,t)\,dz \\ &\quad + \sum_{i\in\Lambda_2}\int_{\mathbf{R}^n} w_i\big(|u'|^2 - |u'-w|^2\big)(\cdot,t)\,dz \\ &= P_1 + P_2.\end{aligned} \tag{3.30}$$

To estimate P_2 we observe that $c(n)\gamma r_i^2 \ge s$, when $i \in \Lambda_2$, and argue as in (3.22) to find that

$$\begin{aligned}|P_2| &= \Big|\sum_{i\in\Lambda_2}\int_{\mathbf{R}^n} w_i(2u'w - w^2)(\cdot,t)\,dz\Big| \\ &\le 2\Big|\sum_{i\in\Lambda_2}\int_{\mathbf{R}^n} u'ww_i(\cdot,t)\,dz\Big| + cs^{-1}\sum_{i\in\Lambda_2}\sum_{j\in\mu_i} a(Q_j,|u'|)^2|Q_i\cap S| \\ &\le 2\Big|\sum_{i\in\Lambda_2}\int_{\mathbf{R}^n} u'ww_i(\cdot,t)\,dz\Big| + cs^{-1}\int_{Q^+}|u-a(Q)|^2\,dz\,d\tau.\end{aligned} \tag{3.31}$$

To estimate the last sum in this display we put

$$\phi_j = \widetilde{\phi}\widetilde{w}_j w_i \psi_1, \qquad j = 1,2,\dots,N,$$

where ψ_1 is defined following (3.28) and use $(0,\dots,\phi_j,\dots,0)$ as a test function in (2.7) for $j = 1,2,\dots,N$. Arguing as in (3.22) and Lemma 3.23 we get

$$|P_2| \le c\widehat{\lambda}^2\lambda^{p-2}|\mathbf{R}^{n+1}\setminus E(\lambda)| + cs^{-1}\int_{Q^+}|u-a(Q)|^2\,dz\,d\tau. \tag{3.32}$$

To estimate P_1 set $a_i = a(Q_i,u')$ and write

$$\begin{aligned}P_1 &= \sum_{i\in\Lambda}\int_{\mathbf{R}^n} w_i\big(|u'|^2 - |u'-w|^2\big)(\cdot,t)\,dz \\ &= \sum_{i\in\Lambda}\int_{\mathbf{R}^n} w_i\big(|u'|^2 - |u'-a_i|^2\big)(\cdot,t)\,dz - \sum_{i\in\Lambda}\int_{\mathbf{R}^n} w_i|w-a_i|^2(\cdot,t)\,dz \\ &\quad + 2\sum_{i\in\Lambda}\int_{\mathbf{R}^n} w_i\langle u'-a_i, w-a_i\rangle(\cdot,t)\,dz \\ &= L_1 + L_2 + L_3.\end{aligned} \tag{3.33}$$

To handle L_1 we use (3.24) to obtain

$$\begin{aligned}L_1 &= \sum_{i\in\Lambda}\int_{\mathbf{R}^n} w_i\big(|u'^2| - |u'-a_i|^2\big)(\cdot,t)\,dz \\ &= \sum_{i\in\Lambda}\int_{\mathbf{R}^n}\big(2\langle a_i, u'-a_i\rangle + |a_i|^2\big)w_i(\cdot,t)\,dz\,dt \\ &\ge -c\widehat{\lambda}\lambda^{p-2}\sum_{i\in\Lambda}|a_i|r_i^{-1}|Q_i| + \sum_{i\in\Lambda}|a_i|^2\int_{\mathbf{R}^n} w_i(\cdot,t)\,dz \\ &= L_{11} + L_{12}.\end{aligned} \tag{3.34}$$

We note that if $w_i \neq 0$ on $\mathbf{R}^n \times \{t\}$, then there exists a Whitney rectangle Q_j with $w_j \geq c(n,p)^{-1}$ on $Q_j \cap (\mathbf{R}^n \times \{t\})$ and $\operatorname{supp} w_j \cap \operatorname{supp} w_i \neq \emptyset$. Either $j \in \Lambda$ or $j \in \Lambda_2$. Let Λ' denote the set of those i's for which $Q_j \in \Lambda$. In this case we see from the same argument as in (3.17) and (3.18) that

$$|a(Q_i, u') - a(Q_j, u')| \leq c\widehat{\lambda} \min\{\rho, r_i\}.$$

Using these observations we find for some $c_6 \geq 1$ that

$$L_{12} \geq \lambda^{p-2} c_6^{-1} \sum_{i \in \Lambda'} r_i^{-2} |a_i|^2 |Q_i| - c\widehat{\lambda}^2 \lambda^{p-2} \sum_{i \in \Lambda} |Q_i|.$$

To estimate L_{11} observe that if $i \in \Lambda \setminus \Lambda'$, then $c(n)\gamma r_i^2 \geq s$, so we have

$$\widehat{\lambda}\lambda^{p-2} |a_i| r_i^{-1} |Q_i| \leq c s^{-1} |a_i|^2 |Q_i| + c\widehat{\lambda}^2 \lambda^{p-2} |Q_i|$$

while if $i \in \Lambda'$, then

$$\widehat{\lambda}\lambda^{p-2} |a_i| r_i^{-1} |Q_i| \leq c(n)\delta \lambda^{p-2} r_i^{-2} |a_i|^2 |Q_i| + \delta^{-1} \widehat{\lambda}^2 \lambda^{p-2} |Q_i|.$$

Choosing $\delta > 0$ sufficiently small and summing the above inequalities, we see for some $c \geq 1$ that

$$\begin{aligned} L_{11} \geq &- c_6^{-1} \lambda^{p-2} \sum_{i \in \Lambda'} r_i^{-2} |a_i|^2 |Q_i| - c\widehat{\lambda}^2 \lambda^{p-2} |\mathbf{R}^{n+1} \setminus E(\lambda)| \\ &- c s^{-1} \int_{Q^+} |u - a(Q)|^2 \, dz \, d\tau. \end{aligned}$$

Putting these inequalities for L_{11} and L_{12} in (3.34) we conclude for $c \geq 1$ large enough that

$$L_1 \geq -c\widehat{\lambda}^2 \lambda^{p-2} |\mathbf{R}^{n+1} \setminus E(\lambda)| - c s^{-1} \int_{Q^+} |u - a(Q)|^2 \, dz \, d\tau.$$

Moreover, from (3.27) we deduce that

$$L_2 \geq -c\widehat{\lambda}^2 \lambda^{p-2} |\mathbf{R}^{n+1} \setminus E(\lambda)|$$

and from (3.25) we have

$$L_3 \geq -c\widehat{\lambda}^2 \lambda^{p-2} |\mathbf{R}^{n+1} \setminus E(\lambda)|.$$

Using these inequalities in (3.33) we conclude first that

$$P_1 \geq -c^{-1} \widehat{\lambda}^2 \lambda^{p-2} |\mathbf{R}^{n+1} \setminus E(\lambda)| - c s^{-1} \int_{Q^+} |u - a(Q)|^2 \, dz \, d\tau$$

and thereupon from (3.32) and (3.30) that Lemma 3.29 is true. $\square$

4. Proof of the main result

We continue under the asumptions and notation introduced in Section 3. Recall that $Q = Q_{\rho,s}(z,\tau)$, $\widehat{Q} = Q_{10^3\widehat{\rho},10^6\widehat{s}}(z,\tau)$, $\widehat{\rho} \le \rho \le 10^2\widehat{\rho}$, and $\widehat{s} \le s \le 10^4\widehat{s}$. Let

$$\lambda_3^{2-p} = s\rho^{-2} \tag{4.1}$$

for some $c_7 = c_7(p,n) \ge 1$ and assume that

$$\begin{aligned} &c_7^{-1}\lambda_3^{p-\beta} \\ &\le \fint_Q |\nabla u|^{p-\beta}\,dx\,dt + \alpha(p)\fint_Q \left(s^{-1}\lambda_3^{-\beta}|u-a(Q)|^2 + h^{p-\beta}\right)dx\,dt \\ &\le c_7\fint_{\widehat{Q}} |\nabla u|^{p-\beta}\,dx\,dt + c_7\alpha(p)\fint_{\widehat{Q}} \left(s^{-1}\lambda_3^{-\beta}|u-a(\widehat{Q})|^2 l^{-\beta} + h^{p-\beta}\right)dx\,dt \\ &\le c_7^2\lambda_3^{p-\beta}, \end{aligned} \tag{4.2}$$

where $\alpha(p) = 1$ for $2n/(n+2) < p < 2$ and $\alpha(p) = 0$ for $p \ge 2$. For fixed $\lambda \ge \lambda_2$ and $\varepsilon > 0$ small we construct $v(\cdot,\varepsilon,\lambda)$ as in Section 3 and put

$$\phi_j = (v_j\,\widetilde{\phi}\psi_1)_\varepsilon \qquad j = 1,2,\ldots,N,$$

where $\psi_1 \in C_0^\infty(t_1-\eta, t_2+\eta)$. We use $(0,\ldots,\phi_j,\ldots,0)$ as a test function in (2.7). If $\phi = (\phi_1,\ldots,\phi_N)$, we obtain using the same notation as in (3.6) that

$$\begin{aligned} J_1 =& \int_{\mathbf{R}^{n+1}} \left\langle \frac{\partial\widetilde{u}}{\partial t}, v\right\rangle \psi_1\,dx\,dt \\ =& \int_{\mathbf{R}^{n+1}} \langle (u-a(Q))_\varepsilon, v\rangle \psi_1 \frac{\partial\widetilde{\phi}}{\partial t}\,dx\,dt - \int \left(\langle A(\cdot,\nabla u),\nabla\phi\rangle - B\phi\right)dx\,dt \\ =& J_2 + J_3. \end{aligned} \tag{4.3}$$

We observe from Lemma 3.13 (d) and (f) that

$$\begin{aligned} J_1 =& \int_{\mathbf{R}^{n+1}\setminus\widehat{E}(\lambda)} \left\langle \frac{\partial\widetilde{u}}{\partial t}, v-\widetilde{u}\right\rangle \psi_1\,dx\,dt + \frac{1}{2}\int_{\mathbf{R}^{n+1}} \frac{\partial}{\partial t}|\widetilde{u}|^2\psi_1\,dx\,dt \\ =& -\frac{1}{2}\int_{\mathbf{R}^{n+1}} \frac{\partial}{\partial t}|v-\widetilde{u}|^2\psi_1\,dx\,dt + \frac{1}{2}\int_{\mathbf{R}^{n+1}} \frac{\partial}{\partial t}|\widetilde{u}|^2\psi_1\,dx\,dt \\ &+ \int_{\mathbf{R}^{n+1}\setminus\widehat{E}(\lambda)} \left\langle \frac{\partial v}{\partial t}, v-\widetilde{u}\right\rangle \psi_1\,dx\,dt. \end{aligned} \tag{4.4}$$

when $|t_i - \tau| < 6s - \eta$, $i = 1,2$. Letting $\eta \to 0$ in (4.4) we deduce that

$$J_1 = \frac{1}{2}\int_{\mathbf{R}^n} \left(|\widetilde{u}|^2 - |\widetilde{u}-v|^2\right)(x,\cdot)\,dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_{\mathbf{R}^n} \left\langle \frac{\partial v}{\partial t}, v-\widetilde{u}\right\rangle dx\,dt$$

for almost every $t_1 < t_2$, with $|t_i - \tau| \le 6s$, $i = 1,2$. The last integral in this display can be estimated using Lemma 3.13 (d). Thus

$$\begin{aligned} J_1 =& \frac{1}{2}\int_{\mathbf{R}^n} \left(|\widetilde{u}|^2 - |\widetilde{u}-v|^2\right)(x,\cdot)\,dx\Big|_{t_1}^{t_2} + e_1 \\ =& J(t_2) - J(t_1) + e_1, \end{aligned} \tag{4.5}$$

where

$$|e_1| \le c\widehat{\lambda}^2 \lambda^{p-2} |\mathbf{R}^{n+1} \setminus E(\lambda)| + cs^{-1} \int_{Q^+} |u - a(Q)|^2 \, dx \, dt = \widehat{J}.$$

Letting $\eta \to 0$ in J_2 and J_3 we deduce from Lemma 3.13 (e) that

$$J_2 \to \int_{t_1}^{t_2} \int_{\mathbf{R}^n} \langle (u - a(Q))_\varepsilon, v \rangle \frac{\partial \widetilde{\phi}}{\partial t} \, dx \, dt = J_2^* \tag{4.6}$$

and

$$\begin{aligned} J_3 \to & - \int_{t_1}^{t_2} \int_{\mathbf{R}^n} \big(\langle A(\cdot, \nabla u)_\varepsilon, \nabla (v\widetilde{\phi}) \rangle - B_\varepsilon v \widetilde{\phi} \big) \, dx \, dt \\ = & - \int_{E(\lambda) \cap \{(x,t):\ t_1 < t < t_2\}} \big(\langle A(\cdot, \nabla u)_\varepsilon, \nabla (\widetilde{u}\widetilde{\phi}) \rangle - B_\varepsilon \widetilde{u} \widetilde{\phi} \big) \, dx \, dt + e_3 \\ = & J_3^* + e_3, \end{aligned} \tag{4.7}$$

where $|e_3| \le \widehat{J}$ for c sufficiently large.

We now let $\varepsilon \to 0$ through a properly chosen sequence. For almost every $t_1 < t_2$ with $|t_i - \tau| \le 6s$, $i = 1, 2$, we see that $J(t_i) \to J'(t_i)$ where

$$J'(t_i) = \frac{1}{2} \int_{\mathbf{R}^n} \big(|u'|^2 - |u' - w|^2 \big)(x, t_i) \, dx.$$

We choose t_1, $\tau - 6s < t_1 < \tau - 4s$, so that

$$J'(t_1) \le 100 s^{-1} \int_{Q^+} |u - a(Q)|^2 \, dx \, dt. \tag{4.8}$$

Next from Lemma 3.29 we see for almost every t_2 that

$$J'(t_2) = \frac{1}{2} \int_{\mathbf{R}^n \cap \{x:\ (x,t_2) \in \widehat{E}(\lambda)\}} |u'|^2(\cdot, t_2) \, dx + e_1' = J''(t_2) + e_1' \tag{4.9}$$

and $-e_1' \le \widehat{J}$. We also have $J_2^* \to J_2'$ and $J_3^* \to J_3'$, where

$$|J_2'| = \left| \frac{1}{2} \int_{\mathbf{R}^{n+1}} \langle u - a(Q), w \rangle \frac{\partial \widetilde{\phi}}{\partial t} \, dx \, dt \right| \le \widehat{J}, \tag{4.10}$$

and

$$\begin{aligned} J_3' = & - \int_{E(\lambda) \cap \{(x,t):\ t_1 < t < t_2\}} \big(\langle A(\cdot, \nabla u), \nabla (u'\widetilde{\phi}) \rangle - B u' \widetilde{\phi} \big) \, dx \, dt + e_3' \\ = & - J_3'' + e_3', \end{aligned} \tag{4.11}$$

where $|e_3'| \le \widehat{J}$. Combining (4.3)-(4.11) we conclude for c sufficiently large that

$$J''(t_2) + J_3'' \le \widehat{J}. \tag{4.12}$$

Put

$$l(x,t) = M\big((|\nabla u| + |h|)^{\widehat{\xi}}\chi_{Q^+}\big)^{1/\widehat{\xi}}(x,t),$$

for $(x,t)\in \mathbf{R}^{n+1}$. We multiply both sides of (4.12) by $\lambda^{-(1+\beta)}$ and integrate with respect to λ over (λ_4,∞) where $\lambda_2 \le \lambda_4$. It is easily seen that for almost every t_1 and t_2 we can interchange the order of integration. For the term corresponding to $J''(t_2)$ we obtain

$$\begin{aligned}\beta^{-1}K(t_2) =& \int_{\lambda_4}^{\infty} \lambda^{-(1+\beta)} J''(t_2)\, d\lambda \\ =&\frac{1}{2}\beta^{-1}\lambda_4^{-\beta}\int_{\mathbf{R}^n\cap\{x:\ (x,t_2)\in E(\lambda_4)\}} |u'|^2(\cdot,t_2)\,dx \\ &+\frac{1}{2}\int_{\{x:\ (x,t_2)\notin E(\lambda_4)\}} |u'|^2(\cdot,t_2)\Big(\int_{l(\cdot,t_2)}^{\infty}\lambda^{-1-\beta}\,d\lambda\Big)\,dx \\ =&\frac{1}{2}\beta^{-1}\int_{\mathbf{R}^n} m(\cdot,t_2)^{-\beta}|u'|^2(\cdot,t_2)\,dx,\end{aligned}\tag{4.13}$$

where $m = \max\{\lambda_4, l\}$. Similarly,

$$\begin{aligned}\beta^{-1}K_1 =& \int_{\lambda_4}^{\infty}\lambda^{-(1+\beta)}J_3''\,d\lambda \\ =&\beta^{-1}\int_{t_1}^{t_2}\int_{\mathbf{R}^n} m^{-\beta}\big(\langle A(\cdot,\nabla u),\nabla(u'\widetilde{\phi})\rangle - Bu'\widetilde{\phi}\big)\,dx\,dt.\end{aligned}\tag{4.14}$$

We now consider two cases. First suppose for λ_2 and λ_3 as in (3.4) and (4.1) respectively, that

$$\lambda_2 \le \delta\lambda_3,\tag{4.15}$$

where $0<\delta<10^{-3}$ will be chosen after the proof of Lemma 4.19 to depend only on the constants listed in Theorem 2.8. To estimate the term involving $\widehat{J}$ we note from (4.1) and the definition of $\widehat{\lambda}$ in Lemma 3.5 that $\widehat{\lambda}\le c\lambda$ on (λ_2,∞) when $p\ge 2$ while $\widehat{\lambda}\le c\lambda^{p-1}\lambda_3^{2-p} + c\lambda$ on this interval when $2n/(n+2) < p < 2$. If (4.15) holds we put $\lambda_4 = \delta\lambda_3$ and use the above fact to obtain from the Hardy-Littlewood maximal theorem that

$$\begin{aligned}\int_{\lambda_4}^{\infty}\lambda^{p-3-\beta}\widehat{\lambda}^2|\mathbf{R}^{n+1}\setminus E(\lambda)|\,d\lambda &\le c(\delta^{p-2}+1)\int_{\mathbf{R}^{n+1}} l^{p-\beta}\,dx\,dt \\ &\le c(\delta^{p-2}+1)|Q^+|\lambda_3^{p-\beta}.\end{aligned}\tag{4.16}$$

Inequality (4.16) implies that

$$\begin{aligned}\int_{\delta\lambda_3}^{\infty}\lambda^{-(1+\beta)}\widehat{J}\,d\lambda \le& c(\delta^{p-2}+1)|Q^+|\lambda_3^{p-\beta} \\ &+ c\beta^{-1}(\delta\lambda_3)^{-\beta}s^{-1}\int_{Q^+}|u-a(Q)|^2\,dx\,dt.\end{aligned}\tag{4.17}$$

Combining (4.12)-(4.14) and (4.17) we get provided $\delta^{4(2-p)} \geq \beta$ and $\beta > 0$ is sufficiently small

$$K(t_2) + K_1 \leq c\lambda_3^{-\beta} s^{-1} \int_{Q^+} |u - a(Q)|^2 \, dx \, dt + e, \tag{4.18}$$

where $0 \leq e \leq \beta^{1/2} \lambda_3^{p-\beta} |Q^+|$.

We use (4.18) to prove a form of Caccioppoli type estimate tailored to our situation.

Lemma 4.19. *Let u be a very weak solution to (1.1) for $p > 2n/(n+2)$ and suppose that (4.1), (4.2) ard (4.15) are valid. Then there exists $\beta > 0$ and $c \geq 1$ with the same dependence as the constants in Theorem 2.8 such that*

$$\begin{aligned} |Q^+|\lambda_3^{p-\beta} + &\operatorname*{ess\,sup}_{t_2 \in (\tau - 4s, \tau + 6s)} \int_{D_{3\rho}(z)} m^{-\beta} |u - a(Q)|^2(\cdot, t_2) \, dx \\ \leq & cs^{-1}\lambda_3^{-\beta} \int_{Q^+} |u - a(Q)|^2 \, dx \, dt + c\rho^{-p} \int_{Q^+} |u - a(Q)|^p m^{-\beta} \, dx \, dt \\ &+ c \int_{Q^+} h^{p-\beta} \, dx \, dt. \end{aligned} \tag{4.20}$$

Proof. From (2.5), the fact that $\widetilde{\phi}$ is constant on $Q_{3\rho,12s}(z,\tau)$ with support in $Q_{4\rho,16s}(z,\tau)$, and $\tau - 6s < t_1 < \tau - 4s$, we deduce for t_2, $\tau - 4s < t_2 < \tau + 6s$ that

$$\begin{aligned} cK_1 \geq & \int_{\tau-4s}^{t_2} \int_{D_{3\rho}(z)} m^{-\beta} |\nabla u|^p \, dx \, dt \\ & - c^2 \rho^{-1} \left| \int_{Q_{4\rho,16s}(z,\tau)} m^{-\beta} |u - a(Q)| (|\nabla u| + h)^{p-1} \, dx \, dt \right| \\ =& K_2(t_2) - K_3. \end{aligned} \tag{4.21}$$

Here $c \geq 1$ depends only on p, n, c_1, c_2 and c_3. Thus,

$$\begin{aligned} &\operatorname*{ess\,sup}_{(\tau-4s,\tau+6s)} K(t_2) + K_2(\tau + 6s) \\ &\qquad \leq c\Big(\lambda_3^{-\beta} s^{-1} \int_{Q^+} |u - a(Q)|^2 \, dx \, dt + e + K_3\Big). \end{aligned} \tag{4.22}$$

Let

$$\begin{aligned} E = \big\{(x,t) \in D_{3\rho}(z) \times (\tau - 4s, \tau + 6s) \colon |\nabla u|(x,t) \geq \beta l(x,t)\big\} \\ \cap \big\{(x,t) \colon l(x,t) \geq \delta\lambda_3\big\}. \end{aligned}$$

Then

$$K_2(\tau + 6s) \geq \beta^{-\beta} \int_E |\nabla u|^{p-\beta} \, dx \, dt = K_4.$$

We may suppose that $\beta > 0$ is so small that $\beta^{-\beta} \geq 1/2$. Then from (4.2) and the Hardy–Littlewood maximal theorem we see that

$$\begin{aligned}\int_{\tau-4s}^{\tau+6s} \int_{D_{3\rho}(z)} &|\nabla u|^{p-\beta}\, dx\, dt \\ &\leq 2K_4 + c\delta^{p-\beta}\lambda_3^{p-\beta}|Q^+| + c\beta^{p-\beta}\int_{\mathbf{R}^{n+1}} l^{p-\beta}\, dx\, dt \\ &\leq 2K_4 + c\delta^{p-\beta}\lambda_3^{p-\beta}|Q^+|\end{aligned}$$

since $\beta \leq \delta^{4(2-p)}$. Thus if $\delta \leq \delta_0$ and $\delta_0 > 0$ is small enough (depending on the constants listed in Theorem 2.8), we deduce from (4.2) and the above inequality that

$$\begin{aligned}K_4 \geq &\frac{1}{4}c_7^{-1}\lambda_3^{p-\delta}|Q^+| - c\int_{Q^+} h^{p-\beta}\, dx\, dt \\ &- cs^{-1}\lambda_3^{-\beta}\int_{Q^+} |u-a(Q)|^2\, dx\, dt.\end{aligned} \tag{4.23}$$

From Young's inequality, and the fact that $Q^+ = Q_{6\rho,36s}(z,\tau)$, we also obtain

$$|K_3| \leq \frac{1}{32}c_7^{-1}\lambda_3^{p-\beta}|Q^+| + c\rho^{-p}\int_{Q^+} |u-a(Q)|^p m^{-\beta}\, dx\, dt. \tag{4.24}$$

Putting (4.24), (4.23) into (4.22) we find that Lemma 4.19 is true. □

Fix $\delta > 0$ so small that Lemma 4.19 is true. Next we use Lemma 4.19 to prove a Sobolev type estimate for the very weak solution.

Lemma 4.25. *If the hypotheses of Lemma 4.19 are satisfied,* $\widetilde{p} = \max\{p, 2\}$ *and*

$$\widetilde{q} = \frac{n\widetilde{p}p}{(n+2)p - \beta(2+\widetilde{p})},$$

then for $\beta > 0$ *sufficiently small there exists* $\widetilde{\rho}$, $\rho < \tilde{\rho} < 2\rho$, *such that whenever* $0 < \varepsilon < 10^{-5}$, *we have*

$$\begin{aligned}\int_{Q_{\widetilde{\rho},s}(z)} &|u(\cdot,t) - I_{\widetilde{\rho}}(t,u)|^{\widetilde{p}}\, dx\, dt \\ &\leq \rho^{\widetilde{p}}|Q^+|\Big(c(\varepsilon)\lambda_3^{\widetilde{p}-\widetilde{q}}\int_{Q_{\widetilde{\rho},s}(z)} |\nabla u|^{\widetilde{q}}\, dx\, dt + \varepsilon\lambda_3^{\widetilde{p}}\Big).\end{aligned}$$

for some $c(\varepsilon) \geq 1$.

Proof. Choose $\widetilde{\rho}$, $\rho < \widetilde{\rho} < 2\rho$, such that Lemma 3.1 is valid with $\widehat{\rho}$ replaced by $\widetilde{\rho}$. Set $\sigma = 2(p-\beta)/p$ and define q by $q(1+\sigma/n) = \widetilde{p}$.

Set

$$q = \frac{np(p-\beta)}{(n+\beta)p - \beta^2},$$

and $\sigma = 2(p-\beta)/p$.

We proceed as in the proof of Lemma 3.3 in [KLe]. Let $\psi \in C_0^\infty(Q_{2\widetilde{\rho},2s}(z,\tau))$ be a cutoff function such that $\psi = 1$ on $Q_{\widetilde{\rho},s}(z,\tau)$ and $|\nabla\psi| \le 10/\widetilde{\rho}$. Let

$$v(x,t) = |u(x,t) - I_{\widetilde{\rho}}(t)|\psi(x,t).$$

Denote $\rho^* = 2\widetilde{\rho}$. Hölder's inequality implies that

$$\begin{aligned} J &= \int_{D_{\rho^*}(z)} v(x,t)^{q(1+\sigma/n)}\,dx \\ &\le \Big(\int_{D_{\rho^*}(z)} v(x,t)^\sigma\,dx\Big)^{1/n}\Big(\int_{D_{\rho^*}(z)} v(x,t)^{(q+(\sigma/n)(q-1))n/(n-1)}\,dx\Big)^{(n-1)/n}. \end{aligned}$$

We use Sobolev's theorem for functions in $W^{1,1}(D_{\rho^*}(z))$ to deduce that there is constant $c = c(n)$ such that

$$\begin{aligned} &\Big(\int_{D_{\rho^*}(z)} v(x,t)^{(q+(\sigma/n)(q-1))n/(n-1)}\,dx\Big)^{(n-1)/n} \\ &\qquad\le c\int_{D_{\rho^*}(z)} v(x,t)^{(q-1)(1+\sigma/n)}|\nabla v(x,t)|\,dx \\ &\qquad\le c\Big(\int_{D_{\rho^*}(z)} v(x,t)^{q(1+\sigma/n)}\,dx\Big)^{(q-1)/q}\Big(\int_{D_{\rho^*}(z)} |\nabla v(x,t)|^q\,dx\Big)^{1/q}. \end{aligned}$$

Thus

$$J \le cJ^{(q-1)/q}\Big(\int_{D_{\rho^*}(z)} |\nabla v(x,t)|^q\,dx\Big)^{1/p}\Big(\int_{D_{\rho^*}(z)} v(x,t)^\sigma\,dx\Big)^{1/n}.$$

The same argument as in the proof of Lemma 3.3 in [KLe] gives

$$\Big(\int_{D_{\rho^*}(z)} |\nabla v(x,t)|^q\,dx\Big)^{1/q} \le c\Big(\int_{D_{\rho^*}(z)} |\nabla u(x,t)|^q\,dx\Big)^{1/q}$$

and

$$\begin{aligned} \Big(\int_{D_{\rho^*}(z)} v(x,t)^\sigma\,dx\Big)^{1/n} &\le c\Big(\int_{D_{\rho^*}(z)} |u(x,t) - I_\rho(t)|^\sigma\,dx\Big)^{1/n} \\ &\le c\Big(\int_{D_{\rho^*}(z)} |u(x,t) - I_{\rho^*}(t)|^2\,dx\Big)^{1/n}. \end{aligned}$$

Collecting the obtained estimates we arrive at

$$J \le c\int_{D_{\rho^*}(z)} |\nabla u(x,t)|^q\,dx\Big(\int_{D_{\rho^*}(z)} |u(x,t) - I_{\rho^*}(t)|^\sigma\,dx\Big)^{q/n}. \tag{4.26}$$

We note that the right hand side of (4.20) is smaller than $c\lambda_3^{p-\beta}|Q^+|$. If $2n/(n+2) < p \le 2$, then this note is a direct consequence of (4.1), (4.2) and Hölder's inequality. If $p > 2$ and $I_{6\rho}$ is defined relative to (z, τ), then this note follows from the Sobolev inequality,

$$|u(x,t) - I_{6\rho}(t)| \le cM(|\nabla u|\chi_{Q^+})(x,t) \le c\,l(x,t), \qquad (x,t) \in Q^+,$$

Lemma 3.1, (4.1), and (4.2) (see (3.8), (3.9) in [KLe]). Using this note, Lemma 4.19, Hölder's inequality, and the definitions of σ and q we see that

$$\begin{aligned}
&\Big(\int_{D_{\widetilde{\rho}}(z)} |u(x,t) - I_{\widetilde{\rho}}(t)|^{\sigma}\, dx\Big)^{q/n} \\
&\qquad \le \Big(\int_{D_{3\rho}(z)} |u(x,t) - I_{3\rho}(t)|^2 m^{-\beta}\, dx\Big)^{q\sigma/(2n)} \\
&\qquad\quad \cdot \Big(\int_{D_{3\rho}(z)} m^{p-\beta}\, dx\Big)^{(1-\sigma/2)(q/n)} \\
&\qquad \le c(\lambda_3^{p-\beta}|Q^+|)^{q\sigma/(2n)}\Big(\int_{D_{3\rho}(z)} m^{p-\beta}\, dx\Big)^{\beta q/(np)}.
\end{aligned} \tag{4.27}$$

Putting (4.27) into (4.26) and integrating over $(s-\tau, s+\tau)$ we get using the definition of $q, \widetilde{q}, \sigma$ and Hölder's inequality that

$$\begin{aligned}
&\int_{Q_{\widetilde{\rho},s}(z)} v^{\widetilde{p}}\, dx\, dt \le c\big(\lambda_3^{p-\beta}|Q^+|\big)^{q\sigma/(2n)} \\
&\qquad \cdot \int_{s-\tau}^{s+\tau} \Big(\int_{D_{3\rho}(z)} m^{p-\beta}(\cdot,t)\, dx\Big)^{\beta q/(np)} \Big(\int_{D_{3\rho}(z)} |\nabla u|^{q}(\cdot,t)\, dx\Big)\, dt \\
&\le (\lambda_3^{p-\beta}|Q^+|)^{q/n}\rho^{\beta q/p}\Big(\int_{Q_{\widetilde{\rho},s}(z)} |\nabla u|^{\widetilde{q}}\, dx\, dt\Big)^{q/\widetilde{q}} \\
&\le \rho^{\widetilde{p}}|Q^+|\Big(c(\varepsilon)\lambda_3^{\widetilde{p}-\widetilde{q}} \fint_{Q_{\widetilde{\rho},s}(z)} |\nabla u|^{\widetilde{q}}\, dx\, dt + \varepsilon\lambda_3^{\widetilde{p}}\Big),
\end{aligned} \tag{4.28}$$

where to get the last line we have also used Young's inequality for small $\widehat{\varepsilon}$ in the form

$$ab \le \frac{(\widehat{\varepsilon}a)^r}{r} + \frac{(r-1)(b/\widehat{\varepsilon})^{r/(r-1)}}{r}$$

with

$$\begin{aligned}
a &= (\lambda_3^{p-\beta}|Q^+|)^{q/n}\rho^{\beta q/p}(\lambda_3\rho)^{-\widetilde{p}q/\widetilde{q}}\lambda_3^{q}, \\
r &= \widetilde{q}/(\widetilde{q}-q) = np/\beta q, \\
b &= \Big((\lambda_3\rho)^{\widetilde{p}}\int_{Q_{\widetilde{\rho},s}(z)} (|\nabla u|/\lambda_3)^{\widetilde{q}}\, dx\, dt\Big)^{q/\widetilde{q}},
\end{aligned} \tag{4.29}$$

as well as (4.1), (4.2). The proof of Lemma 4.25 is now complete. □

Lemma 4.30. *Let u be a very weak solution to* (1.1) *for $p > 2n/(n+2)$ and suppose that* (4.1) *and* (4.2) *are valid. Then there exists $\beta > 0$ and $c \ge 1$ with the same dependence as the constants in Theorem 2.8 such that*

$$\lambda_3^{p-\beta} \le c\Big(\fint_{Q_{6\widehat{\rho},36\widehat{s}}(z,\tau)} |\nabla u|^{\overline{q}}\,dx\,dt\Big)^{(p-\beta)/\overline{q}} + \fint_{Q_{12\widehat{\rho},36\widehat{s}}(z,\tau)} h^{p-\beta}\,dx\,dt,$$

where $\overline{q} = \max\{p-1/2, \widetilde{q}\}$ when $p \ge 2$ and $\overline{q} = \max\{(1+p)/2, \widetilde{q}\}$ when $2n/(n+2) < p < 2$.

Proof. Note that if (4.15) is false, then Lemma 4.30 is trivially true. Thus we assume (4.15) is true. To prove Lemma 4.30 we can copy the proofs of Lemmas 3.4 and 3.20 in [KLe] with minor changes except that we now replace p by $p-\beta$ and use Lemma 4.19 (with $\rho, s = \widehat{\rho}, \widehat{s}$), as well as Lemma 4.25 (with $\rho, s = 6\widehat{\rho}, 36\widehat{s}$), in place of Lemma 3.2, (3.11) and (3.24) in [KLe]. We omit the details. □

Proof of Theorem 2.8. Using Lemma 4.30 in place of Lemmas 3.4 and 3.20 in [KLe] we can repeat the argument given in the proof of Propositions 4.1 and 4.14 in [KLe] with p replaced by $p-\beta$. The covering argument used in the proof of these propositions guarantees the existence of $c_7, \widehat{\rho}, \widehat{s}$ for which (4.1), (4.2) hold. Also since the constants in Lemma 4.30 are independent of β for β sufficiently small, it is clear from the proof of Propositions 4.1 and 4.14 in [KLe] that

$$|\nabla u| \in L^{p+\beta}(Q_{\rho,s}(z,\tau))$$

for $\beta > 0$ small enough with the same dependence as in Theorem 2.8. Again we omit the details. □

4.31 Remark. We do not know if one can replace $L^2(\Omega \times (S_1, T_1))$ in (2.1) by $L^{2-\beta}(\Omega \times (S_1, T_1))$ for small $\beta > 0$ and still get the same conclusion in Theorem 2.8 for $2n/(n+2) < p < 2$. Although this seems plausible it would for example require a different estimate of the error term in Lemma 3.29. This query is false when $p = 2n/(n+2)$ as can be seen from the example

$$u(x,t) = e^{kt}|x|^{-n/2},$$

where $k = -(n/2)^{-2n/(n+2)}$. It is easily checked that u satisfies the parabolic $2n/(n+2)$ Laplace equation

$$\frac{\partial u}{\partial t} = \operatorname{div}(|\nabla u|^{-4/(n+2)}\nabla u)$$

for $x \in \mathbf{R}^n \setminus \{0\}$. Moreover, u satisfies (2.7) and the weakened form of (2.1) when $p = 2n/(n+2)$. One also easily sees for $0 < k' < k$, that

$$\widehat{u}(x,t) = -e^{k't}|x|^{-n/2}(\log|x|)^{-1}$$

is a weak subsolution to the above equation near (0,0) (in the sense defined in Section 2) but $u \notin L^{2+\beta}(Q_{\rho,\rho}(0,0))$ for any small $\beta, \rho > 0$.

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