Boundary Behaviour for p Harmonic Functions in Lipschitz and Starlike Lipschitz Ring Domains

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Abstract

In this paper we prove new results for p harmonic functions, $p \neq 2$, $1 < p < \infty$, in Lipschitz and starlike Lipschitz ring domains. In particular we prove the boundary Harnack inequality, Theorem 1, for the ratio of two positive p harmonic functions vanishing on a portion of a Lipschitz domain, with constants only depending on p, n and the Lipschitz constant of the domain. For p capacitary functions, in starlike Lipschitz ring domains, we prove an even stronger result, Theorem 2, showing that the ratio is Hölder continuous up to the boundary. Moreover, for p capacitary functions in starlike Lipschitz ring domains we prove, Theorem 3 and Theorem 4, appropriate extensions to $p \neq 2$, $1 < p < \infty$, of famous results of Dahlberg [D] and Jerison-Kenig [JK1] on the Poisson kernel associated to the Laplace operator (i.e. $p = 2$).

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1. Introduction. In this paper we prove a number of new results concerning the boundary behaviour of p capacitary functions, $p \neq 2$ and $1 < p < \infty$, in starlike Lipschitz ring domains. Using our results we are also able to prove the boundary Harnack inequality for the ratio of two positive p harmonic functions, vanishing on a portion of the boundary of a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$. The constants in the inequality only depend on p, n and the Lipschitz constant of Ω. To put these results into perspective we note that the boundary Harnack inequality for harmonic functions (i.e. $p = 2$) in a Lipschitz domain was first introduced in [K] and later proved independently by [A], [D], [W]. This inequality was generalized in [JK], for $p = 2$, to nontangentially accessible domains (NTA domains). In these settings it was also proved that the ratio of two positive harmonic functions, vanishing on a portion of the boundary, is Hölder continuous up to the boundary. The importance of these two results - the boundary Harnack inequality and Hölder continuity up to the boundary for quotients of harmonic functions - to potential theory, boundary value problems and free boundary problems in Lipschitz domains and beyond, for the Laplace operator and more general elliptic second order operators, can hardly be overstated. To be specific concerning areas where the above results are crucial we mention work of B. Dahlberg [D] as well as Jerison and Kenig [JK1] on harmonic measure and the Poisson kernel in Lipschitz and C^1 domains, the program of Caffarelli [C1,C2,C3] for the analysis of elliptic free boundary problems and the program carried out in the papers [AC], [J], [KT], [KT1], [KT2] and [KT3], on free boundary regularity and regularity of the Poisson kernel below the continous threshold.

Analogues of these results for the p Laplacian are easily stated but until now their proofs have eluded the experts, primarily because this operator is nonlinear when $p \neq 2$. In fact the results and techniques of this paper define a starting point for far reaching developments concerning the p Laplace operator in Lipschitz domains and beyond. In this paper, which is the first in a sequel, we lay the groundwork for further developments by proving for p capacitary functions in starlike Lipschitz ring domains: (a) the boundary Harnack inequality and Hölder continuity of quotients up to the boundary (Theorem 2), and (b) analogues of results of Dahlberg [D] (Theorem 3) and Jerison and Kenig [JK] (Theorem 4). The boundary Harnack inequality is then (Theorem 1), extended to general bounded Lipschitz domains and to general positive p harmonic functions vanishing on a portion of the boundary through comparison with appropriate p capacitary functions. Hence an in depth analysis of p capacitary functions in starlike Lipschitz ring domains is the main focus of this paper.

To proceed and to state our results we need to introduce some notation. Points in Euclidean n space $\mathbb{R}^n$ are denoted by $x = (x_1, \dots, x_n)$ or (x', x_n) where $x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$. We let $\bar{E}, \partial E$, diam E, be the closure, boundary, diameter, of the set $E \subset \mathbb{R}^n$ and we define $d(y, E)$ to equal the distance from $y \in \mathbb{R}^n$ to E. $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^n$ and we let $|x| = \langle x, x \rangle^{1/2}$ be the Euclidean norm of x. $B(x, r) = \{y \in \mathbb{R}^n : |x - y| < r\}$ is defined whenever $x \in \mathbb{R}^n$, $r > 0$, and dx denotes Lebesgue n measure on $\mathbb{R}^n$. If $O \subset \mathbb{R}^n$ is open and $1 \leq q \leq \infty$, then by $W^{1,q}(O)$, we denote the space of equivalence classes of functions f with distributional gradient $\nabla f = (f_{x_1}, \dots, f_{x_n})$, both of which are q th power integrable on O. Let $\|f\|_{1,q} = \|f\|_q + \| \, |\nabla f| \, \|_q$ be the norm in $W^{1,q}(O)$ where $\| \cdot \|_q$ denotes the usual Lebesgue q norm in O. Next let $C_0^\infty(O)$ be infinitely differentiable functions with compact support in O and let $W_0^{1,q}(O)$ be the closure of $C_0^\infty(O)$ in the norm of $W^{1,q}(O)$.

Given G a bounded domain (i.e, a connected open set) and $1 < p < \infty$, we say that $\hat{u}$ is p

harmonic in G provided $\hat{u} \in W^{1,p}(G)$ and

$$\int |\nabla \hat{u}|^{p-2} \langle \nabla \hat{u}, \nabla \theta \rangle \, dx = 0 \tag{1.1}$$

whenever $\theta \in W_0^{1,p}(G)$. Observe that if $\hat{u}$ is smooth and $\nabla \hat{u} \neq 0$ in G, then

$$\nabla \cdot (|\nabla \hat{u}|^{p-2} \nabla \hat{u}) \equiv 0 \text{ in } G \tag{1.2}$$

so $\hat{u}$ is a classical solution in G to the p Laplace partial differential equation. Here, as in the sequel, $\nabla \cdot$ is the divergence operator. We note that $\phi : E \to \mathbb{R}$ is said to be Lipschitz on E provided there exists $b, 0 < b < \infty$, such that

$$|\phi(z) - \phi(w)| \, \leq \, b \, |z - w|, \text{ whenever } z, w \in E. \tag{1.3}$$

The infimum of all b such that (1.3) holds is called the Lipschitz norm of ϕ on E, denoted $\|\phi\hat{\|}_E$. It is well known that if $E = \mathbb{R}^{n-1}$, then ϕ is differentiable on $\mathbb{R}^{n-1}$ and $\|\phi\hat{\|}_{\mathbb{R}^{n-1}} = \| \, |\nabla \phi| \, \|_\infty$. Finally let $e_i, 1 \leq i \leq n$, denote the point in $\mathbb{R}^n$ with one in the i th coordinate position and zeroes elsewhere. We now formulate our result on the boundary Harnack inequality for positive p harmonic functions in Lipschitz domains.

Theorem 1. *Let $G = \{y = (y', y_n) \in \mathbb{R}^n : y_n > \phi(y')\}$ where ϕ is Lipschitz on $\mathbb{R}^{n-1}$. Given $p, 1 < p < \infty$, $w = (w', \phi(w')) \in \partial G$, and $r > 0$, suppose that $\tilde{u}, \tilde{v}$ are positive p harmonic functions in $G \cap B(w, r)$, that $\tilde{u}, \tilde{v}$ are continuous in $\bar{G} \cap B(w, r)$, $\tilde{u}(w + \frac{r}{4} e_n) = \tilde{v}(w + \frac{r}{4} e_n) = 1$ and that $\tilde{u}, \tilde{v} = 0$ on $\partial G \cap B(w, r)$. Then there exists $c_1, 1 \leq c_1 < \infty$, depending only on p, n, and $\| \, |\nabla \phi| \, \|_\infty$ such that*

$$\frac{\tilde{u}(y)}{\tilde{v}(y)} \leq c_1 \text{ whenever } y \in G \cap B(w, r/c_1) \, .$$

The conclusion of Theorem 1 is known as a boundary Harnack inequality and as mentioned above the boundary Harnack inequality for harmonic functions (i.e $p = 2$) in a Lipschitz domain was first introduced in [K], later proved independently by [A], [D], [W], and generalized in [JK] to NTA-domains. For $p \neq 2$, and ϕ sufficiently smooth, we note that Theorem 1 follows from barrier type estimates and the boundary maximum principle for p harmonic functions (see [AKSZ]). However constants then depend on a certain smoothness norm of ϕ. We also remark that Theorem 1 is not new in $\mathbb{R}^2$. In fact in [BL] it is shown that the conclusion of Theorem 1 is valid whenever w lies on a quasicircle. Their proof however works only in two dimensions. Thus Theorem 1 is new for $p \neq 2, 1 < p < \infty, n > 2$.

In the setting of starlike Lipschitz ring domains we are able to prove a refined version of Theorem 1 including the Hölder continuity of quotients of solutions. To formulate our results we have to introduce some more notation. A bounded domain $\Omega \subset \mathbb{R}^n$ is said to be starlike Lipschitz with respect to $\hat{x} \in \Omega$ provided

$$\partial\Omega = \{\hat{x} + R(\omega)\omega : \omega \in \partial B(0,1)\} \text{ where } \log R : \partial B(0,1) \to \mathbb{R} \text{ is Lipschitz on } \partial B(0,1).$$

We say that D is a starlike Lipschitz ring domain with center $\hat{x}$ provided $D = \Omega \setminus \bar{\Omega}'$ where Ω, Ω' are starlike Lipschitz domains with center $\hat{x}$ and $\bar{\Omega}' \subset \Omega$. Let R, R' be the graph functions for $\partial\Omega, \partial\Omega'$.

We shall refer to $\| \log \hat{R} \|_{\partial B(0,1)} + \| \log \hat{R'} \|_{\partial B(0,1)}$ as the Lipschitz constant for D. Observe that this constant is invariant under scaling about $\hat{x}$ and also that diam $\Omega \approx d(\hat{x}, \partial\Omega)$, diam $\Omega' \approx d(\hat{x}, \partial\Omega')$, where $A \approx B$ means A/B is bounded above and below by constants which depend only on p, n, and the Lipschitz constant for D. If p is fixed, $1 < p < \infty$, let $\hat{u} = \hat{u}(\cdot, p)$ be the p capacitary function for D. That is $\hat{u} \equiv 1$ on $\partial\Omega'$, $\hat{u} \equiv 0$ on $\partial\Omega$ in the sense of $W_0^{1,p}(\Omega)$ and $\hat{u}$ is p harmonic in D. It is well known that $\hat{u}$ is unique and

$$\int_D |\nabla \hat{u}|^p \, dx = \inf\{ \int_D |\nabla \theta|^p \, dx\} \tag{1.4}$$

where the infimum is taken over all $\theta \in C_0^\infty(\Omega)$ with $\theta \equiv 1$ on $\bar{\Omega}'$. We are able to prove the following theorem on the boundary behaviour of p capacitary functions.

Theorem 2. *Let $\hat{D}_1, \hat{D}_2$ be starlike Lipschitz ring domains with centers, $\hat{x}, \hat{y}$, respectively. For fixed $p, 1 < p < \infty$, let $\hat{u}_i$ be the p capacitary function for $\hat{D}_i$, and put $\tilde{u}_i = min(\hat{u}_i, 1 - \hat{u}_i)$ for $i = 1, 2$. Assume also that $w \in \partial\hat{D}_1 \cap \partial\hat{D}_2, \hat{x}, \hat{y} \notin B(w, 16r)$,*

$$B(w, 2r) \cap \hat{D}_1 = B(w, 2r) \cap \hat{D}_2,$$

and $\bar{B}(w, 8r)$ does not contain points in either both bounded components or both unbounded components of $\mathbb{R}^n \setminus \hat{D}_i$ for $i = 1, 2$. Then there exist $\alpha, c_2, 0 < \alpha \le 1 \le c_2 < \infty$, depending only on p, n, and the Lipschitz constants for $\hat{D}_1, \hat{D}_2$, such that if $w_1, w_2, \in \bar{B}(w, r) \cap \hat{D}_1$, then

$$\left| \frac{\tilde{u}_1(w_1)}{\tilde{u}_2(w_1)} - \frac{\tilde{u}_1(w_2)}{\tilde{u}_2(w_2)} \right| \le c_2 \frac{\tilde{u}_1(a_r(w))}{\tilde{u}_2(a_r(w))} \left(\frac{|w_1 - w_2|}{r} \right)^\alpha,$$

where $a_r(w)$ is a point in $\bar{B}(w, r) \cap \hat{D}_1$ with $d(a_r(w), \partial\hat{D}_1) = \sup\{ d(y, \partial\hat{D}_1) : y \in \bar{B}(w, r) \cap \hat{D}_1 \}$.

Theorem 1 is proved at the end of section 4. As noted earlier, the proof of this theorem uses Theorem 2 and a comparison argument involving p capacitary functions. Thus we briefly outline the proof of Theorem 2. We start by noting that if $\hat{u}(\cdot, \lambda), \lambda \in [0, 1]$, is p harmonic in a domain G, $\nabla\hat{u}(x, \lambda)$ is nonzero for $x \in G$, and if $\hat{u}$ is sufficiently smooth in x, λ, then $\zeta = \frac{\partial \hat{u}}{\partial \lambda}(\cdot, \lambda)$ satisfies, at x, the partial differential equation

$$L\zeta \; = \; \nabla \cdot [(p-2)|\nabla\hat{u}|^{p-4} \langle \nabla\hat{u}, \nabla\zeta \rangle \, \nabla\hat{u} + |\nabla\hat{u}|^{p-2} \nabla\zeta] = 0. \tag{1.5}$$

This follows from differentiating (1.2) for $\hat{u}$ with respect to λ. In (1.5) we have written $\nabla\hat{u}$ for $\nabla\hat{u}(\cdot, \lambda)$. Clearly,

$$L \, \hat{u}(x, \cdot) \; = \; (p-1) \, \nabla \cdot \left[\, |\nabla\hat{u}|^{p-2} \, \nabla\hat{u}(x, \cdot) \, \right] = 0. \tag{1.6}$$

(1.5) can be written in the form

$$L\zeta \; = \; \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left[\, b_{ij}(x) \zeta_{x_j}(x) \, \right] = 0, \tag{1.7}$$

where at $x \in G$,

$$b_{ij}(x) = |\nabla\hat{u}|^{p-4} [(p-2)\hat{u}_{x_i}\hat{u}_{x_j} + \delta_{ij}|\nabla\hat{u}|^2](x), \; 1 \le i, j \le n, \tag{1.8}$$

and δ_{ij} is the Kronecker δ. Again we have written $\nabla\hat{u}$ for $\nabla\hat{u}(\cdot,\lambda)$. The first key observation in the proof of Theorem 2 is that $\hat{u}(\cdot,\lambda)$, $\frac{\partial\hat{u}}{\partial\lambda}(\cdot,\lambda)$, both satisfy the divergence form partial differential equation (1.7).

To continue our outline of the proof of Theorem 2, the proof uses a delicate deformation technique for starlike Lipschitz ring domains. To describe this technique and to simplify matters, we consider the following special case of Theorem 2. Let $\hat{u}_i$ be the p capacitary functions for starlike Lipschitz ring domains, $\hat{D}_i$, with $\hat{D}_i = \hat{\Omega}_i \setminus \bar{B}(\hat{x},\rho), i = 1,2, w \in \partial\hat{\Omega}_1 \cap \partial\hat{\Omega}_2$, and

$$d(\hat{x},\partial\hat{\Omega}_i)/4 \leq \rho \leq d(\hat{x},\partial\hat{\Omega}_i)/2 \text{ for } i = 1,2.$$

Let $\hat{R}_i, i = 1,2$, be the corresponding graph functions for $\partial\hat{\Omega}_i$ and assume that $\hat{R}_i, i = 1,2$, is infinitely differentiable on the manifold $\partial B(0,1)$. Put $\hat{R}(\tau) = \hat{R}_2^\tau \hat{R}_1^{1-\tau}, 0 \leq \tau \leq 1$, and let $\hat{\Omega}(\tau)$ be the starlike Lipschitz domain with center $\hat{x}$, graph function $\hat{R}(\tau)$, while $\hat{D}(\tau) = \hat{\Omega}(\tau) \setminus \bar{B}(\hat{x},\rho)$ is the corresponding ring domain. Let $\hat{u}(\cdot,\tau), \tau \in [0,1]$, be the p capacitary function for $\hat{D}(\tau)$ so that $\hat{u}(\cdot,0) = \hat{u}_1$, $\hat{u}(\cdot,1) = \hat{u}_2$. In Lemma 2.5 we show that

$$|\nabla\hat{u}(x,\tau)| \approx \frac{\hat{u}(x,\tau)}{d(x,\partial\hat{\Omega}(\tau))} \text{ whenever } x \in \hat{D}(\tau). \tag{1.9}$$

This fact and Schauder type arguments imply (see Lemma 4.5) that $\hat{u}(x,\tau)$ is smooth in x,τ whenever $x \in \bigcup \hat{D}(\tau)$. Hence $\{\hat{u}(x,\tau)\}, \tau \in [0,1]$, is a smooth deformation of $\hat{u}_1(x)$ to $\hat{u}_2(x)$ and (1.5)-(1.8) hold with λ replaced by τ. Using this deduction we get

$$\log\left(\frac{\hat{u}_2(x)}{\hat{u}_1(x)}\right) = \int_0^1 \frac{\hat{u}_\tau(x,\tau)}{\hat{u}(x,\tau)}\,d\tau. \tag{1.10}$$

It follows, from the assumptions in Theorem 2, that

$$\hat{D}(\tau) \cap B(w,2r) = \hat{D}_1 \cap B(w,2r) \text{ for all } \tau \in [0,1]. \tag{1.11}$$

Furthermore it turns out, if for example $\hat{R}_2 \geq \hat{R}_1$, that $\hat{u}_\tau > 0$ in $\hat{D}(\tau)$ and $\hat{u}_\tau = 0$ continuously on $B(w,2r) \cap \partial\hat{D}_1$. Therefore we see, in view of (1.10) and the first key observation, that in order to prove the above simplified version of Theorem 2, it suffices to show that $\hat{u}_\tau/\hat{u}$ is Hölder continuous in $\hat{D}_1 \cap B(w,r)$ with constants independent of $\tau, 0 < \tau < 1$. Thus the proof of Theorem 2 is, in this case, reduced to proving a boundary Harnack inequality for positive solutions to (1.5) vanishing on $B(w,2r) \cap \partial\hat{D}_1$. To prove such an inequality we note that if p is near enough 2, $p \neq 2$, then we can use (1.9) and argue as in [L] to deduce first that $|\nabla\hat{u}(\cdot,\tau)|^{p-2}$ extends to an A_2 weight on $\mathbb{R}^n$ and second apply results from [FKS], [FJK], and [FJK1] to get the desired boundary Harnack inequality. Thus in this case one first gets Hölder continuity of $\hat{u}_\tau(\cdot,\tau)/\hat{u}(\cdot,\tau)$ and then of $\hat{u}_1/\hat{u}_2$. In the general case, $1 < p < \infty, p \neq 2$, we must work harder, as simple examples show that $h = |\nabla\hat{u}|^{p-2}(\cdot,\tau)$ need not be an A_2 weight. To get around this difficulty we use some Rellich type inequalities (see Lemmas 2.39, 2.45, 2.54) and a theorem of Kenig - Pipher [KP] (see Theorem 3.11) to show directly in the spirit of Jerison-Kenig (see Lemma 3.13) that $\hat{u}_\tau(\cdot,\tau)/\hat{u}(\cdot,\tau)$ is Hölder continuous in $\hat{D}_1 \cap B(w,r)$.

Finally we note that we currently cannot prove Hölder continuity in Theorem 1, using a similar variational type argument (as in Theorem 2), because we cannot prove a boundary Harnack inequality for the resulting partial differential equation satisfied by $\hat{u}_\tau, \hat{u}$. At the very least it appears that one needs to know that inequalities similar to (1.9) hold for $\tilde{u}, \tilde{v}$ in $B(w, r/c) \cap G$ for some large c depending only on p, n, and the Lipschitz constant for ϕ.

Next we formulate results on non-tangential limits, at the boundary, for gradients of p capacitary functions in starlike Lipschitz ring domains. In particular we state generalizations of work by Dahlberg [D] (Theorem 3), and results of Jerison and Kenig [JK1] (Theorem 4). To do this we shall need some more notation. Let $D = \Omega \setminus \bar{\Omega}'$ be a starlike Lipschitz ring domain with center $\hat{x}$ and as previously, let R, R' be the graph functions for $\partial\Omega, \partial\Omega'$. Let $w \in \partial D, r > 0$, and suppose that

$$\hat{x} \notin B(w, 8r), \text{ as well as, either } B(w, 8r) \cap \partial\Omega' = \emptyset \text{ or } B(w, 8r) \cap \partial\Omega = \emptyset. \tag{1.12}$$

We say that $B(w, 8r) \cap \partial D$ is C^1 provided $\tilde{R}$ is continuously differentiable on $\{(y - \hat{x})/|y - \hat{x}| : y \in B(w, 8r) \cap \partial D\}$, where $\tilde{R} = R'$ if $B(w, 8r) \cap \partial\Omega = \emptyset$ and $\tilde{R} = R$ when $B(w, 8r) \cap \partial\Omega' = \emptyset$. Given $b > 1$ and $x \in \partial D \cap B(w, 2r)$, let $\Gamma(x) = \{y \in D \cap B(w, 8r) : |y - x| < b\, d(y, \partial D)\}$. We note from elementary geometry that if b is large enough (depending on the Lipschitz constant for D), then $\Gamma(x)$ contains the inside of a truncated cone with vertex x, axis parallel to $\hat{x} - x$, angle opening $\theta = \theta(b) > 0$, and height r. Fix b so that this property holds for all $x \in \partial D \cap B(w, 2r)$. Given a measurable function k on $D \cap B(w, 8r)$ define the nontangential maximal function $N(k) : \partial D \cap B(w, 2r) \to \mathbb{R}$ of k by

$$N(k)(x) = \sup_{y \in \Gamma(x)} |k|(y) \quad \text{whenever } x \in \partial D \cap B(w, 2r). \tag{1.13}$$

Let $H^m, 1 \leq m \leq n$, denote m dimensional Hausdorff measure (see [Ma] for a definition) and let $L^q[\partial D \cap B(w, 2r)]$, $1 \leq q \leq \infty$, be the usual space of q th power H^{n-1} integrable functions on $\partial D \cap B(w, 2r)$. Given a measurable function f on $\partial D \cap B(w, 2r)$ we say that f is of bounded mean oscillation on $\partial D \cap B(w, r)$ $(f \in BMO(\partial D \cap B(w, r)))$ if for all $x \in \partial D \cap B(w, r)$ and $0 < s \leq r$, there exists $0 < A < \infty$ satisfying

$$\int_{B(x,s) \cap \partial D} |f - f_B| \, dH^{n-1} \leq A\, s^{n-1}. \tag{1.14}$$

Here f_B denotes the average of f on $B(x, s) \cap \partial D$ with respect to H^{n-1} measure. The least such A for which (1.14) holds will be denoted by $\|\tilde{f}\|$. We say that f is in $VMO(\partial D \cap B(w, r))$ provided for each $\epsilon > 0$ there is a $\delta > 0$ such that (1.14) holds with A replaced by ϵ whenever $0 < s < \min(\delta, r)$ and $x \in \partial D \cap B(w, r)$. Using this notation, Lemmas 2.39, 2.45, 2.54, and Theorem 3.11, we prove the following two theorems.

Theorem 3. *Let D be a starlike Lipschitz ring domain with center $\hat{x}$ and r, w as in (1.12). Let u be the p capacitary function for D. Then*

$$\lim_{y \in \Gamma(x), y \to x} \nabla u(y) = \nabla u(x) \quad \textit{for } H^{n-1} \textit{ almost every } x \in \partial D.$$

Furthermore, there exist $1 \leq c < \infty$ and q, $q > p$, depending only on p, n, and $\| \, |\nabla\phi| \, \|_\infty$ such that

$$(a) \qquad N(|\nabla u|) \in L^q(\partial D \cap B(w, 2r))$$

$$(b) \quad \int_{B(w,2r)\cap\partial D} |\nabla u|^q \, dH^{n-1} \leq c\, r^{(n-1)(\frac{p-1-q}{p-1})} \left(\int_{B(w,2r)\cap\partial D} |\nabla u|^{p-1} \, dH^{n-1} \right)^{q/(p-1)}$$

$$(c) \quad \log |\nabla u| \in \; BMO \; (\partial D \cap B(w,r)) \; \textit{with} \; \| \log |\nabla u| \tilde{\|} \leq c.$$

Theorem 4. *If $\partial D \cap B(w, 8r)$ is C^1 and u, D, r, w are as in Theorem 3, then*

$$\log |\nabla u| \; \in \; VMO \; (\partial D \cap B(w, r)).$$

The rest of the paper is organized as follows. In section 2 we state and derive some basic lemmas which will be used in the proof of Theorems 1-4. In section 3 we introduce elliptic measure defined with respect to the partial differential equation in (1.7), (1.8), and derive a boundary Harnack inequality for positive solutions. In section 4 we study variations of capacitary functions in smooth starlike Lipschitz ring domains and prove Theorem 1. In section 5 we prove Theorems 2, 3, 4.

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2. Basic Estimates. Let Ω^+, Ω^- be starlike Lipschitz domains with center $\hat{z}$ and $\bar{\Omega}^- \subset \Omega^+$. Let R^+, R^- be the graph functions for Ω^+, Ω^- and put $\hat{D} = \Omega^+ \setminus \bar{\Omega}^-$. Let β be the Lipschitz constant for $\hat{D}$. In the sequel, unless otherwise stated, c will denote a positive constant ≥ 1 (not necessarily the same at each occurrence), depending only on p, n, and β. In general, $c(a_1, \ldots, a_n)$ denotes a positive constant ≥ 1, which depends on p, n, β and $a_1, \ldots, a_n$, not necessarily the same at each occurrence. Let $\hat{u}$ be the p capacitary function for $\hat{D}$ and put $\hat{u} \equiv 1$ on $\bar{\Omega}^-$ while $\hat{u} \equiv 0$ on $\mathbb{R}^n \setminus \Omega^+$. With $\hat{u}$ now defined on $\mathbb{R}^n$ let $\max\limits_{B(z,s)} \hat{u}$, $\min\limits_{B(z,s)} \hat{u}$ be the essential supremum and infimum of $\hat{u}$ on $B(z, s)$ whenever $B(z, s) \subset \mathbb{R}^n$. We say that (1.12) holds for w, provided $w \in \partial\hat{D}$ and (1.12) is valid with $\hat{x}, \Omega, \Omega'$ replaced by $\hat{z}, \Omega^+, \Omega^-$, respectively. To begin this section, we state some interior and boundary estimates for $\hat{u}$.

Lemma 2.1. *Let $\hat{u}$ be the p capacitary function for $\hat{D}$ and put $\tilde{u} = \min(\hat{u}, 1 - \hat{u})$.*

(a) *If $B(w, 2r) \subset \hat{D}$, or (1.12) holds for w, then*
$$r^{p-n} \int_{B(w,r/2)} |\nabla \hat{u}|^p \, dx \; \leq \; c \, \max_{B(w,r)} \tilde{u}^p \, .$$

(b) *If $B(w, 2r) \subset \hat{D}$, then* $\max\limits_{B(w,r)} \; \tilde{u} \leq c \, \min\limits_{B(w,r)} \tilde{u}$.

(c) *If $B(w, 2r) \subset \hat{D}$, or (1.12) holds for w, then*
$$|\tilde{u}(x) - \tilde{u}(y)| \leq c \left(\tfrac{|x-y|}{r} \right)^{\alpha} \max_{B(w,2r)} \tilde{u} \textit{ whenever } x, y \in B(w, r).$$

Proof: (a) of Lemma 2.1 is a standard subsolution estimate. (b) is a well known Harnack inequality for positive solutions of p Laplacian type . If $B(w, 2r) \subset \hat{D}$, then (c) is a well known interior Hölder continuity estimate for solutions of p Laplacian type (see [S] for these results). If (1.12) holds for w, then (c) follows from simple barrier type estimates. (see also [GZ]). $\square$

Lemma 2.2. *Let $\tilde u$ be as in Lemma 2.1 and suppose (1.12) holds for w. Then* $\max\limits_{B(w,2r)} \tilde u \leq c\,\tilde u(a_r(w))$ *where $a_r(w)$ is a point in $\bar B(w,r)$ with* $d(a_r(w), \partial\hat D) = \sup\{\, d(y,\partial \hat D) : y \in \bar B(w,r) \cap \hat D\,\}$. *Thus*

$$|\tilde u(x) - \tilde u(y)| \leq c \left(\tfrac{|x-y|}{r}\right)^{\alpha} \tilde u(a_r(w)) \text{ whenever } x, y \in B(w,r).$$

Proof: The first inequality in Lemma 2.2 follows from a general argument using Lemma 2.1 often attributed to Carleson (see [CFMS]). However Domar was apparently the first to use this argument (see [AS]). The second inequality follows from the first inequality and Lemma 2.1 (c). $\square$

Lemma 2.3. *$\hat u$ has a representative in $W^{1,p}(\mathbb{R}^n)$ that has Hölder continuous partial derivatives in $\hat D$. That is, for some $\sigma \in (0,1]$ (depending only on p, n) we have*

$$c^{-1}\,|\nabla \hat u(x) - \nabla \hat u(y)| \,\leq\, (|x-y|/r)^{\sigma} \max_{B(w,r)} |\nabla \hat u| \,\leq\, c\,r^{-1}\,(|x-y|/r)^{\sigma} \max_{B(w,2r)} \hat u$$

whenever $x, y \in B(w, r/2)$ and $B(w,4r) \subset \hat D$.

Proof: The proof of Lemma 2.3 can be found in [Di], [L] or [T]. $\square$

Lemma 2.4. *$\hat u$ is C^∞ in $\hat D \setminus \{x : \nabla \hat u(x) = 0\}$. Moreover if R^+, R^-, are infinitely differentiable ($R^+, R^- \in C^\infty[\partial B(0,1)]$), then there exists an open neighborhood N of $\partial \hat D$ such that* $\min\limits_{\hat D \cap N} |\nabla \hat u| > 0$ *and $\hat u$ has a C^∞ extension to the closure of $\hat D \cap N$.*

Proof: If R^+, R^- are infinitely differentiable, then from Lemma 2.3 and a result of Lieberman [Li] it follows that $\nabla \hat u$ has a Hölder γ extension to the closure of $\hat D$ for some $\gamma \in (0,1]$, depending on p, n and the C^2 norm for $\partial \hat D$. Using this result and barriers of the form

$$x \rightarrow A|x-z|^{(p-n)/(p-1)} \text{ for } p \neq n, \quad A \log|x-z| + B \text{ for } p = n,$$

where $z \in \hat D$ and A, B are constants, we conclude that there exists a neighborhood N of $\partial \hat D$ for which $\min\limits_{\hat D \cap N} |\nabla \hat u| > 0$. Second from (1.2) and this conclusion we see that $\hat u$ is a solution to a non-divergence form uniformly elliptic equation with Hölder continuous coefficients in the closure of $\hat D \cap N$. We now use Schauder theory (see [GT, chapters 6, 9]) and a bootstrap type argument to get that $\hat u$ has a C^∞ extension to the closure of $\hat D \cap N$. A similar argument gives that $\hat u$ is infinitely differentiable in a neighborhood of each point $x \in \hat D$ where $\nabla \hat u(x) \neq 0$. $\square$

Lemma 2.5. *Let $\tilde u$ be as in Lemma 2.1 and suppose that (1.12) holds for w. There exists c such that*

(*i*) $0 < |\nabla \hat u(x)| \,\leq\, c \,\langle \frac{\hat z - x}{|\hat z - x|}\,,\, \nabla \hat u(x) \rangle$ *whenever* $x \in \hat D$.

(*ii*) $c^{-1}\, \tilde u(x)/d(x, \partial \hat D) \,\leq\, |\nabla \hat u(x)| \,\leq\, c \tilde u(x)/d(x, \partial \hat D)$ *whenever* $x \in \hat D \cap B(w, 3r)$.

(*iii*) $\max\limits_{B(x,\frac{s}{2})} \sum\limits_{i,j=1}^{n} |\hat u_{y_i y_j}| \,\leq\, c \left(s^{-n} \int_{B(x,3s/4)} \sum\limits_{i,j=1}^{n} |\hat u_{y_i y_j}|^2 \, dy \right)^{1/2} \,\leq\, c^2\, \tilde u(x)/d(x, \partial \hat D)^2$
whenever $x \in \hat D \cap B(w, 2r)$ *and* $0 < s \leq d(x, \partial \hat D)$.

Proof: Since (1.2) is invariant under translations we assume, as we may, that $\hat z = 0$. We also temporarily assume that

$$R^+, R^- \in C^\infty[\partial B(0,1)]. \tag{2.6}$$

Let $\theta(x) = -\langle x, \nabla \hat u(x) \rangle$, $x \in \hat D$. Then from Lemma 2.4 we deduce that θ has a continuous extension to the closure of $\hat D$. We claim for some $\epsilon > 0$ that

$$\theta \,\geq\, \epsilon \text{ in } \hat D. \tag{2.7}$$

To prove claim (2.7) first observe from (2.6) and Lemma 2.4 that there exists a neighborhood N of ∂D such that

$$\theta \geq 2\epsilon \text{ in } \hat D \cap N \tag{2.8}$$

for ϵ small enough. Second, given $\eta > 0, 0 < \eta \leq 1/2$, let $v = v(\cdot, \eta)$ be the weak solution to

$$\nabla \cdot \left[(\eta + |\nabla v|^2)^{(p/2-1)} \nabla v \right] = 0 \text{ in } \hat D \tag{2.9}$$

with boundary values 1 on $\bar\Omega^-$, 0 on $\mathbb{R}^n \setminus \Omega^+$ in the sense of the Sobolev space $W_0^{1,p}$. We note that Lemmas 2.1-2.4 are valid with $\hat u$ replaced by v. Moreover, an examination of the proofs in the references after these lemmas, shows that the constants and Hölder exponents in Lemmas 2.2, 2.3, can for a fixed $p, 1 < p < \infty$, be chosen independent of $\eta \in (0, 1/2]$. Using this fact, we deduce that any $W^{1,p}$ weakly convergent subsequence of $\{v(\cdot, \eta)\}$ converges uniformly as $\eta \to 0$ to a weak solution, say $\tilde v$, to the p Laplacian in $\hat D$ with continuous boundary values 1 on $\partial\Omega^-$ and 0 on $\partial\Omega^+$. From the weak maximum principle for the p Laplace equation, it follows that $\tilde v = \hat u$. Thus, $v(\cdot, \eta), \nabla v(\cdot, \eta) \to \hat u, \nabla \hat u$ as $\eta \to 0$, uniformly on compact subsets of $\hat D$. From Schauder type estimates it then follows for each positive integer k that $\partial^k v \to \partial^k \hat u$ uniformly on compact subsets of $V = \hat D \setminus \{x : \nabla \hat u(x) = 0\}$. Here ∂^k denotes a k th partial. Finally $v(\cdot, \eta)$ is infinitely differentiable in $\hat D$, as follows once again from Schauder theory and the fact that (2.9) is uniformly elliptic in $\hat D$.

Given $\xi \in \mathbb{R}^n, |\xi| = 1$, and $\eta \in (0, 1/2]$ let $f = \langle \nabla v, \xi \rangle$. Differentiating (2.9) in the direction ξ we get

$$L^* f = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(b^*_{ij} f_{x_j} \right) = 0 \tag{2.10}$$

in $\hat D$, where at $x \in \hat D$,

$$b^*_{ij} = (\eta + |\nabla v|^2)^{(p/2-2)} \; [(p-2) v_{x_i} v_{x_j} + \delta_{ij} (\eta + |\nabla v|^2)] \tag{2.11}$$

for $1 \le i,j \le n$. In (2.11), δ_{ij} again denotes the Kronecker δ. We have

$$\min(p-1,1)(\eta+|\nabla v|^2)^{(p/2-1)}|\xi|^2 \le \sum_{i,j=1}^n b^*_{ij}\xi_i\xi_j \le \max(p-1,1)(\eta+|\nabla v|^2)^{(p/2-1)}|\xi|^2 \tag{2.12}$$

at $x \in \hat D$. Let $\theta^* = \theta^*(\cdot,\eta)$ be defined on $\hat D$ by $\theta^*(x) = -\langle \nabla v(x), x\rangle, x \in \hat D$. From (2.9), (2.10), we find at $x \in \hat D$, that

$$L^*(\theta^*) = (p-2)\,\eta\,\nabla\cdot\left[(\eta+|\nabla v|^2)^{(p/2-2)}\,\nabla v\right]. \tag{2.13}$$

Set $\psi(x) = \min(\theta^*(x)-\epsilon,0)$ and let $\zeta \in C_0^\infty(\hat D)$ with $\zeta \equiv 1$ on $\hat D\setminus N$. Observe from (2.8) and uniform convergence of ∇v to $\nabla \hat u$, that for sufficiently small η, say $0<\eta\le\eta_0$, we have $\psi\zeta^2\equiv 0$ in $\hat D\cap\bar N$. From this observation and the definition of ζ we see that $\nabla(\psi\zeta^2) = (\nabla\psi)\,\zeta^2$ for almost every x in $\hat D$ (with respect to Lebesgue n measure). Using this fact, (2.12), (2.13), and integrating by parts we get,

$$\begin{aligned} I &= \min(p-1,1)\int_{\hat D}(\eta+|\nabla v|^2)^{(p/2-1)}|\nabla\psi|^2\zeta^2\,dx \le \sum_{i,j=1}^n\int_{\hat D} b^*_{ij}\psi_{x_i}\theta^*_{x_j}\,\zeta^2\,dx \\ &= (p-2)\eta\int_{\hat D}(\eta+|\nabla v|^2)^{(p/2-2)}\langle\nabla v,\nabla(\psi\zeta^2)\rangle\,dx \equiv J. \end{aligned} \tag{2.14}$$

From Cauchy's inequality, we have

$$J \le c\,I^{1/2}\,K^{1/2} \tag{2.15}$$

where

$$K = \eta^2\int_{\hat D}(\eta+|\nabla v|^2)^{(p/2-3)}|\nabla v|^2\,\zeta^2\,dx.$$

For small $\delta>0$ we write,

$$K = \int_{\{|\nabla\hat u|\le\delta\}\cap\hat D}\ldots + \int_{\{|\nabla\hat u|>\delta\}\cap\hat D}\ldots = K_1+K_2.$$

Now $\lim\limits_{\eta\longrightarrow 0} K_2 = 0$, as we see from uniform convergence of ∇v to $\nabla\hat u$ on the support of ζ. Also, $\limsup\limits_{\eta\longrightarrow 0} K_1 \le A\delta^p$ for some $1<A<\infty$, independent of δ. Since $\delta>0$ is arbitrary, we conclude that $\lim\limits_{\eta\longrightarrow 0} K = 0$. Using this equality, the Fatou lemma, and uniform convergence of $\partial^k v$ to $\partial^k\hat u$ on compact subsets of $V = \hat D\setminus\{x : \hat u(x)=0\}$, we deduce from (2.14), (2.15) that

$$\int_{\{\theta<\epsilon\}\cap V}|\nabla\hat u|^{p-2}\,|\nabla\theta|^2\,\zeta^2dx = 0$$

which implies in view of (2.8) that either $\{\theta<\epsilon\}$ is empty or θ is constant on components of $\{\theta<\epsilon\}\cap V$. The latter possibility cannot occur. In fact from (2.8), the only possible boundary points of such components are points in $\hat D\setminus V$ where $\theta\equiv 0$. Thus $\theta\equiv 0$ in $\{\theta<\epsilon\}\cap D$. However this deduction is impossible, since from continuity of θ, connectivity of $\hat D$, and (2.8) we would have to have $\theta = \epsilon/2$ at some point in $\hat D$. Thus claim (2.7) is valid.

To continue the proof of Lemma 2.5, observe from assumption (2.6) and Lemma 2.4 that for c large enough (depending only on β),

$$g(x) \,=\, c\theta(x) \,-\, |x|\,|\nabla \hat{u}(x)| > 0 \tag{2.16}$$

for $x \in \partial \hat{D}$. Here we have also used the fact that $\nabla \hat{u}$ is normal to tangent planes through points of $\partial \hat{D}$ and starlike Lipschitzness of $\partial \hat{D}$. We assert that (2.16) also holds in $\hat{D}$, which clearly implies Lemma 2.5 (i). To prove our assertion, let L be the operator in (1.7), (1.8) defined relative to $\hat{u}$ (to get L replace v by $\hat{u}$ and put $\eta = 0$ in the definition of L^* in (2.10)). We note that $L\theta = 0$ in $\hat{D}$, as follows from Lemma 2.4, (2.7), and either the discussion above (1.6) with $\hat{u}(\cdot, \lambda)$ replaced by $\hat{u}(\lambda x)$ or (2.13) with $\eta = 0$. To show that g cannot have a negative minimum in $\hat{D}$ it suffices to show

$$L(|x||\nabla \hat{u}(x)|) \geq 0 \text{ for all } x \in \hat{D}, \tag{2.17}$$

as we find from (2.7) and a standard argument for uniformly elliptic PDE in divergence form. To this end observe from symmetry and smoothness of $\{b_{ij}(x)\}, 1 \leq i, j \leq n,$ that at $x \in \hat{D}$,

$$\begin{aligned} L(|x||\nabla \hat{u}|) \;\; &= \;\; |\nabla \hat{u}|\, L(|x|) \,+\, |x|\, L(|\nabla \hat{u}|) \\ &\quad + 2\,|x|^{-1}\,|\nabla \hat{u}|^{-1} \left(\sum_{i,j,k=1}^{n} b_{ij}\, x_i\, \hat{u}_{x_k}\, \hat{u}_{x_k x_j} \right). \end{aligned} \tag{2.18}$$

To simplify the calculations we note that solutions to the p Laplacian (see (1.2)) remain solutions under rotations. Moreover, g is invariant under rotations around the origin. Thus we assume, for fixed $x \in \hat{D}$, that

$$\nabla \hat{u} \,=\, |\nabla \hat{u}|\, e_n, \tag{2.19}$$

since otherwise we change coordinate systems. From (1.8) or (2.11) with $\eta = 0, v = \hat{u}$, and (2.19) we see at x that

$$\begin{cases} b_{ij} = 0 \text{ for } i \neq j, \\ b_{ii} = |\nabla \hat{u}|^{p-2} \text{ for } 1 \leq i < n, \\ b_{nn} = (p-1)|\nabla \hat{u}|^{p-2}\,. \end{cases} \tag{2.20}$$

Differentiating (1.8) and using (2.19) we find at x that

$$\begin{cases} \frac{\partial b_{ij}}{\partial x_i} = 0 \text{ for } i \neq j, 1 \leq i, j < n, \\ \frac{\partial b_{ii}}{\partial x_i} = \frac{\partial b_{ni}}{\partial x_n} = (p-2)\,|\nabla \hat{u}|^{p-3}\,\hat{u}_{x_i x_n} \quad \text{when } i < n, \\ \frac{\partial b_{in}}{\partial x_i} = (p-2)\,|\nabla \hat{u}|^{p-3}\,\hat{u}_{x_i x_i} \text{ if } 1 \leq i < n, \\ \frac{\partial b_{nn}}{\partial x_n} = (p-1)(p-2)|\nabla \hat{u}|^{p-3}\,\hat{u}_{x_n x_n} \end{cases} \tag{2.21}$$

We have

$$|\nabla \hat{u}|\, L(|x|) = |\nabla \hat{u}|\, |x|^{-1} \sum_{i,j=1}^{n} \frac{\partial b_{ij}}{\partial x_i}\, x_j + |\nabla \hat{u}|\, |x|^{-3} \sum_{i,j=1}^{n} b_{ij}\, (|x|^2 \delta_{ij} - x_i x_j) \,=\, T_1 \,+\, T_2. \tag{2.22}$$

We note that (1.7) for $\hat{u}$, (2.20), (2.21) imply that

$$(p-1)\hat{u}_{x_n x_n} \,+\, \sum_{i=1}^{n-1} \hat{u}_{x_i x_i} \,=\, 0. \tag{2.23}$$

Using (2.21), (2.23) we get at x,

$$\begin{aligned}
T_1 &= (p-2)\,|\nabla \hat{u}|^{p-2}\,|x|^{-1}\,\left[\sum_{i=1}^{n-1}(2x_i\hat{u}_{x_ix_n} + x_n\,\hat{u}_{x_ix_i}) \;+\; (p-1)x_n\,\hat{u}_{x_nx_n}\right] \\
&= 2(p-2)\,|\nabla \hat{u}|^{p-2}\,|x|^{-1}\sum_{i=1}^{n-1} x_i\hat{u}_{x_ix_n}. \qquad (2.24)
\end{aligned}$$

Also, from (2.20) we deduce

$$\begin{aligned}
T_2 &= |\nabla \hat{u}|^{p-1}\,|x|^{-3}\,\left[\sum_{i=1}^{n-1}(\,|x|^2 - x_i^2\,) + (p-1)\,(|x|^2 - x_n^2)\right] \\
&= |\nabla \hat{u}|^{p-1}\,|x|^{-3}\,[(n+p-3)(\sum_{i=1}^{n-1} x_i^2) + (n-1)x_n^2]. \qquad (2.25)
\end{aligned}$$

Next we note that $\hat{u}_{x_k}$ is a classical solution to (1.7), (1.8), as we see from (2.7), the discussion above (1.5) with $\hat{u}(x,\lambda) = \hat{u}(x+\lambda e_k)$ and the chain rule (see also (2.10)). Using this observation, (2.19), and (2.20) we find at x that

$$\begin{aligned}
|x|\,L(|\nabla \hat{u}|) &= |x| \sum_{i,j,k=1}^{n} \tfrac{\partial}{\partial x_i}\,[\,(\,b_{ij}\,\hat{u}_{x_kx_j})(\hat{u}_{x_k}|\nabla \hat{u}|^{-1})] \\
&= |x|\,|\nabla \hat{u}|^{-1}\sum_{i,j=1}^{n}\sum_{k=1}^{n-1} b_{ij}\hat{u}_{x_kx_i}\hat{u}_{x_kx_j} \qquad (2.26) \\
&= |x||\nabla \hat{u}|^{p-3}\left[\sum_{i,k=1}^{n-1}\hat{u}^2_{x_kx_i} + (p-1)\sum_{k=1}^{n-1}\hat{u}^2_{x_kx_n}\right]
\end{aligned}$$

Finally using (2.20) we see at x that

$$\begin{aligned}
&2\,|x|^{-1}\,|\nabla \hat{u}|^{-1}\left(\sum_{i,j,k=1}^{n} b_{ij}\,x_i\,\hat{u}_{x_k}\,\hat{u}_{x_kx_j}\right) \\
&= 2\,|x|^{-1}\,|\nabla \hat{u}|^{p-2}\left(\sum_{i=1}^{n-1} x_i\,\hat{u}_{x_ix_n} + (p-1)\,x_n\,\hat{u}_{x_nx_n}\right). \qquad (2.27)
\end{aligned}$$

Using (2.22), (2.24) - (2.27) in (2.18) and gathering terms we obtain after some juggling that $|\nabla \hat{u}|^{3-p}\,|x|^3\,L(|x||\nabla \hat{u}|) =$

$$\begin{aligned}
&\left[(n+p-3)\,|\nabla \hat{u}|^2(\sum_{i=1}^{n-1} x_i^2) + 2(p-1)\,|\nabla \hat{u}|\,|x|^2\,(\sum_{i=1}^{n-1} x_i\hat{u}_{x_ix_n}) + (p-1)\,|x|^4\sum_{i=1}^{n-1}\hat{u}^2_{x_ix_n}\right] \qquad (2.28) \\
&+ \left[(n-1)\,|\nabla \hat{u}|^2\,x_n^2 + 2(p-1)|\nabla \hat{u}|\,|x|^2\,x_n\hat{u}_{x_nx_n} + |x|^4\sum_{i,k=1}^{n-1}\hat{u}^2_{x_ix_k}\right].
\end{aligned}$$

To complete the proof of (2.17) we show both terms in brackets in (2.28) are nonnegative. In fact from Schwarz's inequality we see that

$$\left(\sum_{i=1}^{n-1} x_i\, \hat{u}_{x_i x_n}\right)^2 \le \left(\sum_{i=1}^{n-1} \hat{u}^2_{x_i x_n}\right) \cdot \left(\sum_{i=1}^{n-1} x_i^2\right) \tag{2.29}$$

Using (2.29) in the first term in brackets in (2.28) (along with $2ab \le a^2 + b^2$) we deduce that this term is nonnegative. Also from (2.23) and Schwarz's inequality we find

$$(p-1)^2\, \hat{u}^2_{x_n x_n} \le (n-1) \sum_{i=1}^{n-1} \hat{u}^2_{x_i x_i}\,. \tag{2.30}$$

Using (2.30) in the second term in brackets in (2.28) and Schwarz's inequality, we conclude that this term is also nonnegative. Thus (2.17) is true. From our earlier remarks we obtain (2.16) in $\hat{D}$. Hence Lemma 2.5 (i) is true under assumption (2.6).

To continue the proof of Lemma 2.5 we note that the upper bound in Lemma 2.5 (ii) follows from Lemma 2.3. We use (2.16) to prove the lower bound in Lemma 2.5 (ii). We first show that

$$\max_{B(x,s)} \theta \le c \min_{B(x,s)} \theta \text{ whenever } B(x,4s) \subset \hat{D}. \tag{2.31}$$

To prove (2.31) observe from (2.16) and (1.8) that there exists c for which

$$\begin{aligned} c^{-2}\,|x|^{2-p}\,\theta(y)^{p-2}|\xi|^2 &\le c^{-1}\,|\nabla \hat{u}(y)|^{p-2}\,|\xi|^2 \\ &\le \sum_{i,j=1}^{n} b_{ij}(y)\xi_i\,\xi_j \le c\,|\nabla \hat{u}(y)|^{p-2}\,|\xi|^2 \le c^2\,|x|^{2-p}\,\theta(y)^{p-2}\,|\xi|^2 \end{aligned} \tag{2.32}$$

whenever $y \in B(x,2s)$ and $\xi \in \mathbb{R}^n$. Using (2.32) and (1.7) for θ we see that Moser iteration can be applied to powers of θ in the usual way (see [S]) in order to get (2.31). To prove the lower bound in Lemma 2.5 (ii), we consider two cases. First suppose that $\partial\Omega^- \cap B(w,8r) = \emptyset$. From Lemmas 2.1, 2.2 we see that $\tilde{u} \approx \hat{u}$ in $B(w,6r) \cap \hat{D}$ with proportionality constants depending only on p, n, β. If $x \in B(w,3r) \cap \hat{D}$, we can draw a ray l in $\hat{D}$ from x to a point in $\bar{B}(x, d(x,\partial\hat{D})) \cap \partial\Omega^+$ thanks to (1.12) for w. Let y be the first point on l (starting from x) with $\hat{u}(y) = \hat{u}(x)/2$. Then from the mean value theorem of elementary calculus there exists z on the part of l between x, y with

$$\hat{u}(x)/2 = \hat{u}(x) - \hat{u}(y) \le |\nabla \hat{u}(z)|\,|y-x|\,. \tag{2.33}$$

From Lemma 2.2 we deduce the existence of c with

$$y, z \in B[x, (1-c^{-1})d(x,\partial\hat{D})]. \tag{2.34}$$

From (2.34), (2.31), and (2.16) it follows that for some c, $|\nabla \hat{u}(z)| \le c\,|\nabla \hat{u}(x)|$. Using this inequality in (2.33) we conclude that

$$\hat{u}(x) \le c\,|\nabla \hat{u}(x)|\, d(x,\partial\hat{D})\,.$$

Hence the lower bound in Lemma 2.5 (ii) is valid if $w \in \partial\Omega^+$. A similar argument applies if $w \in \partial\Omega^-$. We omit the details. Thus Lemma 2.5 (ii) is valid.

Finally Lemmas 2.5 (ii) implies that the PDE in (1.7), (1.8) is uniformly elliptic in $B(x, 3d(x, \partial\hat{D})/4)$ with Hölder continuous coefficients involving derivatives of $\hat{u}$. Since derivatives of $\hat{u}$ satisfy (1.7) (as mentioned above (2.26)), we can differentiate (1.7) to get a divergence form PDE for second derivatives of u. DiGiorgi or Moser iteration can then be applied to get Lemma 2.5 (iii). One can also obtain Lemma 2.5 (iii) from Schauder type estimates for the nondivergence form PDE satisfied by $\hat{u}$ as in Lemma 2.4. Thus Lemma 2.5 is valid under assumption (2.6).

To complete the proof of Lemma 2.5 we show that assumption (2.6) is unnecessary. For this purpose let $R_m^+, R_m^- \in C^\infty(\partial B(0,1))$ for $m = 1, 2, \ldots$, with $\| \log R_m^* \hat{\|} \leq k\beta$, and $R_m^* \to R^*$ as $m \to \infty$ uniformly on $\partial B(0,1)$ whenever $* \in \{+, -\}$. Here k depends only on n. Let $\hat{D}_m, \hat{u}_m$ be the ring domain and p capacitary function with center $\hat{z}$, corresponding to R_m^+, R_m^-. From Lemmas 2.2, 2.3 we see that $\hat{u}_m, \nabla\hat{u}_m$ converge uniformly on compact subsets of $\hat{D}$ to $\hat{u}, \nabla\hat{u}$. We apply Lemma 2.5 to each $\hat{u}_m$. Since the constants in this lemma and (2.31) are independent of m we conclude that Lemma 2.5 also holds for $\hat{u}$ without hypothesis (2.6). The proof of Lemma 2.5 is now complete. □

Lemma 2.35. *Let $\tilde{u}, \hat{u}, \hat{D}$, be as in Lemma 2.5. There exists unique finite positive Borel measures μ^+, μ^- on $\mathbb{R}^n$ with support in $\partial\Omega^+, \partial\Omega^-$, respectively, such that if $\mu = \mu^+ - \mu^-$, then*

(a) $\displaystyle\int |\nabla\hat{u}|^{p-2} \langle \nabla\hat{u}, \nabla\phi \rangle \, dx = -\int \phi \, d\mu$ *whenever* $\phi \in C_0^\infty(\mathbb{R}^n)$.

(b) *If w satisfies (1.12) then there exists c such that*
$c^{-1} r^{p-n} \tilde{\mu}[B(w, 4r)] \leq \tilde{u}(a_r(w))^{p-1} \leq c r^{p-n} \tilde{\mu}[B(w, r/2)]$,
where $\tilde{\mu} = \mu^+$ if $B(w, 8r) \cap \partial\Omega^- = \emptyset$ and $\tilde{\mu} = \mu^-$ if $B(w, 8r) \cap \partial\Omega^+ = \emptyset$.

Proof: Existence and uniqueness of μ satisfying Lemma 2.35 (a) is easily proved using Lemmas 2.1 and 2.2 (see [BL] or [HKM] for a proof). Assuming Lemma 2.35 (a) one gets the lefthand inequality in Lemma 2.35 (b) by first choosing, $\phi \in C_0^\infty(B(w, 6r))$ with $\phi \equiv 1$ on $B(w, 4r)$ and then using Lemma 2.1. The righthand inequality in Lemma 2.35 (b) is essentially proved in [KZ, Lemma 3.1] (see also [EL, Lemma 1]). □

From Lemma 2.5 it is easily seen that

$$\begin{aligned} &D(t) = \{x : t < \hat{u}(x) < 1\} \text{ and } D'(t) = \{x : 0 < \hat{u}(x) < t\}, 0 < t < 1, \text{ are starlike} \\ &\text{Lipschitz ring domains with center } \hat{z} \text{ and constants depending only on } \beta. \end{aligned} \tag{2.36}$$

Moreover, $(1-t)^{-1} \max\{\hat{u} - t, 0\}$ and $\min\{\hat{u}/t, 1\}$ are the p capacitary functions for $D(t), D'(t)$, respectively . If $B(w, 8r) \cap \Omega^- = \emptyset$, let μ_t, μ^-, be the measures corresponding to $\max\{\hat{u} - t, 0\}$ in $D(t)$, as in Lemma 2.35. We note that

$$d\mu_t(w) = |\nabla\hat{u}|^{p-1}(w) \, dH^{n-1}(w) \text{ for } w \in \{x : \hat{u}(x) = t\}, \tag{2.37}$$

as follows from Lemma 2.35 (a) with μ^+ replaced by μ_t, Lemmas 2.4, 2.5, and integration by parts. A similar argument gives that μ_t, μ^+ are the measures corresponding to $\min(u, t)$ in $D'(t)$. Moreover, since $\hat{u} \in W^{1,p}(\mathbb{R}^n)$, it is easily deduced from Lemma 2.35 (a) that

$$\mu_t \to \mu^+, \mu^- \text{ weakly in the sense of measures as } t \to 0, 1 \text{ respectively .} \tag{2.38}$$

Next we have the following reverse Hölder inequality.

Lemma 2.39. *Let μ, w be as in Lemma 2.35. Then $d\mu/dH^{n-1} = \pm\, k^{p-1}$ on $B(w,8r)\cap\partial\hat{D}$ where $k\geq 0$ with*

$$\int_{B(w,2r)\cap\partial\hat{D}} k^p\, dH^{n-1} \,\leq\, c\, r^{-\frac{n-1}{p-1}}\left(\int_{B(w,2r)\cap\partial\hat{D}} k^{p-1}\, dH^{n-1}\right)^{p/(p-1)}$$

Proof: Again we consider two cases. If $B(w,8r)\cap\Omega^- = \emptyset$, we first show that Lemma 2.39 is valid with μ^+ replaced by μ_t for t sufficiently near 0. Again we assume that $\hat{z} = 0$. For t near 0, and $2r < s < 3r$, we note from Lemmas 2.4, 2.5 that integration by parts can be used to get

$$\begin{aligned}\int_{\partial[B(w,s)\cap D(t)]} \langle\, x,\, \nu\,\rangle\, |\nabla\hat{u}|^p\, dH^{n-1} \,&=\, \int_{B(w,s)\cap D(t)} \nabla\cdot(x\,|\nabla\hat{u}|^p)\, dx \\ &= n\int_{B(w,s)\cap D(t)} |\nabla\hat{u}|^p\; dx \,+\, \sum_{i,j=1}^{n}\int_{B(w,s)\cap D(t)} p\,|\nabla\hat{u}|^{p-2}\, x_j\, \hat{u}_{x_i}\, \hat{u}_{x_i x_j}\, dx\end{aligned} \tag{2.40}$$

where ν denotes the outer unit normal to the boundary of $B(w,s)\cap D(t)$. Now from (1.2) and integration by parts, we deduce

$$\begin{aligned}&\sum_{i,j=1}^{n}\int_{B(w,s)\cap D(t)} p\,|\nabla\hat{u}|^{p-2}\, x_j\, \hat{u}_{x_i}\, \hat{u}_{x_i x_j}\, dx \\ &\quad= \,-p\int_{B(w,s)\cap D(t)} |\nabla\hat{u}|^p\, dx + \, p\int_{\partial[B(w,s)\cap D(t)]} \langle x, \nabla\hat{u}\rangle\langle\nabla\hat{u}, \nu\rangle\, |\nabla\hat{u}|^{p-2}\, dH^{n-1}\,.\end{aligned}$$

Using this equality in (2.40) we get

$$\begin{aligned}I = &\int_{\partial[B(w,s)\cap D(t)]} \langle\, x,\, \nu\,\rangle\, |\nabla\hat{u}|^p\, dH^{n-1} \,-\, p\int_{\partial[B(w,s)\cap D(t)]} \langle x, \nabla\hat{u}\rangle\langle\, \nabla\hat{u},\, \nu\,\rangle\, |\nabla\hat{u}|^{p-2}\, dH^{n-1} \\ &=\, (n-p)\int_{B(w,s)\cap D(t)} |\nabla\hat{u}|^p\, dx\,.\end{aligned} \tag{2.41}$$

We note that $\nu = -\frac{\nabla\hat{u}}{|\nabla\hat{u}|}$ on $\partial D(t)\cap B(w,s)$. Thus from (2.41)

$$(p-1)\int_{\partial D(t)\cap B(w,s)} \langle x, \nabla\hat{u}\rangle\, |\nabla\hat{u}|^{p-1} dH^{n-1} = I + E \tag{2.42}$$

where

$$|E| \,\leq\, (p+1)\int_{\partial B(w,s)\cap D(t)} |x|\, |\nabla\hat{u}|^p dH^{n-1}.$$

Next choose $s \in (2r, 3r)$ so that

$$\int_{\partial B(w,s)\cap D(t)} |\nabla\hat{u}|^p\, dH^{n-1} \,\leq\, 2r^{-1}\int_{B(w,3r)\cap D(t)} |\nabla\hat{u}|^p\, dx.$$

This choice is possible from weak type estimates. Using these inequalities in (2.42), $d(0, \partial\hat{D}) \approx$ diam $\hat{D}$, and (2.5) (i), we deduce

$$\int_{B(w,s)\cap\partial D(t)} |x|\, |\nabla \hat{u}|^p \, dH^{n-1} \,\leq\, c[d(0,\partial\hat{D})/r] \int_{B(w,3r)\cap D(t)} |\nabla \hat{u}|^p \, dx \,+\, |I|. \tag{2.43}$$

Then from (2.43), (2.41), (2.37), Lemmas 2.1, 2.2, 2.35 with $\hat{D}, \hat{u}$, replaced by $D(t)$, $(1-t)^{-1}\max(\hat{u}-t, 0)$, we find that

$$\begin{aligned} c^{-2}\int_{B(w,s)\cap\partial D(t)} |\nabla \hat{u}|^p \, dH^{n-1} \,&\leq\, c^{-1}\, r^{-1}\int_{B(w,3r)\cap D(t)} |\nabla \hat{u}|^p \, dx \\ &\leq\, r^{-\frac{n-1}{p-1}} \left(\int_{B(w,2r)\cap\partial D(t)} |\nabla \hat{u}|^{p-1} \, dH^{n-1} \right)^{p/(p-1)} \,\leq\, c\, r^{-p+n-1}\, \hat{u}(a_{3r}(w))^p \\ &= c\, r^{-\frac{n-1}{p-1}} \, (\mu_t[B(w,2r)])^{p/(p-1)}. \end{aligned} \tag{2.44}$$

In (2.44) we have also used the fact that $d(0, \partial\hat{D}) > r$. Thus Lemma 2.39 holds with $k = |\nabla \hat{u}|$ and $\mu^+, \hat{D}$ replaced by $\mu_t, D(t)$. Using the change of variables formula we can pull back each μ_t to a measure on $\partial B(0,1)$. In view of (2.44) and (2.36) we see that the Radon - Nikodym derivative of each pullback measure with respect to H^{n-1} measure on $\partial B(0,1)$, satisfies a $L^{p/(p-1)}$ reverse Hölder inequality on $\{x/|x| : x \in B(w,2r) \cap \partial D(t)\}$. Moreover $L^{p/(p-1)}$ and Hölder constants depend only on p, n, β, for small $t > 0$. Hence any sequence of these derivatives has a subsequence that converges weakly in $L^{p/(p-1)}$. Using these observations, lower semicontinuity of the norm in $L^{p/(p-1)}$ under weak convergence, (2.38), and letting $t{\to}0$ we get the Radon - Nikodym derivative of the pullback of μ^+ on $B(w,2r) \cap \partial\hat{D}$ satisfies a $L^{p/(p-1)}$ reverse Hölder inequality. Transforming back, we conclude that Lemma 2.39 is true when $B(w,8r) \cap \partial\Omega^- = \emptyset$. If $B(w,8r) \cap \partial\Omega^+ = \emptyset$, we can use the same argument with $D(t)$ replaced by $D'(t)$. Letting $t{\to}1$ and taking limits as above we get Lemma 2.39 in this case also. □

We use Lemma 2.39 to prove the following localization lemma.

Lemma 2.45. *Let $\tilde{u}, \hat{u}$, and $\hat{D}$ be as in Lemma 2.5. Let w satisfy (1.12) and let $w' \in \hat{D}$ be the point on the ray from $\hat{z}$ through w with $|w - w'| = r/4$. There exists c and a starlike Lipschitz domain $\tilde{\Omega} \subset B(w,2r) \cap \hat{D}$ with center at w',*

$$c\, H^{n-1}[\partial\tilde{\Omega} \cap B(w,r) \cap \partial\hat{D}] \geq r^{n-1},$$

and Lipschitz constant $\beta' \leq c(\beta+1)$. Moreover if $x \in \tilde{\Omega}$, then

$$c^{-1} r^{-1}\, \tilde{u}(w') \,\leq\, |\nabla \hat{u}(x)| \,\leq\, c r^{-1}\, \tilde{u}(w').$$

Proof: As usual we consider two cases. First suppose that $B(w,8r) \cap \partial\Omega^- = \emptyset$. From Lipschitzness of $\partial\hat{D}$ and basic geometry we deduce the existence of c' (depending only on p, n, β) such that if

$$\begin{aligned} &\hat{R}(\omega) = |x - w'| \text{ when } \omega = \tfrac{x-w'}{|x-w'|}, x \in B(w, r/c') \cap \partial\hat{D}, \text{ then } \log\hat{R} \text{ is} \\ &\text{Lipschitz on } E = \{\tfrac{x-w'}{|x-w'|} : x \in B(w, r/c') \cap \partial\hat{D}\} \text{ with } \|\log\hat{R}\hat{\|}_E \leq c'(\beta+1). \end{aligned} \tag{2.46}$$

One proof of (2.46) is to observe that it is equivalent to the statement that the cones $\Gamma(x), x \in B(w, r/c') \cap \partial \hat{D}$, defined above (1.13), each contain w'. (2.46) can also be deduced from basic trigonometry using starlike Lipschitzness of $\hat{D}$ with respect to $\hat{z}$. Let $r' = r/c'$ and for $x \in B(w, r') \cap \partial \hat{D}$ set

$$M_1(x) = \inf_{0<s<r'} s^{1-n} \int_{B(x,s)\cap\partial\hat{D}} k^{p-1}\, dH^{n-1}\,,$$

$$M_2(x) = \sup_{0<s<r'} s^{1-n} \int_{B(x,s)\cap\partial\hat{D}} k^{p-1}\, dH^{n-1}\,.$$

We claim there exists a compact set $F \subset B(w, r'/4) \cap \partial\hat{D}$ and c'' (depending only on p, n, β) with

$$c'' M_1 > (r')^{1-p} \hat{u}(a_{r'}(w))^{p-1} \text{ on } F \text{ and } c'' H^{n-1}(F) > (r')^{n-1}. \tag{2.47}$$

In fact if

$$\epsilon = (1/c'')\,(r')^{1-p} \hat{u}(a_{r'}(w))^{p-1}$$

$$G = \{x \in B(w, r'/4) \cap \partial\hat{D} : M_1(x) \le \epsilon\},$$

then by a standard covering argument there exists $\{B(x_i, r_i)\}$ with $x_i \in G$, $0 < r_i \le r'$, $G \subset \bigcup_i B(x_i, r_i)$ and $\{B(x_i, r_i/10)\}$ pairwise disjoint. Also,

$$\int_{B(x_i,r_i)\cap\partial\hat{D}} k^{p-1}\, dH^{n-1} \le 2\epsilon\, r_i^{n-1} \text{ for each } i.$$

Using these facts and $H^{n-1}[B(x_i, r_i/10) \cap \partial\hat{D}] \approx r_i^{n-1}$, we get

$$\begin{aligned} \int_G k^{p-1}\, dH^{n-1} &\le \sum_i \int_{B(x_i,r_i)\cap\partial\hat{D}} k^{p-1}\, dH^{n-1} \\ &\le 2\epsilon \sum_i r_i^{n-1} \le c\,\epsilon\,(r')^{n-1}. \end{aligned} \tag{2.48}$$

On the other hand if $\Lambda = B(w, r'/4) \cap \partial\hat{D} \setminus G$, then from Lemmas 2.39, 2.35, with r replaced by $r'/8$, and Hölder's inequality, we get for some c,

$$\begin{aligned} \int_\Lambda k^{p-1}\, dH^{n-1} &\le H^{n-1}(\Lambda)^{1/p} \cdot \left(\int_{B(w,r'/4)\cap\partial\hat{D}} k^p dH^{n-1} \right)^{1-1/p} \\ &\le c\,[(r')^{1-n}\, H^{n-1}(\Lambda)\,]^{1/p} \cdot \int_{B(w,r'/4)\cap\partial\hat{D}} k^{p-1} dH^{n-1} \\ &\le c^2\, [(r')^{1-n} H^{n-1}(\Lambda)]^{1/p}\, (r')^{n-p}\, \hat{u}(a_{r'}(w))^{p-1}\,. \end{aligned} \tag{2.49}$$

Since

$$(r')^{n-p}\, \hat{u}(a_{r'}(w))^{p-1} \le c \int_{B(w,r'/4)\cap\partial\hat{D}} k^{p-1}\, dH^{n-1}$$

we can add (2.48), (2.49) to get after division by $(r')^{n-p}\,\hat u(a_{r'}(w))^{p-1}$ that for some c,

$$c^{-1} \,\leq\, [(r')^{1-n} H^{n-1}(\Lambda)]^{1/p} \,+\, 1/c''. \tag{2.50}$$

Clearly (2.50) implies (2.47) with F replaced by Λ. A standard measure theory argument then shows that we can replace Λ by suitable F compact, $F \subset \Lambda$. Thus (2.47) is valid.

Fix c'' so that (2.47) holds. To continue, we use the Hardy-Littlewood Maximal theorem (see [St, ch 1]) and Lemmas 2.35, 2.39, to find F' compact, $F' \subset F$ and $\bar c \geq c''$ such that

$$M_2 \,\leq\, \bar c\,(r')^{1-p}\,\hat u(a_{r'}(w))^{p-1} \;\text{ on } F' \text{ and } \bar c\, H^{n-1}(F') \,\geq\, (r')^{n-1}\,.$$

Combining this inequality with (2.47) and using Harnack's inequality (Lemma 2.1 (b)), we have for some c,

$$\begin{aligned} &c^{-1} r^{1-p} \hat u(w')^{p-1} \,\leq s^{1-n} \int_{B(x,s)\cap\partial\hat D} k^{p-1}\, dH^{n-1} \,\leq\, c r^{1-p} \hat u(w')^{p-1} \\ &\text{for } x \in F' \text{ and } 0 < s < r' \text{ where } H^{n-1}(F') \geq c^{-1} r^{n-1}. \end{aligned} \tag{2.51}$$

Next we draw all rays from points in $\bar B(w', r/c)$ to points in F'. Let $\tilde\Omega$ be the interior of this set. Using (2.46) it is easily seen for c large enough that $\tilde\Omega$ is starlike Lipschitz with center at w' and Lipschitz constant $\leq c(\beta+1)$. Also from (2.51) we see that

$$c\, H^{n-1}[\partial\tilde\Omega \cap B(w,r) \cap \partial\hat D] \geq r^{n-1}.$$

If $x \in \tilde\Omega$ and $d(x, \partial\hat D) \geq r'/10^5$, then from Lemma 2.5 (ii) and Harnack's inequality, we find that

$$c^{-1} r^{-1}\, \hat u(w') \,\leq\, |\nabla \hat u(x)| \,\leq\, c r^{-1}\, \hat u(w'). \tag{2.52}$$

Otherwise there exists $\tilde x \in F'$ with $|x-\tilde x| \leq c d(x, \partial\hat D)$. Choose s, $0 < s < r'$, such that $s \leq |x - \tilde x| \leq c\, s$. Using Lemmas 2.35, 2.39, 2.5 (ii), (2.51), and Harnack's inequality we conclude that

$$|\nabla \hat u(x)|^{p-1} \,\approx\, [\hat u(x)/d(x,\partial\hat D)]^{p-1} \approx s^{1-n} \int_{B(\tilde x,s)\cap\partial\hat D} k^{p-1} dH^{n-1} \approx\, r^{1-p}\, \hat u(w')^{p-1} \tag{2.53}$$

where all proportionality constants depend only on p, n, β. Thus (2.52) holds in this case also. The proof of Lemma 2.45 is now complete when $B(w, 8r) \cap \partial\Omega^- = \emptyset$. An essentially identical argument holds if $B(w, 8r) \cap \partial\Omega^+ = \emptyset$. $\Box$

Finally in this section we use Lemmas 2.45, 2.5 to show that if

$$d\tilde\sigma(y) = d(y, \partial\tilde\Omega) \max_{B(y, \frac12 d(y,\partial\tilde\Omega))} \{|\nabla \hat u|^{2p-6} \sum_{i,j=1}^{n} \hat u^2_{x_i x_j}\}\, dy$$

when $y \in \tilde\Omega$, then $\tilde\sigma$ is a Carleson measure on $\tilde\Omega$. More specifically we prove the following lemma.

Lemma 2.54. *Let $w, w', r, \tilde\Omega$ be as in Lemma 2.45. If $z \in \partial\tilde\Omega$ and $0 < s < r$, then*

$$\tilde\sigma(B(z,s) \cap \tilde\Omega) \,\leq\, c\, s^{n-1}\, (\tilde u(w')/r)^{2p-4}.$$

Proof: We only give the proof when $B(w, 8r) \cap \Omega^- = \emptyset$. Let $\tilde{R}$ be the graph function for $\tilde{\Omega}$. We let for, $0 < \epsilon < \frac{1}{100} \min_{\partial B(0,1)} \tilde{R}$, $R = \tilde{R} - \epsilon$ and let Ω be the starlike Lipschitz domain with center w', graph function R. If $z \in \partial\tilde{\Omega}$ and $0 < s \leq \frac{1}{2} d(z, \partial\hat{D})$, we can use Lemmas 2.45, 2.5, and the fact that $d(y, \partial\tilde{\Omega}) \leq d(y, \partial\hat{D})$ for $y \in \tilde{\Omega}$, to get that Lemma 2.54 is valid in this case. Otherwise, we use a simplified version of the argument in [LV], [LV1]. Again, let $L, (b_{ij}), 1 \leq i, j \leq n$, be as in (1.7), (1.8), defined relative to $\hat{u}, \hat{D}$. Since derivatives of $\hat{u}$ are solutions to $L\zeta = 0$ (see the remark above (2.26)), we find at $x \in \hat{D}$,

$$
\begin{aligned}
L(|\nabla \hat{u}|^2) &= \sum_{i,j=1}^{n} \tfrac{\partial}{\partial x_i}[b_{ij} \tfrac{\partial}{\partial x_j}(|\nabla \hat{u}|^2)] \\
&= 2 \sum_{i,j=1}^{n} b_{ij} \hat{u}_{x_i x_j} \, \hat{u}_{x_i x_j} \; \geq \; \min(1, p-1) \, |\nabla \hat{u}|^{p-2} \sum_{i,j=1}^{n} \hat{u}^2_{x_i x_j} .
\end{aligned} \tag{2.55}
$$

Also we note from (2.5) $(ii), (iii)$ and Lemma 2.45 that

$$
\begin{aligned}
&\int_{B(z,s)\cap\Omega} d(y, \partial\Omega) \max_{B(y, \frac{1}{2} d(y,\partial\Omega))} \{ |\nabla \hat{u}|^{2p-6} \sum_{i,j=1}^{n} \hat{u}^2_{x_i x_j} \} \, dy \\
&\quad \leq c \int_{B(z,s)\cap\Omega} d(y, \partial\Omega)^{1-n} \left(\int_{B(y, \frac{3}{4} d(y,\partial\Omega))} |\nabla \hat{u}|^{2p-6} \sum_{i,j=1}^{n} \hat{u}^2_{x_i x_j} dx \right) dy \\
&\quad \leq c \int_{B(z,4s)\cap\Omega} d(y, \partial\Omega) \, |\nabla \hat{u}(y)|^{2p-6} \sum_{i,j=1}^{n} \hat{u}^2_{y_i y_j} \, dy
\end{aligned} \tag{2.56}
$$

where the third integral is obtained from the second integral by interchanging the order of integration. Define σ as above Lemma 2.54 with $\tilde{\Omega}$ replaced by Ω. Using (2.55), (2.56) and once again Lemmas 2.5, 2.45 we see that

$$
\begin{aligned}
\sigma[B(z, s) \cap \Omega] &\leq c \int_{B(z,4s)\cap\Omega} d(y, \partial\Omega) \, |\nabla \hat{u}(y)|^{2p-6} \sum_{i,j=1}^{n} \hat{u}^2_{y_i y_j} \, dy \\
&\leq c \left(\tfrac{\hat{u}(w')}{r} \right)^{(p-5)} \int_{B(z,4s)\cap\Omega} \hat{u}(y) L(|\nabla \hat{u}|^2)(y) dy \, .
\end{aligned} \tag{2.57}
$$

We note that the divergence theorem can be used in $B(z, 4s) \cap \Omega$. Since $L\hat{u} = 0$ in $\hat{D}$ we have

$$
\int_{B(z,4s)\cap\Omega} \hat{u} L(|\nabla \hat{u}|^2) dy \; = \int_{B(z,4s)\cap\Omega} [\, \hat{u} L(|\nabla \hat{u}|^2) - |\nabla \hat{u}|^2 \, L\hat{u} \,] \, dy.
$$

Hence, if we let $\nu = (\nu_1, \dots, \nu_n)$ be the outer unit normal to $B(z, 4s) \cap \Omega$, then

$$
\left| \int_{B(z,4s)\cap\Omega} \hat{u} L(|\nabla \hat{u}|^2) dy \right| \;\; \leq \;\; 2 \left| \int_{\partial[B(z,4s)\cap\Omega]} \sum_{i,j,k=1}^{n} \hat{u} \; \nu_i \, b_{ij} \, \hat{u}_{x_k} \, \hat{u}_{x_k x_j} \, dH^{n-1} \right|
$$

$$
\begin{aligned}
& + \left| \int_{\partial[B(z,4s)\cap\Omega]} \sum_{i,j=1}^{n} |\nabla \hat u|^2 \, \nu_i \, b_{ij} \, \hat u_{x_j} \, dH^{n-1} \right| \\
\leq \quad & cs^{(n-1)} \, r^{-(p+1)} \hat u(w')^{p+1} \qquad (2.58)
\end{aligned}
$$

where we have used $|b_{ij}| \leq p|\nabla \hat u|^{p-2}$, and once again Lemmas 2.5, 2.45 to make estimates. Combining (2.57) - (2.58) letting $\epsilon \to 0$, and using Fatou's lemma we conclude that Lemma 2.54 is true. $\Box$

3. Elliptic Measure and a Boundary Harnack Inequality. To continue the program outlined in section 1, let $\hat u, \hat D, \hat z, \Omega^+, \Omega^-, R^+, R^-, \beta$, be as in section 2. Throughout this section we assume that

$$
R^+, R^- \in C^\infty(\partial B(0,1)). \qquad (3.1)
$$

Let $L, (b_{ij})$, be as in (1.7), (1.8) relative to $\hat u, \hat D$. Then from (3.1), Lemmas 2.4, 2.5, see that (b_{ij}) is symmetric and infinitely differentiable in the closure of $\hat D$. Also,

$$
c^{-1} \, |\nabla u(y)|^{p-2} \, |\xi|^2 \, \leq \, \sum_{i,j=1}^{n} b_{ij}(y) \, \xi_i \, \xi_j \, \leq \, c \, |\xi|^2 \, |\nabla u(y)|^{p-2} \qquad (3.2)
$$

whenever $y \in \hat D$ and $\xi \in \mathbb{R}^n$, so (b_{ij}) is uniformly elliptic in $\hat D$. Let $H \subset \hat D$ be either a starlike Lipschitz domain or $H = \hat D$. Using the above facts and arguing as in [LSW] we deduce the existence of Green's function, $g(\cdot,\cdot) : H \times H \to (0,\infty]$ with the following properties.

(a') $\quad g(x,y) = g(y,x)$ when $x \neq y, x, y \in H$,

(b') $\quad \zeta \, g(\cdot, y) \in W_0^{1,2}[H \setminus \bar B(y,\epsilon)]$ whenever $\bar B(y,\epsilon) \subset H$ and $\zeta \in C_0^\infty(\mathbb{R}^n \setminus \bar B(y,\epsilon))$,

(c') $\quad$ if $x \neq y, x, y \in B[y, \frac{d(y,\partial\hat D)}{2}] \cap H$, then, $g(x,y) \approx \left(\frac{\hat u(y)}{d(y,\partial\hat D)}\right)^{2-p} |x-y|^{2-n}$ $\qquad (3.3)$
for $n > 2$ and for $n = 2$, $g(x,y) \approx \left(\frac{\hat u(y)}{d(y,\partial\hat D)}\right)^{2-p} \log\left(\frac{2d(y,\partial\hat D)}{|x-y|}\right)$,

(d') $\quad$ if $y \in H$, let $g(\cdot,y) \equiv 0$ on $\mathbb{R}^n \setminus H$. Then $g(\cdot,y)$ is continuous on $\mathbb{R}^n \setminus \{y\}$.

In (c') we have used Lemma 2.5 to estimate the ellipticity constants in (3.2). Also all proportionality constants depend only on p, n, and β. For given $y \in H$ extend $g(\cdot,y)$ to a continuous function on $\mathbb{R}^n \setminus \{y\}$ as in (d'). Then there exists a positive Borel measure, $\omega(\cdot,y)$, on $\mathbb{R}^n$ with support in ∂H, $\omega(\partial H, y) = 1$, and the property that

$$
\theta(y) \, = \, \int \sum_{i,j=1}^{n} \theta_{x_i} \, b_{ij} \, g_{x_j}(\cdot,y) \, dx \, + \, \int \theta \, d\omega(\cdot,y) \qquad (3.4)
$$

whenever $\theta \in C_0^\infty(\mathbb{R}^n)$. Observe from (3.4) with $\theta \in C_0^\infty(H \setminus \{y\})$ that $g(\cdot,y)$ is a weak solution to L in $H \setminus \{y\}$. From Schauder or Moser-DiGiorgi type estimates it follows that $g(\cdot,y)$ is an infinitely differentiable solution to L in $H \setminus \{y\}$ which extends to a continuous function in $\mathbb{R}^n \setminus \{y\}$ with

$g(\cdot, y) \equiv 0$ on $\mathbb{R}^n \setminus H$. Using this fact, we deduce the following interior estimates for $g(\cdot, y)$.

Lemma 3.5. *If $B(z, 4s) \subset H \setminus \{y\}$, then,*

$$(a) \quad s^{2-n} \int_{B(z,s/2)} |\nabla g(\cdot, y)|^2 \, dx \, \leq \, c \, \max_{B(z,s)} g(\cdot, y)^2,$$

$$(b) \quad \max_{B(z,s)} g(\cdot, y) \, \leq \, c \, \min_{B(z,s)} g(\cdot, y),$$

(c) *For some* $0 < \hat{\alpha} < 1$, $|g(\zeta, y) - g(\xi, y)| \leq c \left(\frac{|\zeta - \xi|}{s} \right)^{\hat{\alpha}} \max_{B(z,s)} g(\cdot, y)$ *whenever* $\zeta, \xi \in B(z, s)$.

Proof: From Lemma 2.5 and (3.2) we see that the ellipticity constants for L are bounded above and below in $B(z, 2s)$ by constant multiples of $(\frac{u(z)}{d(z,\partial \hat{D})})^{p-2}$ (depending only on p, n, β). Thus Lemma 3.5 follows from standard estimates of the above mentioned authors. $\Box$

Using (3.4), Lemma 3.5, and arguing as in [LSW], one can deduce by way of a method of Perron - Wiener - Brelot, that the continuous Dirichlet problem for L in H has a solution in the sense that given a continous function f on ∂H there exists a strong solution F to L in H with $F = f$ continuously on ∂H. Indeed,

$$F(y) = \int f \, d\omega(\cdot, y) \text{ for } y \in H.$$

If $H = \hat{D}$, then from C^∞ smoothness of $b_{ij}, \partial \hat{D}$, as well as (3.2) and Schauder type arguments, one can show that $g(\cdot, y)$ extends to a C^∞ function in the closure of $\hat{D}$ (away from y). Using this fact and the divergence theorem we deduce that on $\partial \hat{D}$,

$$d\omega(\cdot, y) = |\nabla g|^{-1}(\cdot, y) \sum_{i,j=1}^{n} b_{ij} \, g_{x_i}(\cdot, y) g_{x_j}(\cdot, y) \, dH^{n-1} \approx |\nabla \hat{u}|^{p-2} \, |\nabla g(\cdot, y)| \, dH^{n-1}. \tag{3.6}$$

$\omega(\cdot, y)$ is called elliptic measure for H relative to L and $y \in H$. We say that $\omega(\cdot, y)$ is a doubling measure if, for some $1 \leq C < \infty$,

$$\omega(B(x, 2s), y) \leq C\omega(B(x, s), y) \text{ whenever } x \in \partial H \text{ and } 0 < s \leq \text{ diam } H. \tag{3.7}$$

$\omega(\cdot, y)$ is said to be an A^∞ weight with respect to H^{n-1} on ∂H if $\omega(\cdot, y)$ is a doubling measure and for some $\delta > 0$ there exists $\epsilon = \epsilon(\delta) > 0$ with the property that if E is Borel, $E \subset B(x, s) \cap \partial H$, ($x, s$, as in (3.7)), then

$$\frac{H^{n-1}(E)}{H^{n-1}(B(x, s) \cap \partial H)} \geq \delta \longrightarrow \frac{\omega(E, y)}{\omega(B(x, s), y)} \geq \epsilon. \tag{3.8}$$

(see [CF] for several equivalent characterizations of A^∞). Let $\hat{\omega}, \hat{g}$ denote elliptic measure and Green's function for $\hat{D}$. We prove the following result on the elliptic measure.

Lemma 3.9. *Let $x \in \partial\hat{D}, 0 < \rho$, suppose $\hat{z} \notin B(x, 8\rho)$ and either $B(x, 8\rho) \cap \partial\Omega^+ = \emptyset$ or $B(x, 8\rho) \cap \partial\Omega^- = \emptyset$. Then for some $\eta > 0$ (depending only on p, n, β)*

$$\hat{\omega}(B(x, \rho), a_\rho(x)) \geq \eta.$$

Proof: We note that $a_\rho(x)$ is defined as in Lemma 2.2 with r, w replaced by ρ, x. To prove Lemma 3.9, let $x' \in \hat{D}$ be the point on the ray from $\hat{z}$ through x with $|x' - x| = \rho$. Let $\tilde{\Omega}$ be as in Lemmas 2.45, 2.54, defined relative to x, x', ρ. From the definition of (b_{ij}) in (1.8) we find at $y \in \tilde{\Omega}$ that

$$\sum_{i,j=1}^{n} |\nabla b_{ij}|^2 \ \leq \ c\, |\nabla \hat{u}|^{2p-6} \sum_{i,j=1}^{n} |\hat{u}_{y_i y_j}|^2.$$

From this display and Lemma 2.54 we conclude that if

$$d\gamma(y) = \sum_{i,j=1}^{n} d(y, \partial\tilde{\Omega}) \max_{\bar{B}(y, \frac{1}{2} d(y, \tilde{\Omega}))} |\nabla b_{ij}|^2 \, dy \text{ for } y \in \tilde{\Omega},$$

then γ is a Carleson measure on $\tilde{\Omega}$ with

$$\sup_{0<s<\rho} \frac{\gamma(B(z, s) \cap \tilde{\Omega})}{s^{n-1}} \ \leq \ c \left(\frac{u(x')}{\rho} \right)^{2p-4} \quad \text{whenever } z \in \partial\tilde{\Omega}. \tag{3.10}$$

Let $\tilde{\omega}(\cdot, x')$ be elliptic measure for $\tilde{\Omega}$ defined relative to L, x'. We shall need the following theorem of [KP] taylored to our situation (see also [HL, ch 3]).

Theorem 3.11. *If (3.10) holds then $\tilde{\omega}(\cdot, x')$ is an A^∞ weight with respect to H^{n-1} with constants C in (3.7) and δ, ϵ in (3.8) depending only on p, n, β.*

Proof: We note that the dependence of the constants in Theorem 3.11 can be easily checked since if

$$b'_{ij} = \left(\frac{u(x')}{\rho} \right)^{2-p} b_{ij}$$

then $\tilde{\omega}(\cdot, x')$ is obviously a solution in $\tilde{\Omega}$ to the PDE one gets from replacing (b_{ij}) by (b'_{ij}) in the definition of L. Moreover ellipticity constants in (3.2) and the Carleson norm in (3.10) taken relative to (b'_{ij}) depend only on p, n, β. $\Box$

To get Lemma 3.9 from Theorem 3.11, we first observe from Lemma 2.45 that

$$H^{n-1}(B(x, \rho) \cap \partial\tilde{\Omega} \cap \partial\hat{D}) \geq \rho^{n-1}/c.$$

In particular there exist $\tilde{x} \in \partial\tilde{\Omega}$ and $\tilde{c} > 1$ such that $B(\tilde{x}, \rho/\tilde{c}) \subset B(x, \rho)$ and

$$H^{n-1}(B(\tilde{x}, \rho/\tilde{c}) \cap \partial\tilde{\Omega} \cap \partial\hat{D}) \geq \rho^{n-1}/c.$$

Applying Theorem 3.11 we deduce from the definition of A^∞ in (3.8) that

$$\tilde{\omega}(B(\tilde{x}, \rho/\tilde{c}) \cap \partial\hat{D}, x') \geq c^{-2} \tilde{\omega}(B(\tilde{x}, \rho/\tilde{c}), x') \geq c^{-3},$$

where the last inequality follows from the doubling property of $\tilde{\omega}$ and the fact that $\tilde{\omega}(\partial\tilde{\Omega}, x') = 1$. Thus from the maximum principle we have for some c

$$\hat{\omega}(B(x,\rho) \cap \partial\tilde{\Omega}, x') \geq \tilde{\omega}(B(x,\rho) \cap \partial\hat{D}, x') \geq c^{-1}.$$

We note that Harnack's inequality holds for $\hat{\omega}(\cdot, y)$ with constants depending only on p, n, β by the same reasoning as in Lemma 3.5 (b). From Harnack's inequality, we see that x' can be replaced by $a_\rho(x)$ in the above inequality. Hence Lemma 3.9 is true. $\Box$

If x, ρ are as in Lemma 3.9 we claim that

$$\hat{\omega}(B(x,\rho), y) \geq c^{-1} \text{ whenever } y \in B(x, \rho/4) \cap \hat{D}. \tag{3.12}$$

In fact given y as in (3.12) choose $\hat{y} \in \partial\hat{D}$ with $|y - \hat{y}| = d(y, \partial\hat{D})$. Put $\hat{\rho} = 2|y - \hat{y}|$ and observe that Lemma 3.9 can be used with x, ρ replaced by $\hat{y}, \hat{\rho}$. Doing this and using Harnack's inequality we get claim (3.12). Our goal is to prove a boundary Harnack inequality for L. Using the argument in [HL, ch 2, sec 3] or [BL1, Lemma 2.2], it will turn out that (3.12), (3.3) (c'), and Harnack's inequality are enough to give this inequality.

Lemma 3.13. *Let $w \in \partial\hat{D}, r > 0$. Suppose that $\hat{z} \notin B(w, 8r)$ and either $B(w, 8r) \cap \partial\Omega^- = \emptyset$ or $B(w, 8r) \cap \partial\Omega^+ = \emptyset$. Let h_1, h_2 be positive solutions to L in $\hat{D} \cap B(w, r)$ with $h_i = 0, i = 1, 2$, continuously on $\partial\hat{D} \cap B(w, r)$. There exists $\hat{c}$ (depending only on p, n, β), such that if $\hat{r} = r/\hat{c}$, then*

$$\hat{c}^{-1}\,\frac{h_1(a_{\hat{r}}(w))}{h_2(a_{\hat{r}}(w))} \leq \frac{h_1(y)}{h_2(y)} \leq \hat{c}\,\frac{h_1(a_{\hat{r}}(w))}{h_2(a_{\hat{r}}(w))}$$

whenever $y \in B(w, \hat{r}) \cap \hat{D}$.

Proof: Let $w' \in \hat{D}$ be the point on the ray from $\hat{z}$ through w with $|w - w'| = r/4$. We assume, as we may, that $h_i(w') = 1, i = 1, 2$, since otherwise we divide h_i by these constants for $i = 1, 2$. We note that interior Hölder continuity estimates and Harnack's inequality can be stated for h_1, h_2 as in Lemma 3.5 with $g(\cdot, y)$ replaced by h_1, h_2. From Harnack's inequality and starlike Lipschitzness of $\hat{D}$ we see that in order to prove Lemma 3.13 it suffices to show for some c that

$$c^{-1} \leq h_1(y)/h_2(y) \leq c \text{ for } y \in B(w, r/c). \tag{3.14}$$

To prove (3.14), we note as in (2.46) that if $c' \geq 10^{10}$ is large enough (depending only on p, n, β), then the domain $\hat{\Omega} \subset B(w, r) \cap \hat{D}$, formed by drawing all open line segments from points in $\partial\hat{D} \cap B(w, r/c')$ to points in $B(w', r/c')$ is a starlike Lipschitz domain with center w' and Lipschitz constant $\leq c(\beta + 1)$. Moreover if $\rho = r/(c')^2$, then

$$B(w, 1000\rho) \cap \hat{D} = B(w, 1000\rho) \cap \hat{\Omega}. \tag{3.15}$$

With c' now fixed let g' denote Green's function for $\hat{\Omega}$. If $y \in B(w, \rho/100) \cap \hat{D}$, choose $\tilde{y} \in \partial\hat{D} \cap \partial\hat{\Omega}$ with $|y - \tilde{y}| = d(y, \partial\hat{\Omega})$. This choice is possible thanks to (3.15). Let $\rho_m = 2^{-m}\rho, \hat{\rho}_m = 2^{-1/2}\rho_m$, for $m = 0, 1, 2, \ldots$. and $G_j = \bar{B}(\tilde{y}, \rho_j) \cap \partial\hat{D} \setminus B(\tilde{y}, \rho_{j+1})$ for $j = 0, 1, 2, \ldots$ From (3.15), (3.3) (c'), (3.12) with x replaced by points in $\partial B(\tilde{y}, \hat{\rho}_0) \cap \partial \hat{D}$, Harnack's inequality, and the weak boundary maximum principle for L, we see that

$$c\, h_1(\cdot) \geq [\hat{u}(a_\rho(\tilde{y}))]^{p-2}\, r^{n-p}\, g'(\cdot, a_\rho(\tilde{y})) \text{ and } h_2(\cdot) \leq c\, \hat{\omega}(G_0, \cdot) \tag{3.16}$$

in $[B(\tilde{y}, \rho/2) \setminus B(a_\rho(\tilde{y}), \frac{d[a_\rho(\tilde{y}), \partial\hat{\Omega})]}{1000})] \cap \hat{\Omega}$. From (3.16) we conclude that to prove $h_1(y)/h_2(y) \geq c^{-1}$ it suffices to show that

$$\hat{\omega}(G_0, y) \leq c\ [\hat{u}(a_\rho(\tilde{y}))]^{p-2} r^{n-p} g'(y, a_\rho(\tilde{y})) \tag{3.17}$$

(3.17) is a consequence of the following claim. Given $\epsilon > 0$, sufficiently small (depending on p, n, β) there exists $c(\epsilon)$ (depending only on ϵ, p, n, β) such that

$$\hat{\omega}(G_{j-1}, \cdot)\ \leq\ \epsilon\, \hat{\omega}(G_j, \cdot)\ +\ c(\epsilon)\ [\hat{u}(a_{\rho_j}(\tilde{y}))]^{p-2}\ \rho_j^{n-p}\, g'(\cdot, a_{\rho_j}(\tilde{y})) \text{ in } B(\tilde{y}, \rho_{j+1}) \cap \hat{\Omega} \tag{3.18}$$

whenever $j = 1, 2, \ldots$. To prove (3.18) for a fixed j, let $\epsilon' < 10^{-10}$, and let N be the largest positive integer $\leq\ 1/\epsilon'$. Set

$$E_{ij} = \{\zeta \in G_j : (1 + \tfrac{i-1}{100N})\hat{\rho}_j\ \leq\ |\zeta - \tilde{y}| < (1 + \tfrac{i}{100N})\hat{\rho}_j\} \text{ for } i = 1, 2, \ldots, N.$$

Then

$$\{E_{ij}\}_1^N \text{ are pairwise disjoint, } \bigcup_{i=1}^{N} E_{ij} \subset G_j \text{ and } \hat{\omega}(E_{ij}, \cdot) \geq c^{-1} \tag{3.19}$$

on

$$F_{ij} = \bigcup \{B(\zeta, \tfrac{\epsilon' \rho_j}{10000}) \cap \hat{\Omega} : \zeta \in E_{ij} \text{ and } (1 + \tfrac{(i-3/4)}{100N})\hat{\rho}_j \leq |\zeta - \tilde{y}| < (1 + \tfrac{(i-1/4)}{100N})\hat{\rho}_j\},$$

as we see from (3.12). From (3.19) we find that if

$$z \in H_{ij} = \Big(\hat{\Omega} \cap \partial B[\tilde{y}, (1 + \tfrac{i-1/2}{100N})\hat{\rho}_j]\Big) \setminus F_{ij},$$

then $d(z, \partial\hat{\Omega}) \geq \epsilon' \rho_j / c$. Using this fact, Harnack's inequality, and (3.3) (c') once again we find for some $c(\epsilon')$ that

$$c(\epsilon')\ [\hat{u}(a_{\rho_j}(\tilde{y}))]^{p-2} \rho_j^{n-p}\, g'(\cdot, a_{\rho_j}(\tilde{y})) \geq 1 \text{ on } H_{ij} \tag{3.20}$$

for $1 \leq i \leq N, j = 0, 1, \ldots$ Adding (3.19), (3.20) we get that

$$\hat{\omega}(G_{j-1}, \cdot)\ \leq\ 1\ \leq\ c\, \hat{\omega}(E_{ij}, \cdot)\ +\ c(\epsilon')\, \hat{u}[a_{\rho_j}(\tilde{y})]^{p-2} \rho_j^{n-p}\, g'(\cdot, a_{\rho_j}(\tilde{y})) \tag{3.21}$$

on $\partial B[\tilde{y}, (1 + \frac{i-1/2}{100N})\hat{\rho}_j] \cap \hat{\Omega}$. Using the boundary maximum principle for L we deduce that (3.21) is valid in $B(\tilde{y}, \rho_{j+1}) \cap \hat{\Omega}$. Summing (3.21) for a fixed j from $i = 1$ to N, we obtain from (3.19) (after division by N) that

$$\hat{\omega}(G_{j-1}, \cdot) \leq\ (c/N)\, \hat{\omega}(G_j, \cdot)\ +\ c(\epsilon')\, [\hat{u}(a_{\rho_j}(\tilde{y}))]^{p-2} \rho_j^{n-p}\, g'(\cdot, a_{\rho_j}(\tilde{y})) \tag{3.22}$$

in $B(\tilde{y}, \rho_{j+1}) \cap \hat{\Omega}$. Choose ϵ' so that $c/N < \epsilon$ or equivalently $c\epsilon' = \epsilon$ for some c. Rewriting (3.22) in terms of ϵ we obtain claim (3.18) provided $0 < \epsilon < 1/c$ and c is large enough.

To prove (3.17) let k be the positive integer with $y \in B(\tilde{y}, \rho_{k+1}) \setminus \bar{B}(\tilde{y}, \rho_{k+2})$. Iterating (3.18) from 1 to k we get

$$\hat{\omega}(G_0, y) \leq \epsilon^k \, \hat{\omega}(G_k, y) \, + \, c(\epsilon) \sum_{j=1}^{k} \epsilon^{j-1} \, [\hat{u}(a_{\rho_j}(\tilde{y}))]^{p-2} \, \rho_j^{n-p} \, g'(y, a_{\rho_j}(\tilde{y})) \, . \tag{3.23}$$

Observe from (3.3) (a') that Lemma 3.5 holds with $g(\cdot, y)$ replaced by $g(y, \cdot)$. Thus if

$$y \notin \bigcup_i B[a_{\rho_i}(\tilde{y}), \tfrac{d(a_{\rho_i}(\tilde{y}), \partial\hat{\Omega})}{100}], \tag{3.24}$$

then from Harnack's inequality applied to $g'(y, \cdot), \hat{u}$, we deduce

$$\begin{aligned} &\epsilon^{j-1} \, [\hat{u}(a_{\rho_j}(\tilde{y}))]^{p-2} \, \rho_j^{n-p} \, g'(y, a_{\rho_j}(\tilde{y})) \\ &\qquad \leq \, c \, (1/2)^j \, [\hat{u}(a_\rho(\tilde{y}))]^{p-2} \, \rho^{n-p} \, g'(y, a_\rho(\tilde{y})) \end{aligned} \tag{3.25}$$

for $j = 1, 2, \ldots,$ provided ϵ is sufficiently small (depending only on p, n, β). Likewise

$$\epsilon^k \hat{\omega}(G_k, y) \leq (1/2) \hat{\omega}(G_0, y) \tag{3.26}$$

for small ϵ. Fix $\epsilon > 0$ so that (3.25), (3.26) are true. Using (3.25), (3.26) in (3.23) we obtain after some arithmetic that (3.17) is valid when (3.24) holds. Otherwise we note the existence of y^* satisfying (3.24) (with y replaced by y^*) and with $|y^* - y| \approx d(y^*, \partial\hat{\Omega}) \approx \rho_k$. Then (3.17) is valid for y^* and from Harnack's inequality it follows that (3.17) is valid for y. From the remark above (3.17) we get the lower bound in (3.14). Interchanging h_1, h_2 we also get the upper bound in (3.14). This proves Lemma 3.13. □

We end this section by proving the following lemma.

Lemma 3.27. *Let $w \in \partial\hat{D}, r, \hat{r}$ and h_1, h_2 be as in Lemma 3.13. There exists λ, $0 < \lambda < 1$, and c^* (depending only on p, n, β) such that*

$$\left| \frac{h_1(y)}{h_2(y)} - \frac{h_1(y')}{h_2(y')} \right| \leq c^* \left(\frac{|y - y'|}{r} \right)^\lambda \frac{h_1(a_{\hat{r}}(w))}{h_2(a_{\hat{r}}(w))}$$

whenever $y, y' \in B(w, \hat{r}/4)$.

Proof: Given $\tilde{y} \in \partial\hat{D} \cap B(w, \hat{r}/2)$ set

$$M(s) = M(s, \tilde{y}) = \sup_{B(\tilde{y}, s)} \tfrac{h_1}{h_2}(y)$$

$$m(s) = m(s, \tilde{y}) = \inf_{B(\tilde{y}, s)} \tfrac{h_1}{h_2}(y)$$

when $0 < s < \hat{r}/2$, where $\hat{c}$ is the constant in Lemma 3.13. Then $h_1 - m(s)h_2 \geq 0$ and $M(s)h_2 - h_1 \geq 0$ in $B(\tilde{y}, s) \cap \hat{D}$. Observe from Harnack's inequality that each of these functions is either positive or identically zero in $B(\tilde{y},s)\cap\hat{D}$. If both functions are positive then from Lemma 3.13 with h_1,h_2 replaced by $h_1-m(s)h_2,h_2$, and $h_2M(s)-h_1,h_2$, respectively, we find that if $\hat{c}$ is the constant in Lemma 3.13 and $\hat{s}=s/\hat{c}$, then

$$\begin{aligned} &(\alpha)\quad M(\hat{s})-m(s)\,\le\,\hat{c}\,[m(\hat{s})-m(s)]\\ &(\beta)\quad M(s)-m(\hat{s})\,\le\,\hat{c}\,[M(s)-M(\hat{s})]. \end{aligned} \tag{3.28}$$

If $\psi(t)=\psi(t,\tilde{y})=M(t)-m(t)$, then adding (3.28) $(\alpha),(\beta)$ we get

$$\psi(s)+\psi(\hat{s})\,\le\,\hat{c}[\psi(s)-\psi(\hat{s})]$$

or

$$\psi(\hat{s})\,\le\,\tfrac{\hat{c}-1}{\hat{c}+1}\,\psi(s). \tag{3.29}$$

Clearly this inequality is also valid if either of the above functions vanishes. If $y,y'\in B(w,\hat{r}/4)\cap\hat{D}$ and $|y-y'|>\frac{\hat{r}}{1000}$, then from Lemma 3.13 we see that Lemma 3.27 holds. If $|y-y'|\le\min[\frac{\hat{r}}{1000},d(y,\partial\hat{D})/2]$ we can use Harnack's inequality and interior Hölder continuity estimates for h_1,h_2 to get Lemma 3.27 in this case. Otherwise, choose $\tilde{y}\in\partial\hat{D}\cap B(w,\hat{r}/2)$ with $|y-\tilde{y}|=d(y,\partial\hat{D})$. Iterating (3.29) starting from $s=\hat{r}/4$ and finishing with $s\approx 4|y-y'|$, we deduce for some $\lambda,0<\lambda<1$, and c,

$$\begin{aligned} \left|\tfrac{h_1(y)}{h_2(y)}-\tfrac{h_1(y')}{h_2(y')}\right| &\le\, M(2|y-y'|,\tilde{y})-m(2|y-y'|,\tilde{y})\\ &\le\, c\left(\frac{|y-y'|}{r}\right)^{\lambda}\frac{h_1(a_{\hat{r}}(w))}{h_2(a_{\hat{r}}(w))}\,. \end{aligned}$$

Hence Lemma 3.27 holds in all cases. □

Remark. Let $\hat{D},w,r,\hat{r}$, be as in Lemma 3.27 and suppose that $\hat{\zeta}\in\hat{D}\setminus B(w,8r)$. If h_1,h_2 are positive solutions to L in $\hat{D}\setminus\{\hat{\zeta}\}$, then Lemmas 3.13, 3.27 are valid with $\hat{r}$ replaced by r. Indeed, we can first apply Lemmas 3.13, 3.27, with w,r replaced by $\tilde{y},\hat{s}=r/10$, whenever $\tilde{y}\in\partial\hat{D}\cap B(w,r)$. Using $h_i(a_{\hat{s}}(\tilde{y}))\approx h_i(a_{\hat{r}}(w)),i=1,2$, we then get Lemmas 3.13, 3.27 with $\hat{r}$ replaced by r.

4. Deformation of Ring Domains and Proof of Theorem 1. Let $\Omega_i^-,\Omega_i^+,i=1,2$, be starlike Lipschitz domains with center $\hat{z}$ and $\bar{\Omega}_i^-\subset\Omega_i^+$ for $i=1,2$. Let $R_i^+,R_i^-,i=1,2$, be the corresponding graph functions and let $D_i=\Omega_i^+\setminus\bar{\Omega}_i^-,i=1,2$. For fixed $p,1<p<\infty$, let u,v, be the p capacitary functions for D_1,D_2, respectively. We assume that

$$R_i^+,R_i^-\in C_0^\infty(\partial B(0,1))\text{ for }i=1,2. \tag{4.1}$$

and also that $w\in\partial D_1\cap\partial D_2,r>0$, with

$$\begin{aligned} &(a)\quad B(w,8r)\cap D_1=B(w,8r)\cap D_2\\ &(b)\quad \text{Either }\bar{B}(w,8r)\cap[\Omega_1^-\cup\Omega_2^-]=\emptyset\text{ or }\bar{B}(w,8r)\cap[\mathbb{R}^n\setminus\Omega_1^+\cup\mathbb{R}^n\setminus\Omega_2^+]=\emptyset. \end{aligned} \tag{4.2}$$

We first study properties of the deformation touted above (1.10). To this end we define for $0\le\tau<1$ and $\omega\in\partial B(0,1)$,

$$R^+(\tau,\omega)=[R_2^+(\omega)]^\tau\;[R_1^+(\omega)]^{1-\tau}\,,\;R^-(\tau,\omega)=[R_2^-(\omega)]^\tau\;[R_1^-(\omega)]^{1-\tau}\,.$$

Let $\Omega^+(\tau), \Omega^-(\tau)$, be starlike Lipschitz domains with center $\hat{z}$ and graph functions, $R^+(\tau,\cdot), R^-(\tau,\cdot)$, while $D(\tau) = \Omega^+(\tau) \setminus \bar{\Omega}^-(\tau)$ is the corresponding ring domain. For fixed $p, 1 < p < \infty$, let $u(\cdot,\tau), \tau \in [0,1]$, be the p capacitary function for $D(\tau)$. Extend $u(\cdot,\tau)$ to $\mathbb{R}^n$ by putting $u(\cdot,\tau) \equiv 0$ on $\mathbb{R}^n \setminus \Omega^+(\tau)$ and $u(\cdot,\tau) \equiv 1$ on $\bar{\Omega}^-(\tau)$. From Lemma 2.1 (c) we see that $\{u(\cdot,\tau), 0 \le \tau \le 1\}$ are Hölder continuous on $\mathbb{R}^n$ with exponent independent of $\tau \in [0,1]$. From Lemma 2.5 and the same argument as in the proof of Lemma 2.4, we see that $u(\cdot,\tau), \tau \in [0,1]$, has an infinitely differentiable extension to $\bar{D}(\tau)$. Moreover, a check of the argument shows that there exists $\delta, \epsilon > 0$, for a given nonnegative integer k and fixed $s \in [0,1]$, such that

$$
\begin{aligned}
&(-) \quad \min_{\bar{D}(\tau)} |\nabla u(\cdot,\tau)| > \epsilon \text{ provided } \tau \in (s-\delta, s+\delta),\\
&(--) \quad \text{the } C^k \text{ norm of } u(\cdot,\tau) \text{ in } \bar{D}(\tau) \text{ is uniformly bounded for } \tau \in (s-\delta, s+\delta).
\end{aligned}
\tag{4.3}
$$

Using Ascoli's theorem on $\mathbb{R}^n$, (4.3), and uniqueness of $u(\cdot,\tau)$ we see that if $\partial^l u(\cdot,\tau)$ denotes an arbitrary l th partial derivative of $u(\cdot,\tau)$ in $D(\tau)$ (with respect to x), then for $s \in [0,1]$,

$$
\lim_{\tau \longrightarrow s} \partial^l u(\cdot,\tau) = \partial^l u(\cdot,s) \text{ as } \tau \longrightarrow s, \text{ uniformly on compact subsets of } D(s). \tag{4.4}
$$

Next we consider regularity in τ.

Lemma 4.5. *Let $u(\cdot,\cdot)$ be as above. Then $u_\tau(x,\tau) = \frac{\partial}{\partial\tau} u(x,\tau)$ exists continuously whenever $(x,\tau) \in \bigcup_{t\in[0,1]} D(t) \times \{t\}$. Moreover for $\tau \in [0,1]$, $u_\tau(\cdot,\tau)$ extends continuously to $\bar{D}(\tau)$ and*

(α) *$u_\tau(\cdot,\tau)$ is a solution to (1.7) in $D(\tau)$ with b_{ij} defined relative to $u(\cdot,\tau)$,*

(β) *$u_\tau(y,\tau) = -\langle y-\hat{z}, \nabla u(y,\tau)\rangle \log(R_2'/R_1')(\frac{y-\hat{z}}{|y-\hat{z}|})$ when $y \in \partial D(\tau)$, where $' = +$ if $y \in \partial\Omega^+(\tau)$ and $' = -$ if $y \in \partial\Omega^-(\tau)$.*

(γ) $\log\left(\frac{v(x)-a}{u(x)-a}\right) = \int_0^1 \frac{u_\tau(x,\tau)}{u(x,\tau)-a}\, d\tau$ *whenever $a = 0,1$ and $x \in B(w,8r) \cap D_1$.*

Proof: We assume, as we may, that $\hat{z} = 0$. If $\tau = 0,1$ in Lemma 4.5, then u_τ should be interpreted as a limit of one sided difference quotients. To begin, if $\xi = (\xi_1,\dots,\xi_n), w = (w_1,\dots,w_n) \in \mathbb{R}^n \setminus \{0\}$, and $1 \le i \le n$ we note that

$$
\begin{aligned}
|\xi|^{p-2}\,\xi_i - |w|^{p-2}\,w_i &= \int_0^1 \frac{d}{d\lambda}\{|\lambda\xi + (1-\lambda)w|^{p-2}\,[\lambda\xi_i + (1-\lambda)w_i]\}d\lambda\\
&= \sum_{j=1}^n (\xi - w)_j \left(\int_0^1 a_{ij}[\lambda\xi + (1-\lambda)w]d\lambda\right),
\end{aligned}
$$

where for $1 \le i,j \le n$,

$$
a_{ij}(\eta) = |\eta|^{p-4}\,[(p-2)\eta_i\,\eta_j \;+\; \delta_{ij}\,|\eta|^2] \text{ for } \eta \in \mathbb{R}^n \setminus \{0\}. \tag{4.6}
$$

In this display δ_{ij}, once again, denotes the Kronecker delta. Using (4.3), (4.4), and (1.2), we deduce for given $s \in [0,1]$ that if

$$U(x) = U(x,\tau,s) = \frac{u(x,s) - u(x,\tau)}{s-\tau},$$

$$A_{ij}(x) = A_{ij}(x,\tau,s) = \int_0^1 a_{ij}[\lambda \nabla u(x,s) + (1-\lambda)\nabla u(x,\tau)]\, d\lambda,$$

then, whenever $x \in D(\tau) \cap D(s), \tau \in [s-\delta, s+\delta]$, and $1 \leq i,j \leq n$,

$$\tilde{L}U(x) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}[A_{ij}(x)\, U_{x_j}] = 0 \text{ on } D(\tau) \cap D(s). \tag{4.7}$$

Moreover, if $x \in D(\tau) \cap D(s)$, then it is easily seen that

$$\begin{aligned} &(*) \quad c^{-1}|\xi|^2||\nabla u(x,s)| + |\nabla u(x,\tau)||^{p-2} \leq \sum_{i,j=1}^{n} A_{ij}(x)\xi_i\xi_j \text{ whenever } \xi \in \mathbb{R}^n \setminus \{0\}, \\ &(**) \quad \sum_{i,j=1}^{n} |A_{ij}(x)| \leq c||\nabla u(x,s)| + |\nabla u(x,\tau)||^{p-2}, \end{aligned} \tag{4.8}$$

where c depends only on p, n.

Given $\epsilon_1 > 0$ small, choose $\delta_1, \delta_2 > 0$ so small that

$$\begin{aligned} &|\nabla u(t\omega,\tau) - \nabla u[(1-\delta_1)R^+(s,\omega)\,\omega,\tau]| \\ &\quad + |\nabla u(\hat{t}\omega,\tau) - \nabla u[(1+\delta_1)R^-(s,\omega)\,\omega,\tau]| \leq \epsilon_1 \end{aligned} \tag{4.9}$$

whenever $t \in [(1-\delta_1)R^+(s,\omega), R^+(\tau,\omega)]$, $\hat{t} \in [R^-(\tau,\omega), (1+\delta_1)R^-(s,\omega)]$, and $\omega \in \partial B(0,1), \tau \in (s-\delta_2, s+\delta_2)$. Also, we require

$$|\nabla u(y,s) - \nabla u(y,\tau)| < \epsilon_1 \text{ on } \Gamma_1 \cup \Gamma_2 \text{ if } \tau \in (s-\delta_2, s+\delta_2) \tag{4.10}$$

where

$$\Gamma_1 = \{(1-\delta_1)R^+(s,\omega)\,\omega : \omega \in \partial B(0,1)\}$$

$$\Gamma_2 = \{(1+\delta_1)R^-(s,\omega)\,\omega : \omega \in \partial B(0,1)\}.$$

This choice of $\delta_1, \delta_2 > 0$ is possible thanks to (4.3), (4.4). Let $y \in \partial[D(s) \cap D(\tau)]$, for some $\tau \in (s-\delta_2, s+\delta_2)$ with $y/|y| = \omega$. Let $a_1 = \min\{R'(\tau,\omega), R'(s,\omega)\}$ and $a_2 = \max\{R'(\tau,\omega), R'(s,\omega)\}$ where $' = +$ if y lies in the unbounded component of $D(\tau) \cap D(s)$ while $' = -$ if y is in the bounded component of $D(\tau) \cap D(s)$. We note that

$$U(y,\tau,s) = \pm \int_{a_1}^{a_2} \frac{\partial}{\partial t} U(t\omega,\tau,s)dt$$

where the - sign is taken if $' = +$ and the + sign if $' = -$. Using this note (4.9), (4.10), and the chain rule, we deduce that

$$U(y,\tau,s) = \tfrac{R'(\tau,\omega)-R'(s,\omega)}{s-\tau}\,\{e + [R'(s,\omega)]^{-1}\,\langle\nabla u(y,s)\,,\,y\rangle\} \tag{4.11}$$

where $|e| \leq 3\epsilon_1$. Since

$$R'_s(s,\omega) = [R'(s,\omega)]\,\log(R'_2/R'_1)(\omega) \text{ for } \ ' \in \{+,-\},$$

we conclude from (4.11) that $U(\cdot,\tau,s)$ converges uniformly on $\partial[D(\tau)\cap D(s)]$ to the righthand side of Lemma 4.5 (β) as $\tau{\rightarrow}s$ in the sense that given $\epsilon_2 > 0$, there is a $\delta_3 > 0$ such that

$$|U(y,\tau,s) + \langle y, \nabla u(y,s)\rangle\,\log(R'_2/R'_1)(\tfrac{y}{|y|})| \leq \epsilon_2 \tag{4.12}$$

for all $y \in \partial[D(\tau)\cap D(s)]$ and $\tau \in (s-\delta_3, s+\delta_3)$. Let ψ be a classical solution to

$$L\psi = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}[b_{ij}(x)\frac{\partial\psi}{\partial x_j}] \;=\; 0 \text{ in } D(s) \tag{4.13}$$

with continuous boundary values

$$\psi(y) \;=\; -\langle y, \nabla u(y,s)\rangle\,\log(R'_2/R'_1)(\tfrac{y}{|y|}) \text{ on } \partial D(s). \tag{4.14}$$

Here $b_{ij}(x) = a_{ij}(\nabla u(x,s))$, $1 \leq i,j \leq n, x \in D(s)$, and a_{ij} are as in (4.6). Since (b_{ij}) are uniformly elliptic it follows from the usual minimizing argument that

$$\int_{D(s)} |\nabla\psi|^2\,dx \;\leq\; M < \infty. \tag{4.15}$$

To prove Lemma 4.5 $(\alpha),(\beta)$, we need to show that $\psi = u_s(\cdot,s)$. To this end, we observe from (4.3), (4.4), (4.8) that $A_{ij}(\cdot,s,t), 1 \leq i,j \leq n$, are uniformly elliptic and $C^l, l = 1,2,\ldots$ with ellipticity and norm constants in $D(s)\cap D(\tau)$ that are independent of $\tau \in (s-\delta_3, s+\delta_3)$. Also, from (4.12), (4.7), (4.8), and the boundary maximum principle for $\tilde{L}$ we see that the functions $U(\cdot,\tau,s)$ are uniformly bounded in $D(s)\cap D(\tau), \tau \in (s-\delta_3, s+\delta_3)$.

Using the above facts and Schauder theory, we find that if $\lim_{k\longrightarrow\infty} \tau_k = s$, then a subsequence of $\{U(\cdot,\tau_k,s)\}$ converges in the C^l norm on each compact subset of $D(s)$ for $l = 0,1,2,\ldots$. Let V be such a limit. From these remarks and (4.7), (4.8), we deduce that $LV = 0$ in $D(s)$ where L is as in (4.13). Let

$$\phi_1(x) = \max(U(x,\tau,s) - \psi(x) - 2\epsilon_2, 0)$$

$$\phi_2(x) = \min(U(x,\tau,s) - \psi(x) + 2\epsilon_2, 0)$$

when $x \in D(s)\cap D(\tau)$ and $\tau \in (s-\delta_3, s+\delta_3)$. From (4.12) we see that $\phi_k, k = 1,2$, vanishes continuously on $\partial[D(s)\cap D(\tau)]$. This fact and (4.7) imply

$$\int_{D(s)\cap D(\tau)} \sum_{i,j=1}^{n} A_{ij}\,U_{x_j}\,(\phi_k)_{x_i}\,dx = 0 = \int_{D(s)\cap D(\tau)} \sum_{i,j=1}^{n} b_{ij}\,\psi_{x_j}\,(\phi_k)_{x_i}\,dx$$

for $k = 1, 2$. This equality, (4.3), and (4.8) yield for some $0 < K < \infty$, depending on various smoothness constants but independent of $s, \tau \in (s - \delta_3, s + \delta_3)$, that

$$K^{-1} \int_{D(s) \cap D(\tau) \cap \{|U - \psi| > 2\epsilon_2\}} |\nabla U(\cdot, \tau, s) - \nabla \psi|^2 dx \leq \int_{D(s) \cap D(\tau)} \sum_{i,j=1}^{n} A_{ij} \left(U_{x_j} - \psi_{x_j}\right) (\phi_1 + \phi_2)_{x_i} \, dx$$

$$= \int_{D(s) \cap D(\tau)} \sum_{i,j=1}^{n} (b_{ij} - A_{ij}) \psi_{x_j} \, (\phi_1 + \phi_2)_{x_i} \, dx.$$

Using Cauchy's inequaily with $\epsilon' s$, (4.3), (4.4), (4.8), (4.15), it follows for some K_1 having the same dependence as K that

$$\int_{D(s) \cap D(\tau) \cap \{|U - \psi| > 2\epsilon_2\}} |\nabla U(\cdot, \tau, s) - \nabla \psi|^2 \, dx \leq K_1 \int_{D(s) \cap D(\tau)} \sum_{i,j=1}^{n} |b_{ij} - A_{ij}|^2 \, |\nabla \psi|^2 dx \rightarrow 0$$

as $\tau \rightarrow s$. Using Fatou's Lemma and Poincàre's inequality, we conclude that

$$\int_{D(s) \cap D(\tau) \cap \{|V - \psi| > 2\epsilon_2\}} |V - \psi|^2 \, dx \leq \liminf_{k \longrightarrow \infty} \int_{D(s) \cap D(\tau) \cap \{|U - \psi| > 2\epsilon_2\}} |U(\cdot, \tau_k, s) - \psi|^2 \, dx = 0. \quad (4.16)$$

Since $\epsilon_2 > 0$ is arbitrary we deduce from (4.16) that $\psi \equiv V$. Since every subsequence of $\{U(\cdot, \tau, s)\}$ must converge to ψ we conclude that $u_s \equiv \psi$. Thus Lemma 4.5 $(\alpha), (\beta)$, hold. Continuity of u_s as in Lemma 4.5, is a consequence of (4.3), (4.4), Lemma 4.5 $(\alpha), (\beta)$, and Schauder theory. Finally Lemma 4.5 (γ) follows from continuity of u_s, the fact that $0 < u(\cdot, s) < 1$ in $D(s)$, the fundamental theorem of calculus, and the observation that (4.2) yields, $x \in D(s), s \in [0, 1]$, whenever $x \in B(w, 8r) \cap D_1$. The proof of Lemma 4.5 is now complete. □

We now consider some applications of Lemma 4.5. We show that under certain deformations, a boundary Harnack inequality applies to the integrand in Lemma 4.5 (γ) for each fixed τ. Thus we will essentially obtain Theorem 2 in certain special cases. To prove Theorem 2, we shall deform $\hat{D}_1$ into $\hat{D}_2$ by a series of such deformations, ending in section 5. At the end of this section we pause to deduce Theorem 1 from Lemma 4.34. To begin suppose that $D = \Omega^+ \setminus \Omega^-$ is a starlike Lipschitz ring domain with Lipschitz constant β, center $\hat{z}$, and graph functions R^+, R^-. Let u be the p capacitary function for D. As in Theorem 2 we assume for $w \in \partial D, r > 0$, that

$$\hat{z} \notin B(w, 16r) \text{ and either } \bar{B}(w, 8r) \cap \Omega^- = \emptyset \text{ or } \bar{B}(w, 8r) \cap \mathbb{R}^n \setminus \Omega^+ = \emptyset. \quad (4.17)$$

We prove

Lemma 4.18. *Given $p, 1 < p < \infty$, let D, R^+, R^-, u, be as above. Let $w, r, \hat{z}$ satisfy (4.17) and suppose that $w \in \partial\Omega^+$. If $R^+, R^- \in C^\infty(\partial B(0, 1))$, then there exists a starlike Lipschitz ring domain $D^* = \Omega^+ \setminus B(\hat{z}, \rho)$ with $\rho \approx d(\hat{z}, \partial\Omega^+)$ and the following properties:*

(*i*) $\quad \bar{B}(w,8r)\cap B(\hat{z},2\rho)=\emptyset \quad and \quad B(\hat{z},2\rho)\subset\Omega^+.$

(*j*) *If u^* denotes the p capacitary function for D^* then for some c,*
$$c^{-1}\frac{u^*(a_r(w))}{u(a_r(w))}\leq\frac{u^*(z)}{u(z)}\leq c\frac{u^*(a_r(w))}{u(a_r(w))} \text{ whenever } z\in B(w,r/2)\cap D.$$

(*k*) *Let u^* be as in (j) and $w_1,w_2\in B(w,r/2)\cap D$. Then there exist $c,\lambda, 0<\lambda<1$, such that*
$$\left|\frac{u^*(w_1)/u(w_1)}{u^*(w_2)/u(w_2)}-1\right|+\left|\frac{u(w_1)/u^*(w_1)}{u(w_2)/u^*(w_2)}-1\right|\leq c\left(\frac{|w_1-w_2|}{r}\right)^{\lambda}.$$

Moreover if (i) holds with ρ replaced by $\tilde\rho$ and $\tilde u$ is the p capacitary function for $\tilde D=\Omega^+\setminus B(\hat z,\tilde\rho)$, then $(j),(k)$ are valid with either u,D or u^*,D^* replaced by $\tilde u,\tilde D$.

Proof: In Lemma 4.18, λ, c and the proportionality constant for $\rho/d(\hat z,\partial\Omega^+)$, depends only on p,n,β. In the proof of Lemma 4.18 we assume, as usual, that $\hat z=0$. We consider two cases. First suppose that
$$\bar\Omega^-\cap\partial B(0,d(0,\partial\Omega^+)/4)\neq\emptyset. \tag{4.19}$$
In this case since Ω^- is starlike Lipschitz, we can choose c', depending only on p,n,β, to be the largest number such that if $\rho=d(0,\partial\Omega^+)/c'$, then
$$B(0,2\rho)\subset\Omega^-. \tag{4.20}$$
(i) of Lemma 4.18 follows from (4.17), (4.20). Let $R_2^-(\omega)=\rho\,\omega$ when $\omega\in\partial B(0,1)$ and set $R_i^+=R^+$ for $i=1,2$, $R_1^-=R^-$. Applying the deformation at the beginning of this section, with $D_1=D, D_2=D^*$, we get $u(\cdot,\tau)$, the p capacitary function for $D(\tau)=\Omega^+(\tau)\setminus\bar\Omega^-(\tau)$. From (4.20) and starlike Lipschitzness of Ω^+ we see that $2\leq R^-/\rho\leq c$. This inequality, Lemma 4.5 (β), and Lemma 2.5 (i) imply that $u_\tau\equiv 0$ on $\partial\Omega^+$ and
$$0<-u_\tau\leq -c\,\langle x,\nabla u(x)\rangle \tag{4.21}$$
for $x\in\partial\Omega^-(\tau)$. Since u_τ, $\langle x,\nabla u(x,\tau)\rangle$, both satisfy (1.7), (1.8) with (b_{ij}) defined relative to $u(\cdot,\tau)$ in $D(\tau)$ it follows from the weak maximum principle for this divergence form equation that (4.21) also holds in $D(\tau)$. Let
$$r(\tau)=\sup\{s:\bar B(w,2s)\cap\Omega^-(\tau)=\emptyset\}.$$
From (4.20), (4.17), and the definition of $R(\tau,\cdot)$, we see that r is increasing and continuous on [0,1] with $r(0)\geq 4r$, $r(1)\geq\rho$. Next we note that Lemmas 3.13, 3.27 can be applied with h_1,h_2 replaced by $-u_\tau(\cdot,\tau),u(\cdot,\tau)$ and $\hat D\cap B(w,8r)$ by $D(\tau)\cap B(w,2r(\tau))$. Using this note, the Remark at the end of section 3, and Lemma 4.5 (γ) we get for $w_1,w_2\in B(w,r)$,
$$\begin{aligned}\left|\log\left(\frac{u^*(w_1)}{u(w_1)}\right)-\log\left(\frac{u^*(w_2)}{u(w_2)}\right)\right|&\leq\int_0^1\left|\frac{u_\tau(w_1,\tau)}{u(w_1,\tau)}-\frac{u_\tau(w_2,\tau)}{u(w_2,\tau)}\right|d\tau\\ &\leq c\left(\frac{|w_1-w_2|}{r}\right)^{\lambda}\int_0^1\frac{-u_\tau(a_{r(\tau)}(w),\tau)}{u(a_{r(\tau)}(w),\tau)}\,d\tau,\end{aligned}\tag{4.22}$$

where $a_{r(\tau)}(w)$ is defined relative to $D(\tau)$. If $r(0) \geq \rho/1000$, we deduce from Lemma 2.5 that

$$\frac{-u_\tau(a_{r(\tau)}(w),\tau)}{u(a_{r(\tau)}(w),\tau)} \leq c \tag{4.23}$$

Using (4.23) in (4.22) we find for $w_1, w_2 \in B(w,r)$ that

$$\left|\log\left(\frac{u^*(w_1)}{u(w_1)}\right) - \log\left(\frac{u^*(w_2)}{u(w_2)}\right)\right| \leq c\left(\frac{|w_1-w_2|}{r}\right)^\lambda. \tag{4.24}$$

Choosing $w_1 = a_r(w) \in D \cap B(w,r)$, and exponentiating (4.24) we conclude that (j) of Lemma 4.18 is valid. (k) follows from (j), (4.24), and the fact that $|\log t|/|t-1|$ is bounded below for $t \in (0,c]$ by a constant depending only on p,n,β. If $r(0) < \rho/1000$ we choose, $\tau_1,\tau_2,\dots,\tau_m \in [0,1]$, so that $\tau_i, 1 \leq i \leq m$, is the value in [0,1] with $r(\tau_i) = 2^i r(0), 1 \leq i \leq m$, and $\rho/1000 \leq r(\tau_m) < \rho/500$. Also let $\tau_{m+1} = 1$ and note that $\tau_1 = 0$ by definition. We claim that

$$0 < \tau_{i+1} - \tau_i \leq cr(\tau_i)/\rho \text{ for } 1 \leq i \leq m. \tag{4.25}$$

To prove this claim observe from the definition of $r(\tau)$ that there exists $\omega_i, 1 \leq i \leq m$, such that $\omega_i \in \partial B(0,1)$ and $R^-(\tau_i,\omega_i)\omega_i \in \partial B(w, r(\tau_i))$ for $i = 1,2,\dots m$. From the definition of τ_i we see that

$$R^-(\tau_i,\omega_i)/R^-(\tau_{i+1},\omega_{i+1}) \leq e^{cr(\tau_i)/\rho}. \tag{4.26}$$

On the other hand from the definition of $R^-(\tau,\cdot)$ and the definition of τ_i we deduce that

$$\frac{R^-(\tau_i,\omega_i)}{R^-(\tau_{i+1},\omega_{i+1})} = [R^-(\omega_{i+1})/\rho]^{(\tau_{i+1}-\tau_i)} \cdot [R^-(\omega_{i+1})/R^-(\omega_i)]^{(\tau_i-1)} \geq e^{-cr(\tau_i)/\rho}\ 2^{(\tau_{i+1}-\tau_i)}. \tag{4.27}$$

Combining (4.26), (4.27) and then taking logarithms, we get claim (4.25).

Observe from Lemma 2.5 and (4.21) that

$$\frac{-u_\tau(a_{r(\tau_i)}(w),\tau)}{u(a_{r(\tau_i)(w),\tau})} \leq \frac{c\rho}{r(\tau_i)} \text{ whenever } \tau \in [\tau_i,\tau_{i+1}], 1 \leq i \leq m. \tag{4.28}$$

Applying Lemmas 3.13, 3.27, as above, and using (4.25), (4.28) we find for $w_1, w_2 \in B(w,r) \cap D$ that

$$\begin{aligned}\int_{\tau_i}^{\tau_{i+1}} \left|\frac{u_\tau(w_1,\tau)}{u(w_1,\tau)} - \frac{u_\tau(w_2,\tau)}{u(w_2,\tau)}\right| d\tau &\leq c\left(\frac{|w_1-w_2|}{r(\tau_i)}\right)^\lambda \int_{\tau_i}^{\tau_{i+1}} \frac{-u_\tau(a_{r(\tau_i)}(w),\tau)}{u(a_{r(\tau_i)}(w),\tau)}\, d\tau \\ &\leq c\, 2^{-i\lambda}\left(\frac{|w_1-w_2|}{r(0)}\right)^\lambda \leq c\, 2^{-i\lambda}\left(\frac{|w_1-w_2|}{r}\right)^\lambda.\end{aligned} \tag{4.29}$$

From (4.29) and Lemma 4.5 (γ) we conclude for $w_1, w_2 \in B(w,r)$ that

$$\begin{aligned}\left|\log\left(\frac{u^*(w_1)}{u(w_1)}\right) - \log\left(\frac{u^*(w_2)}{u(w_2)}\right)\right| &\leq \sum_{i=1}^m \int_{\tau_i}^{\tau_{i+1}} \left|\frac{u_\tau(w_1,\tau)}{u(w_1,\tau)} - \frac{u_\tau(w_2,\tau)}{u(w_2,\tau)}\right| d\tau \\ &\leq c[(\sum_{i=1}^{m-1} 2^{-i\lambda}) + 1]\left(\frac{|w_1-w_2|}{r}\right)^\lambda.\end{aligned} \tag{4.30}$$

From (4.30) we see first that (4.24) is valid and second that Lemma 4.18 holds for our choice of ρ when (4.19) is true.

If (4.19) is false, choose t to be the largest number in [0,1] with

$$\{x : u(x) \geq t\} \cap \partial B(0, d(0, \partial\Omega^+)/4) \neq \emptyset. \tag{4.31}$$

Then from Lemma 2.5 we see that $\min(u/t, 1)$ is the p capacitary function for the starlike Lipschitz ring domain, $\tilde{D} = \{x : 0 < u(x) < t\}$ with Lipschitz constant depending only on p, n, β. Let ρ be the largest number so that $B(0, 2\rho) \subset \{x : u(x) > t\}$. We can now repeat our earlier argument with $\tilde{D}, u/t$ replacing u, D. Doing this we get Lemma 4.18 for u/t. Clearly we can replace u/t by u in $(j), (k)$ of this lemma. The proof of the first part of Lemma 4.18 is now complete.

If $\tilde{\rho}$ is as in Lemma 4.18, we can use Lemma 4.5 to deform D^* into $\tilde{D}$. Using the above argument with either $B(\hat{z}, \rho)$ or $B(\hat{z}, \tilde{\rho})$ replacing Ω^- we get Lemma 4.18 for $u^*, \tilde{u}$. Clearly Lemma 4.18 for u, u^* and $u^*, \tilde{u}$ imply Lemma 4.18 for $u, \tilde{u}$. $\Box$

Next we state an analogue of Lemma 4.18 when $w \in \partial\Omega^-$.

Lemma 4.32. *Given $p, 1 < p < \infty$, let D, R^+, R^-, u, be as in Lemma 4.18. Let $w, r, \hat{z}$ satisfy (4.17) and suppose that $w \in \partial\Omega^-$. If $R^+, R^- \in C^\infty(\partial B(0,1))$, then there exists a starlike Lipschitz ring domain $D^* = B(\hat{z}, \hat{\rho}) \setminus \Omega^-$ with $\hat{\rho} \approx d(\hat{z}, D)$ and the following properties:*

(i') $\bar{B}(w, 8r) \cap B(\hat{z}, \hat{\rho}/2) = \emptyset$ *and* $\Omega^- \subset B(\hat{z}, \hat{\rho}/2)$.

(j') *If u^* denotes the p capacitary function for D^* and $u_+ = 1 - u, u_* = 1 - u^*$, then for some c,*
$$c^{-1}\frac{u_*(a_r(w))}{u_+(a_r(w))} \leq \frac{u_*(z)}{u_+(z)} \leq c\frac{u_*(a_r(w))}{u_+(a_r(w))} \text{ whenever } z \in B(w, r/2) \cap D.$$

(k') *Let u_*, u_+, be as in (j') and $w_1, w_2 \in B(w, r/2) \cap D$. Then there exist $c, \lambda, 0 < \lambda < 1$, such that*
$$\left|\frac{u_*(w_1)/u_+(w_1)}{u_*(w_2)/u_+(w_2)} - 1\right| + \left|\frac{u_+(w_1)/u_*(w_1)}{u_+(w_2)/u_*(w_2)} - 1\right| \leq c\left(\frac{|w_1 - w_2|}{r}\right)^\lambda.$$

Moreover if (i') holds with ρ replaced by $\tilde{\rho}$ and $\tilde{u}$ is the p capacitary function for $\tilde{D} = B(\hat{z}, \tilde{\rho}) \setminus \Omega^-$, then $(j'), (k')$ are valid with either u_+, D or u_*, D^* replaced by $1 - \tilde{u}, \tilde{D}$. Constants have the same dependence as in Lemma 4.18.

Proof: Let $s^- = \max_{\partial B(0,1)} R^-$ and $s^+ = \max_{\partial B(0,1)} R^+$. Again one considers two cases. If

$$\bar{B}(\hat{z}, 4s^-) \cap \partial\Omega^+ \neq \emptyset \tag{4.33}$$

put $\hat{\rho} = 2s^+$ and define D^* as in Lemma 4.32. Arguing as in the case when (4.19) of Lemma 4.18 held, we obtain $(i'), (j'), (k')$ of Lemma 4.32. If (4.33) is false we choose t to be the smallest number in [0,1] with $\{x : 0 < u(x) \leq t\} \cap \bar{B}(\hat{z}, 4s^-) \neq \emptyset$. In this case we put $\hat{D} = \{x : t < u(x) < 1\}, \hat{u} = \max\{(u-t)/(1-t), 0\}$ and note from Lemma 2.5 that $\hat{D}$ is starlike Lipschitz with Lipschitz constant depending only on p, n, β. Let $\tilde{s} = \max\{|x - \hat{z}| : x \in \partial\hat{D}\}$ and set $\hat{\rho} = 2\tilde{s}$. Define D^*, u^* as previously relative to $\hat{\rho}$. Proceeding as in the first case with u_+, D, replaced by $1 - \hat{u} = \frac{1-u}{1-t}, \hat{D}$, we get Lemma 4.32 for $1 - \hat{u}, u_*$. Since $(j'), (k')$ are unchanged upon multiplying a given function by a constant, we conclude that the first part of Lemma 4.32 holds in this case also. To prove the second part of Lemma 4.32 we use Lemma 4.5 to deform D^* into $\tilde{D}$ and argue as in the first part of this lemma. $\square$

Lemma 4.34. *Let $\Omega_i^+, i = 1, 2,$ be starlike Lipschitz domains with center $\hat{z}$ and C^∞ graph functions, R_i^+. Let $D_i = \Omega_i^+ \setminus \bar{B}(\hat{z}, \rho_1)$ and suppose for fixed $p, 1 < p < \infty,$ that u_i is the p capacitary function for $D_i, i = 1, 2$. If $w \in \partial\Omega_1^+ \cap \partial\Omega_2^+, r, \rho_1 > 0,$ assume that*

$$B(w, 8r) \cap \Omega_1^+ = B(w, 8r) \cap \Omega_2^+, \quad B(w, 8r) \cap B(\hat{z}, 2\rho_1) = \emptyset, \quad B(\hat{z}, 2\rho_1) \subset \Omega_i^+ \text{ for } i = 1, 2.$$

Then $(j), (k)$ of Lemma 4.18 are valid with u, u^* replaced by u_1, u_2. c, λ depend only on p, n and the Lipschitz constants for $D_i, i = 1, 2$.

Proof: As usual we assume that $\hat{z} = 0$. We employ the same strategy as in the proof of Lemma 4.18. Applying the deformation at the beginning of section 4, we get $u(\cdot, \tau), \tau \in (0, 1)$, satisfying Lemma 4.5. For fixed $\tau \in [0, 1]$ let $g(\cdot, y)$ be Green's function for $D(\tau)$ and the operator L in (1.7) with pole at $y \in D(\tau)$. Here (b_{ij}) are defined relative to $u(\cdot, \tau)$. Then $u(\cdot, \tau), g(\cdot, y)$ have C^∞ extensions to $\partial D(\tau)$ and since $\nabla g, \nabla u$ are both parallel to the inner normal at a given point of $\partial D(\tau)$, we have

$$\sum_{i,j=1}^{n} b_{ij} g_{x_i} g_{x_j} = (p-1)\, |\nabla u|^{p-2}\, |\nabla g(\cdot, y)|^2 \text{ on } \partial D(\tau). \tag{4.35}$$

Let

$$E_j = \partial\Omega^+(\tau) \cap [B(w, 2^j r) \setminus B(w, 2^{j-1} r)] \text{ for } j = 4, 5, \ldots, m,$$

where m is the first positive integer with $E_j = \emptyset$ for $j > m$. If $y \in D(\tau)$, let

$$I_j(y) = -(p-1) \int_{E_j} \langle x, \nabla u(x, \tau)\rangle \, \log(R_2^+/R_1^+)(\tfrac{x}{|x|})\, |\nabla u(x, \tau)|^{p-2}\, |\nabla g(x, y)|\, dH^{n-1}(x).$$

From (4.35), (3.6), and Lemma 4.5 we see that

$$u_\tau(y, \tau) = \sum_{j=4}^{m} I_j(y) \text{ whenever } y \in D(\tau). \tag{4.36}$$

Let β_i be the Lipschitz constant for $D_i, i = 1, 2$, and set $\hat{r} = \sup\{s : B(w, 2s) \cap \bar{B}(0, \rho_1) = \emptyset\}$. We note that

$$\hat{r} \approx d(0, \partial\Omega^+(\tau)) \approx \text{ diam } D(\tau) \tag{4.37}$$

with proportionality constants depending only on β_1, β_2, p, n. From (4.37) and Lipschitzness of $\log R_i^+, i = 1, 2$, we find that

$$|\log(R_2^+/R_1^+)| \le c\, 2^j r/\hat{r} \text{ on } E_j, 4 \le j \le m. \tag{4.38}$$

Let k be the largest positive integer with $E_k \subset B(w, \hat{r})$. From Lemmas 3.13, 3.27, and (3.3) (c'), we observe for $\zeta \in D(\tau)$ near a point of $E_j, 4 \le j \le k$, and $w_1, w_2 \in B(w, r) \cap D_1$, as in the Remark at the end of section 3, that

$$\left|\frac{g(\zeta,w_1)}{u(\zeta,\tau)\,u(w_1,\tau)}-\frac{g(\zeta,w_2)}{u(\zeta,\tau)\,u(w_2,\tau)}\right| \leq c\left(\frac{|w_1-w_2|}{2^j\,r}\right)^{\lambda}\frac{g(\zeta,a_{2^jr}(w))}{u(\zeta,\tau)\,u(a_{2^jr}(w),\tau)}$$
$$\leq c\left(\frac{|w_1-w_2|}{2^jr}\right)^{\lambda}(2^{j(p-n)}\,r^{p-n})\,u(a_{2^jr}(w),\tau)^{-p}\,. \tag{4.39}$$

A similar argument yields (4.39) with $a_{2^jr}(w)$ replaced by $a_{\hat r}(w)$ when $k<j\leq m$. Letting $\zeta\to x\in E_j$ we obtain from (4.39) and smoothness of $u(\cdot,\tau), g$ that

$$\left|\frac{|\nabla g(x,w_1)|}{|\nabla u(x,\tau)|\,u(w_1,\tau)}-\frac{|\nabla g(x,w_2)|}{|\nabla u(x,\tau)|\,u(w_2,\tau)}\right|$$
$$\leq c\left(\frac{|w_1-w_2|}{2^jr}\right)^{\lambda}(2^jr)^{p-n}\,u(a_{2^jr}(w),\tau)^{-p} \tag{4.40}$$

when $4\leq j\leq k$. If $k<j\leq m$, we replace $a_{2^jr}(w)$ by $a_{\hat r}(w)$ in (4.40). We can now use (4.40), (4.38), (4.37) to estimate I_j. If $4\leq j\leq k$ and $w_1,w_2\in B(w,r)\cap D$ we find

$$\left|\frac{I_j(w_1)}{u(w_1,\tau)}-\frac{I_j(w_2)}{u(w_2,\tau)}\right|$$
$$\leq c\,2^jr\int_{E_j}|\nabla u(x,\tau)|^p\left|\frac{|\nabla g(x,w_1)|}{|\nabla u(x,\tau)|\,u(w_1,\tau)}-\frac{|\nabla g(x,w_2)|}{|\nabla u(x,\tau)|\,u(w_2,\tau)}\right|dH^{n-1}(x) \tag{4.41}$$
$$\leq c\left(\int_{E_j}|\nabla u|^pdx\right)\left(\frac{|w_1-w_2|}{2^jr}\right)^{\lambda}(2^{j(p+1-n)}\,r^{p+1-n})\,u(a_{2^jr}(w),\tau)^{-p}\,.$$

From Lemma 2.39 and Lemma 2.35 (b) we deduce for $1\leq j\leq k$ that

$$\int_{E_j}|\nabla u|^p\,dH^{n-1}\;\leq\;c\,u(a_{2^jr}(w),\tau))^p\,(2^jr)^{n-1-p}. \tag{4.42}$$

Putting (4.42) in (4.41) it follows that

$$\left|\frac{I_j(w_1)}{u(w_1,\tau)}-\frac{I_j(w_2)}{u(w_2,\tau)}\right|\leq c2^{-j\lambda}\left(\frac{|w_1-w_2|}{r}\right)^{\lambda}. \tag{4.43}$$

A similar argument yields that (4.43) also holds for $k<j\leq m$. Note that $m-k\leq c$, thanks to (4.37). From this note, (4.36), (4.43) we see that

$$\left|\frac{u_\tau(w_1,\tau)}{u(w_1,\tau)}-\frac{u_\tau(w_2,\tau)}{u(w_2,\tau)}\right|\leq\sum_{j=4}^{m}\left|\frac{I_j(w_1)}{u(w_1,\tau)}-\frac{I_j(w_2)}{u(w_2,\tau)}\right|\leq c\left(\frac{|w_1-w_2|}{r}\right)^{\lambda}. \tag{4.44}$$

Using (4.44) and (4.5) (γ) as in (4.22) we conclude that for $w_1,w_2\in B(w,r)\cap D$,

$$\left|\log\left(\frac{u_2(w_1)}{u_1(w_1)}\right)-\log\left(\frac{u_2(w_2)}{u_1(w_2)}\right)\right|\leq c\Big(\frac{|w_1-w_2|}{r}\Big)^{\lambda}. \tag{4.45}$$

(4.45) is easily seen to imply Lemma 4.34 (see the discussion after (4.24)). $\Box$

Next we state a version of Lemma 4.34 when $w \in \partial\Omega_1^- \cap \partial\Omega_2^-$.

Lemma 4.46. *Let $\Omega_i^-, i = 1, 2$, be starlike Lipschitz ring domains with center $\hat z$ and C^∞ graph functions, R_i^-. Let $D_i = B(0, \rho_1) \setminus \bar\Omega_i^-$ and suppose for fixed p, $1 < p < \infty$, that u_i is the p capacitary function for $D_i, i = 1, 2$. If $w \in \partial\Omega_1^- \cap \partial\Omega_2^-, r, \rho_1 > 0$, assume that*

$$B(w, 8r) \cap \Omega_1^- = B(w, 8r) \cap \Omega_2^-, \quad \hat z \notin B(w, 16r), \quad \Omega_i^- \subset B(\hat z, \rho_1/2), i = 1, 2.$$

Then $(j'), (k')$ of Lemma 4.32 are valid with u_+, u_*, replaced by $1 - u_1, 1 - u_2$. c, λ depend only on p, n and the Lipschitz constants for $D_i, i = 1, 2$.

Proof: The proof of Lemma 4.46 is essentially identical to the proof of Lemma 4.34. We omit the details. $\Box$

Proof of Theorem 1. We now use Lemma 4.34 to prove Theorem 1. First we claim that

$$\text{Lemma 4.34 remains true without the assumption that } R_i^+ \in C^\infty(\partial B(0, 1)), i = 1, 2. \tag{4.47}$$

Indeed, let $\{R_{i,m}^+\}, i = 1, 2, m = 1, 2, \ldots$ be a sequence of $C^\infty(\partial B(0, 1))$ real valued functions with

$$\begin{aligned} &(a) \quad \|R_{i,m}^+\hat{\|}_{\partial B(0,1)} \leq c\beta_i \text{ for } i = 1, 2, m = 1, 2, \ldots \\ &(b) \quad R_{i,m}^+ \to R_i^+ \text{ uniformly on } \partial B(0, 1). \\ &(c) \quad R_{1,m}^+ = R_{2,m}^+ \text{ on } \{\tfrac{x - \hat z}{|x - \hat z|} : x \in B(w, 8r - 1/m)\} \cap \partial D. \end{aligned} \tag{4.48}$$

In (4.48) $(a), c$ depends only on n. Existence of $\{R_{i,m}^+\}$ satisfying (4.48) is easily shown. For example extend $R_i^+, i = 1, 2$ to $\mathbb{R}^n \setminus \{\hat z\}$ by putting $R_i^+(x) = R_i^+(\frac{x - \hat z}{|x - \hat z|})$ when $x \in \mathbb{R}^n \setminus \{\hat z\}$. One can take $\log R_{i,m}^+ = \log R_i^+ * \theta_{\epsilon_m}$ for certain $\epsilon_m > 0, m = 1, 2, \ldots,$ with $\lim_{m \to \infty} \epsilon_m = 0$. Here $\theta \in C_0^\infty[B(0, 1)]$ with $\int_{\mathbb{R}^n} \theta \, dx = 1$ and $\theta_\epsilon(x) = \epsilon^{-n} \theta(x/\epsilon), x \in \mathbb{R}^n$. Also, $*$ denotes convolution on $\mathbb{R}^n$. Let $\Omega_{i,m}^+$ be the starlike Lipschitz domain with center $\hat z$ and graph function $R_{i,m}^+$ for $i = 1, 2, m = 1, 2, \ldots$ Put $D_{i,m} = \Omega_{i,m}^+ \setminus B(\hat z, \rho_1)$ and let $u_{i,m}$ be the p capacitary functions for $D_{i,m}$. From (4.48) and Lemma 2.2 we see that $u_{i,m}, D_{i,m}$ converge to u_i, D_i in Hölder and Hausdorff distance norms, as $m \to \infty$. We apply Lemma 4.34 with $u_i, D_i, 8r$ replaced by $u_{i,m}, D_{i,m}, 8r - 1/m$. Since the constants in Lemma 4.34 depend only on p, n and the Lipschitz constants for $D_{i,m}$ we can let $m \to \infty$ and conclude that (4.47) is true.

Let $\tilde u, \tilde v, r, w, \phi, G$ be as in Theorem 1 and let $\tilde w = w + \frac{r}{4} e_n$. As in (2.46), observe that if c' is large enough (depending on p, n and the Lipschitz norm of ϕ), then the domain $\Omega \subset G \cap B(w, r)$ obtained from drawing all open line segments from points in $B(w, r/c') \cap \partial G$ to points in $B(\tilde w, r/c')$ is Lipschitz starlike with center $\tilde w$ and Lipschitz constant $\leq c(\||\nabla\phi|\|_\infty + 1)$, where c depends only on n. Let $r' = \frac{r}{4c'}$ and let u be the p capacitary function for $D = \Omega \setminus \bar B(\tilde w, r'/4)$. Since $\tilde u(\tilde w) = 1 = \tilde v(\tilde w)$, it follows from Harnack's inequality and the weak maximum principle for the p Laplacian that

$$c \min(\tilde u, \tilde v) \geq u \text{ in } D. \tag{4.49}$$

Let

$$R(\omega) = |y - \tilde{w}| \text{ when } \omega = \tfrac{y-\tilde{w}}{|y-\tilde{w}|} \text{ and } y \in B(w, r') \cap \partial G,$$

$$K_i = \{ \tfrac{y-\tilde{w}}{|y-\tilde{w}|} : y \in B(w, 2^{-i} r') \} \text{ for } i = 0, 1, 2,$$

$$L = \sup_{K_0} R\,.$$

From our construction, we observe for some c (depending on p, n, and the Lipschitz constant for ϕ) that

$$\min\{d(K_2, \partial B(0,1) \setminus K_1), d(K_1, \partial B(0,1) \setminus K_0)\} \geq c^{-1}. \tag{4.50}$$

Let $0 \leq \alpha \leq 1$, $\alpha \in C_0^\infty(\mathbb{R}^n)$, with $\alpha \equiv 1$ on K_2 and $\alpha \equiv 0$ on $\partial B(0,1) \setminus K_1$. We can choose α so that

$$|\nabla \alpha| \leq c^{-1} \tag{4.51}$$

thanks to (4.50). Let

$$\log R'(\omega) = \begin{cases} \alpha \log R + (1-\alpha) \log(2L) \text{ when } \omega \in K_0 \\ \log(2L) \text{ when } \omega \in \partial B(0,1) \setminus K_0 \end{cases}$$

Using (4.51) it is easily shown that

$$\| \log R' \hat{\|}_{\partial B(0,1)} \leq c\,(\| \log R \hat{\|}_{\partial K_0} + 1).$$

Let Ω' be the starlike Lipschitz domain with center at $\tilde{w}$ and graph function R'. Let $D' = \Omega' \setminus B(\tilde{w}, r'/4)$ and let u' be the p capacitary function for D'. Then from our construction, the fact that $L \geq cr'$, Lemma 2.2, and Harnack's inequality we deduce first that $c\,u' \geq \max(\tilde{u}, \tilde{v})$ on $G \cap \partial B(w, 3r'/4)$ and second from the weak maximum principle that $c\,u' \geq \max(\tilde{u}, \tilde{v})$ in $G \cap B(w, 3r'/4)$. Using this inequality and (4.49) we conclude that to prove Theorem 1 it suffices to prove this theorem for u, u'. Now from our construction it is easily checked that the hypotheses of Lemma 4.34 are satisfied with u_1, u_2, r replaced by $u, u', r'/100$ so Theorem 1 holds for u, u'. From our earlier reasoning we conclude that Theorem 1 is true. □

5. Proof of Theorems 2, 3, and 4. Let $\hat{D}_i, i = 1, 2$, be as in Theorem 2 with centers $\hat{x}, \hat{y}$ and corresponding p capacitary functions $\hat{u}_i, i = 1, 2$. We note that Lemmas 4.18, 4.34 imply the validity of Theorem 2 when $w \in \partial \hat{D}_i, i = 1, 2$, lies in the unbounded components of $\mathbb{R}^n \setminus \hat{D}_i, i = 1, 2$, and $\hat{x} = \hat{y}$ (i.e, $\hat{D}_i, i = 1, 2$ have the same center). Moreover, Lemmas 4.32, 4.46 imply the validity of Theorem 2 when $w \in \partial \hat{D}_i$ lies in the bounded components of $\mathbb{R}^n \setminus \hat{D}_i, i = 1, 2$, and $\hat{x} = \hat{y}$. Indeed, the C^∞ smoothness assumption on the various graph functions in the above lemmas can be done away with as in (4.48). Also, if w lies in the unbounded components of $\mathbb{R}^n \setminus \hat{D}_i, i = 1, 2$, and $\hat{x} = \hat{y}$, then we can first use Lemma 4.18 to reduce the proof of Theorem 2 to the case when $\hat{D}_i = \hat{\Omega}_i^+ \setminus \bar{B}(\hat{x}, \rho_1)$, $i = 1, 2$, where $\partial \hat{\Omega}_i^+$ is the outer boundary of $\hat{D}_i$. Applying Lemma 4.34 we then get Theorem 2. A similar argument holds when w lies in the bounded components of $\mathbb{R}^n \setminus \hat{D}_i, i = 1, 2$, and $\hat{x} = \hat{y}$. Thus

$$\text{Theorem 2 is valid when } \hat{D}_1, \hat{D}_2 \text{ have the same center.} \tag{5.1}$$

To complete the proof of Theorem 2 we need to consider the case when $\hat{x} \neq \hat{y}$. To this end recall that $w \in \partial \hat{D}_1 \cap \partial \hat{D}_2$ with

$$B(w, 2r) \cap \hat{D}_1 = B(w, 2r) \cap \hat{D}_2. \tag{5.2}$$

Also, $B(w, 8r)$ does not contain any points in either both bounded components or both unbounded components of $\mathbb{R}^n \setminus \hat{D}_i, i = 1, 2$. Once again we consider two cases. If w lies in the unbounded component of $\mathbb{R}^n \setminus \hat{D}_i$ for $i = 1, 2$, let $\hat{R}_i^+$ be the corresponding graph functions for the outer boundary of $\hat{D}_i, i = 1, 2$. Put $\hat{\Omega}_1^+ = \{\hat{x} + t\hat{R}_1^+(\omega)\,\omega : \omega \in \partial B(0,1), 0 \leq t < 1\}$ and $\hat{\Omega}_2^+ = \{\hat{y} + t\hat{R}_2^+(\omega)\,\omega : \omega \in \partial B(0,1), 0 \leq t < 1\}$. From Lemma 4.18 we see that since $w \in \partial\hat{\Omega}_1^+ \cap \partial\hat{\Omega}_2^+$ we may also assume that

$$\hat{D}_1 = \hat{\Omega}_1^+ \setminus \bar{B}(\hat{x}, \rho_1), \hat{D}_2 = \hat{\Omega}_2^+ \setminus \bar{B}(\hat{y}, \rho_2), \text{ where } \rho_1 \approx d(\hat{x}, \partial\hat{\Omega}_1^+), \rho_2 \approx d(\hat{y}, \partial\hat{\Omega}_2^+). \tag{5.3}$$

Moreover,

$$\bar{B}(w, 8r) \cap [B(\hat{x}, 2\rho_1) \cup B(\hat{y}, 2\rho_2)] = \emptyset. \tag{5.4}$$

Given $z \in B(w, r) \cap \partial\hat{D}_i, i = 1, 2$, we claim that it suffices to show there exists c, depending only on p, n, and the Lipschitz constants for $\hat{\Omega}_i^+, i = 1, 2$, such that if $z \in B(w, r) \cap \partial\hat{\Omega}_i^+, i = 1, 2$, then

$$\text{the conclusion of Theorem 2 holds with } w, r \text{ replaced by } z, r/c. \tag{5.5}$$

In fact assuming (5.5) one can cover $B(w, r) \cap \partial\hat{D}_i, i = 1, 2$, by at most c^+ such balls to get Theorem 2 near $B(w, r) \cap \partial\hat{D}_1$. One can then use interior Hölder continuity and Harnack estimates for $\hat{u}_1, \hat{u}_2$ (see Lemma 2.1) to get Theorem 2 for the rest of $B(w, r) \cap \hat{D}_1$. Hence we prove only (5.5). Let $z \in B(w, r) \cap \partial\hat{\Omega}_i^+, i = 1, 2$ and let l denote the open line segment drawn from $\hat{x}$ to z. Let β_1 be the Lipschitz constant for $\hat{D}_1$. If $\zeta \in l$, let $\Omega = \Omega(\zeta)$, be the domain obtained from drawing all open line segments connecting points in $B(\zeta, |\zeta - z|/c')$ to points in $B(z, |\zeta - z|/c') \cap \partial\hat{\Omega}_1^+$. If c' is large enough (depending only on p, n, β_1) and $|\zeta - z| \geq r/8$, it follows from the same reasoning as in (2.46) that Ω is starlike Lipschitz with center at ζ, constant $\leq c(\beta_1 + 1)$, and $\Omega \subset \hat{D}_1$. Let $D(\zeta) = \Omega(\zeta) \setminus B(\zeta, \frac{1}{2}|\zeta - z|/c')$. We note that if $\xi \in B(\zeta, \frac{1}{4}|\zeta - z|/c')$, then

$$D(\zeta) \text{ is also a starlike Lipschitz ring with center at } \xi \text{ and constant } \leq c(\beta_1 + 1). \tag{5.6}$$

Let $\zeta_1 \in l$ be such that $|\zeta_1 - z| = r/8$. From our construction it is easily seen that there exists $0 < \theta < 1/2$ (depending only on p, n, β_1) and a sequence $\{\zeta_i\}_1^N$ with

$$\begin{aligned} &(a) \qquad \frac{|\zeta_{i+1} - z|}{|\zeta_i - z|} \geq 1 + \theta,\ 1 \leq i \leq N - 1, \\ &(b) \qquad \zeta_{i+1} \in B(\zeta_i, \tfrac{1}{4}|\zeta_i - z|/c') \text{ for } 1 \leq i \leq N - 1 \text{ and } \hat{x} \in B(\zeta_N, \tfrac{1}{4}|\zeta_N - z|/c'). \end{aligned} \tag{5.7}$$

Let $v_i, i = 1, 2, \dots, N$ be the p capacitary function for $D(\zeta_i)$. From (5.6) we see that $D(\zeta_i), D(\zeta_{i+1})$ are both starlike Lipschitz with respect to ζ_{i+1}. Also $v_1 \approx v_2 \approx 1$ at a point in $D(\zeta_i) \cap D(\zeta_{i+1})$ with distance $\approx |\zeta_i - z|$ from $\partial\hat{D}_i, i = 1, 2$. From these facts, (5.1), (5.6), and our construction we see that Theorem 2 can be applied with $\hat{u}_1, \hat{u}_2, r, w$ replaced by $v_i, v_{i+1}, \frac{|\zeta_i - z|}{100c'}, z$ to get first, for $c \geq 10^6\, c'$, that $v_j/v_k \leq c$ on $B(z, |\zeta_i - z|/c) \cap \hat{D}_1$ whenever $j, k \in \{i, i+1\}$ and thereupon, from (5.7), that

$$\begin{aligned} &\left| \log\left(\frac{v_{i+1}(w_1)}{v_i(w_1)}\right) - \log\left(\frac{v_{i+1}(w_2)}{v_i(w_2)}\right) \right| \\ &\leq c\left(\frac{|w_1 - w_2|}{|\zeta_i - z|}\right)^{\lambda} \leq c(1+\theta)^{-i\lambda}\left(\frac{|w_1 - w_2|}{r}\right)^{\lambda} \end{aligned} \tag{5.8}$$

whenever $w_1, w_2 \in B(z, r/c) \cap \hat{D}_1$. Summing this inequality we deduce that

$$
\begin{aligned}
&\left|\log\left(\frac{v_N(w_1)}{v_1(w_1)}\right) - \log\left(\frac{v_N(w_2)}{v_1(w_2)}\right)\right| \\
&\quad \le \sum_{i=1}^{N-1}\left|\log\left(\frac{v_{i+1}(w_1)}{v_i(w_1)}\right) - \log\left(\frac{v_{i+1}(w_2)}{v_i(w_2)}\right)\right| \le c\Big(\frac{|w_1 - w_2|}{r}\Big)^{\lambda}.
\end{aligned} \tag{5.9}
$$

In view of (5.1), (5.6), (5.7), we see that Theorem 2 can also be applied to $v_N, \hat{u}_1$. Doing this and using (5.9) we get

$$
\left|\log\left(\frac{\hat{u}_1(w_1)}{v_1(w_1)}\right) - \log\left(\frac{\hat{u}_1(w_2)}{v_1(w_2)}\right)\right| \le c\Big(\frac{|w_1 - w_2|}{r}\Big)^{\lambda} \tag{5.10}
$$

whenever $w_1, w_2 \in B(z, r/c) \cap \hat{D}_1$ and c is large enough, depending only on p, n, β_1.

Let β_2 be the Lipschitz constant for $\hat{D}_2$ and let η be a point in $\hat{D}_2$ on the line segment from $\hat{y}$ to z with $|\eta - z| = r/8$. Let $H(\eta)$ be the domain obtained from drawing all infinite rays with tips in $B(\eta, |\eta - z|/c')$ through points in $B(z, |\eta - z|/c') \cap \partial\hat{\Omega}_2^+$. Let $\Omega(\eta) = H(\eta) \cap \hat{D}_2$. If c' is large enough, it follows as in (2.46) or the construction of $\Omega(\zeta)$, that $\Omega(\eta)$ is the union of all open line segments joining points in $B(\eta, |\eta - z|/c')$ to points in $B(z, |\eta - z|/c') \cap \partial\hat{\Omega}_2^+$. Thus $\Omega(\eta)$ is starlike with respect to centers in $\bar{B}(\eta, \frac{1}{16}r/c')$ and starlike Lipschitz with respect to points in $B(\eta, \frac{1}{32}r/c')$. Again c' and the Lipschitz constant can be chosen to depend only on p, n, β_2. Let $D(\eta) = \Omega(\eta) \setminus B(\eta, \frac{1}{16}r/c')$ and let v be the p capacitary function for $D(\eta)$. Arguing as in the proof of (5.10) we deduce that

$$
\left|\log\left(\frac{\hat{u}_2(w_1)}{v(w_1)}\right) - \log\left(\frac{\hat{u}_2(w_2)}{v(w_2)}\right)\right| \le c\Big(\frac{|w_1 - w_2|}{r}\Big)^{\lambda} \tag{5.11}
$$

whenever $w_1, w_2 \in B(z, r/c) \cap \hat{D}_1$ and c is large enough, depending only on p, n, β_2. With c' now fixed we draw the closed line segment l' connecting η to ζ_1. If $y \in l'$ let $\Omega'(y)$ be the union of all open line segments connecting points in $B(y, \frac{1}{8}r/c')$ to points in $B(z, \frac{1}{8}r/c') \cap \partial\hat{D}_i, i = 1, 2$. We claim that

$$
G = \bigcup_{y \in l'} \Omega'(y) \subset \mathbb{R}^n \setminus [B(z, \tfrac{1}{8}r/c') \cap \partial\Omega_2^+] \tag{5.12}
$$

so that $\Omega(y), y \in l'$, is starlike Lipschitz with respect to any point in $B(y, \frac{1}{32}r/c')$. Indeed, otherwise we first observe that

$$
L = \bigcup_{y' \in l'} B(y', \tfrac{1}{8}r/c') \subset \mathbb{R}^n \setminus [B(z, \tfrac{1}{8}r/c') \cap \partial\Omega_2^+]
$$

since a ray drawn from a point in $B(\eta, \frac{1}{8}r/c')$ through a point in $B(z, \frac{1}{8}r/c') \cap \partial\Omega_2^+$ can never reenter $\hat{D}_2$ once it leaves $\hat{D}_2$ and $B(\zeta_1, \frac{1}{8}r/c') \subset D_2$. Using this observation, we see that if (5.12) is false, then there are points P, Q, R with $P \in L, Q, R \in B(z, \frac{1}{8}r/c') \cap \partial\hat{\Omega}_i^+, i = 1, 2$, such that the line segment l_1 from P to Q contains R. Choose $P_1, \in \Omega(\zeta_1)$ and $P_2 \in \Omega(\eta)$ such that the line segment, l'', from P_1 to P_2 is parallel to l' and contains P. Let l_2 be the line segment from P_1 to Q. Then P_1, P_2, P, Q, R all lie in a two dimensional plane and P_2 lies outside the triangle with vertices P_1, P, R. Draw the infinite ray, P_2 through R. Let σ denote the part of this ray connecting R to ∞.

Then by our choice of $\Omega(\eta)$, we must have $\sigma \subset \mathbb{R}^n \setminus \hat{D}_2$. However clearly σ must intersect the open line segment l^* from P_1 to Q and

$$l^* \subset \Omega(\zeta_1) \subset \hat{D}_1 \cap B(w, 2r) = \hat{D}_2 \cap B(w, 2r)$$

thanks to our construction of $\Omega(\zeta_1)$ and (5.2). We have reached a contradiction. Thus claim (5.12) is true. From (5.12) we that $D(y) = \Omega(y) \setminus B(y, \frac{1}{16} r/c')$ is starlike Lipschitz with center at ξ and constant depending only on p, n, β_1, β_2, whenever $\xi \in B(y, \frac{1}{32} r/c')$ and $y \in l'$. We can now use Theorem 2 at most c times, as in the proof of (5.9) in order to conclude that

$$\left| \log \left(\frac{v_1(w_1)}{v(w_1)} \right) - \log \left(\frac{v_1(w_2)}{v(w_2)} \right) \right| \leq c \left(\frac{|w_1 - w_2|}{r} \right)^{\lambda} \tag{5.13}$$

provided $w_1, w_2 \in B(z, r/c) \cap \hat{D}_1$ and c is large enough (again depending only on p, n, β_1, β_2). Combining (5.10), (5.11), (5.13), we deduce first that (5.5) is true and thereupon, from our earlier remarks, that Theorem 2 is true when w lies in the unbounded components of $\mathbb{R}^n \setminus \hat{D}_i, i = 1, 2$.

Using essentially the same argument we can deduce Theorem 2 when w lies in the bounded components of $\mathbb{R}^n \setminus \hat{D}_i, i = 1, 2$. One small change in the proof is to choose c' so large that rays drawn from $B(\eta, \frac{1}{8} r/c')$ through points in $B(z, \frac{1}{8} r/c') \cap \partial \hat{D}_2$ cannot reenter $\hat{D}_2$ in $B(w, 2r)$. Then $\Omega(\eta) = B(w, 2r) \cap H(\eta) \cap \hat{D}_2$ is the union of line segments joining points in $B(\eta, \frac{1}{8} r/c')$ with $B(w, r/c') \cap \partial \hat{D}_2$ and the proof of (5.13) is unchanged. We omit the rest of the details. The proof of Theorem 2 is now complete. □

Proof of Theorem 3. Let D be the starlike Lipschitz ring domain in Theorem 3 with center $\hat{x}$ and p capacitary function u. Let $\mu = \mu^+ - \mu^-$ be the measure corresponding to u as in Lemma 2.35. Let $w \in \partial D, r$ be as in (1.12) and recall from Lemma 2.39 that $d\mu/dH^{n-1} = \pm k^{p-1}$ on $B(w, 8r) \cap D$ where the + sign is taken if w is in the outer boundary of D while the - sign is taken if w lies in the inner boundary of D. Moreover, applying Lemma 2.39 with w, r replaced by $y \in B(w, 8r) \cap \partial D$ and s so that $B(y, 8s) \subset B(w, 8r)$, we deduce

$$\int_{B(y,s) \cap \partial D} k^p \, dH^{n-1} \leq c \, s^{-\frac{n-1}{p-1}} \left(\int_{B(y,s) \cap \partial D} k^{p-1} \, dH^{n-1} \right)^{p/(p-1)}.$$

The above inequality and Lemma 2.35 (b) imply (see [G]) that for some $q' > p$ (depending only on p, n and the Lipschitz constant for D) that

$$\int_{B(w,3r) \cap \partial D} k^{q'} \, dH^{n-1} \leq c \, r^{-\frac{(n-1)(q'+1-p)}{p-1}} \left(\int_{B(w,3r) \cap \partial D} k^{p-1} \, dH^{n-1} \right)^{q'/(p-1)}. \tag{5.14}$$

Let $y \in \partial D \cap B(w, 2r)$ and $z \in \Gamma(y) \cap B(y, r/4)$, where $\Gamma(y), y \in B(w, 2r) \cap D$, is defined for a fixed constant b above (1.13). From Lemmas 2.35 and 2.5 with w, x replaced by y, z and $s = |z - y|$, we obtain that

$$\begin{aligned} |\nabla u(z)| &\leq c \min\{u(z), 1 - u(z)\}/s \leq c^2 s^{-1} [s^{p-n} |\mu(B(y, s))|]^{1/(p-1)} \\ &= c^2 \, [s^{1-n} \int_{B(y,s) \cap \partial D} k^{p-1} dH^{n-1}]^{1/(p-1)} \leq c^2 \, [M(k^{p-1})(y)]^{1/(p-1)} \end{aligned} \tag{5.15}$$

where

$$M(f)(y) = \sup_{0<s<r/4} [s^{1-n} \int_{B(y,s)} f dH^{n-1}]$$

whenever f is an integrable funcion on $\partial D \cap B(w, 4r)$. Let

$$N_1(|\nabla u|)(y) = \sup_{\Gamma(y)\cap B(y,r/4)} |\nabla u|.$$

Then from (5.14), (5.15), and the Hardy-Littlewood maximal theorem, we get for $q = (q' + p)/2$ that

$$\begin{aligned} \int_{B(w,3r)\cap\partial D} N_1(|\nabla u|)^q \, dH^{n-1} \, &\leq \, c\int_{B(w,3r)\cap\partial D} M(k^{p-1})^{q/(p-1)} \, dH^{n-1} \\ &\leq \, c^2 \, r^{-\frac{(n-1)(q+1-p)}{p-1}} \left(\int_{B(w,3r)\cap\partial D} k^{p-1} \, dH^{n-1} \right)^{q/(p-1)} . \end{aligned} \tag{5.16}$$

Using Lemma 2.3 we see that $|\nabla u(x)| \leq cr^{-1}$ when $x \in \Gamma(y) \setminus B(y, r/2)$ and $y \in \partial D$. Thus $N(|\nabla u|) \leq N_1(|\nabla u|) + cr^{-1}$ on $\partial D \cap B(w, 2r)$. In view of (5.16) it now follows that (a) of Theorem 3 is true.

The proof that ∇u has nontangential limits H^{n-1} almost everywhere is by contradiction. Suppose for a fixed constant, in the definition of $\Gamma(z), z \in \partial D$, that

$$\lim_{y\in\Gamma(z), y\to z} \nabla u(y) \text{ does not exist for } z \in F \subset \partial D \text{ with } H^{n-1}(F) > 0. \tag{5.17}$$

Let $w \in F$ be an H^{n-1} point of density for F. Then $\lim_{t\longrightarrow 0} t^{1-n} H^{n-1}[B(w, t) \cap \partial D \setminus F] = 0$, so if $r > 0$ is small enough, we deduce that

$$cH^{n-1}[\partial\tilde{\Omega} \cap B(w, r) \cap F] \geq r^{n-1}$$

where $\tilde{\Omega} \subset D$ is as in Lemma 2.45. From this lemma, we have $|\nabla u| \approx c$ on $\tilde{\Omega}$. Define $(b_{ij}(x))$ as in (1.8) with $\hat{u}$ replaced by u and let L be the corresponding divergence form elliptic operator defined as in (1.7). Then as noted above (2.26) we have $Lu_{x_k} = 0$ in D for $1 \leq k \leq n$. Moreover L is uniformly elliptic on $\tilde{\Omega}$ and u_{x_k} is bounded on $\tilde{\Omega}$. It now follows from a well known argument (see [CFMS]) that u_{x_k} has nontangential limits at almost every boundary point of $\tilde{\Omega}$ with respect to elliptic measure defined relative to L (see section 3). Here $b, 1 < b < \infty$, is arbitrary in the choice of nontangential cones for $\tilde{\Omega}$. From Theorem 3.11 we conclude that u_{x_k} has nontangential limits at H^{n-1} almost every point in $\partial\tilde{\Omega}$. Since b is arbitrary, and $\partial\tilde{\Omega}$ has tangent planes H^{n-1} a.e, we see that this limit can only exist if (5.17) is false.

Let $\nabla u(w), w \in \partial D$, denote the nontangential limit of ∇u whenever this limit exists. To prove (b) of Theorem 3 we can retrace the argument leading to Lemma 2.39. Indeed in (2.44) we showed that if w is in the outer boundary of D, $D(t) = \{y \in D : t < u(y) < 1\}$, and t is near 0 , then

$$\int_{B(w,2r)\cap\{u=t\}} |\nabla u|^p \, dH^{n-1} \, \leq \, c\, r^{-\frac{n-1}{p-1}} \left(\int_{B(w,2r)\cap\{u=t\}} |\nabla u|^{p-1} \, dH^{n-1} \right)^{p/(p-1)} . \tag{5.18}$$

If w lies in the inner boundary of D, this inequality also holds provided t is near enough 1. From Lemma 2.5 we see that if $\{x : u(x) = t\} = \{R_t(\omega)\omega : \omega \in \partial B(0, 1)\}$, then $\{\log R_t(\cdot)\}$ are uniformly

Lipschitz on $\partial B(0,1)$. Using this fact, the fact that ∇u has radial limits H^{n-1} almost everywhere, Theorem 3 (a), and dominated convergence, we get that (5.18) holds with $\{u=t\}$ replaced by ∂D. Also from (2.38) and Lemma 2.39 we find that

$$\nu \approx |\mu| \text{ where } d\nu = |\nabla u|^{p-1}\, dH^{n-1} \text{ on } \partial D. \tag{5.19}$$

Theorem 3 (b) now follows from (5.14) and (5.19). Theorem 3 (b) implies Theorem 3 (c) (see [CF]). The proof of Theorem 3 is complete. □

Remark: For use in the proof of Theorem 3 we point out that a slightly more sophisticated argument in Theorem 3 yields,

$$\nu = |\mu| \text{ on } \partial D \cap B(w, 2r). \tag{5.20}$$

In fact extend R_t to $\mathbb{R}^n \setminus \{0\}$ by putting $R_t(x) = R_t(\frac{x}{|x|})$. since $u(R_t(\omega)\omega) = t$ for $\omega \in \partial B(0,1)$, we can use the implicit function theorem and the chain rule to deduce for $\omega = (\omega_1, \dots, \omega_n)$ and $x = |x|\omega$ that

$$\nabla \log R_t(\omega) = -\frac{|x|\, \nabla u(x)}{\langle x, \nabla u(x)\rangle} + \frac{x}{|x|}. \tag{5.21}$$

We note that $|\nabla u| \neq 0, H^{n-1}$ almost everywhere, on ∂D as follows from Theorem 3 and [CF]. Using this note, (5.21), Lemma 2.5, and Theorem 3, we see that if $y \in \Gamma(z) \cap \{x : u(x) = t\} \rightarrow z \in \partial D$, then

$$\nabla \log R_t(y/|y|) \rightarrow -\frac{|z| \nabla u(z)}{\langle z, \nabla u(z)\rangle} + \frac{z}{|z|} \tag{5.22}$$

H^{n-1} almost everywhere. Let R be a graph function for either the outer or inner boundary of D. Since $\log R_t \rightarrow \log R$ uniformly on $\partial B(0,1)$ as $t \rightarrow 0$ or 1, it follows that the righthand side of (5.22) is a distributional gradient of $\log R$. Thus it is equal H^{n-1} almost everywhere to the gradient of $\log R$ at $z/|z|$. Using this fact, (5.18), (2.38), the change of variables theorem, Lemma 2.5, Theorem 3, and dominated convergence we get (5.20). Finally note that taking limits in (5.22) it also follows that ∇u is H^{n-1} almost everywhere normal to the tangent planes through points in ∂D. Taking limits in (2.41) we conclude that (2.41) holds with $D(t)$ replaced by D where $\nu = \frac{\nabla u}{|\nabla u|}$, H^{n-1} almost everywhere on ∂D.

Proof of Theorem 4. Let D, w, r, u, p be as in Theorem 3 and suppose also that $\partial D \cap B(w, 8r)$ is C^1. To prove Theorem 4 it suffices by way of a lemma of Sarason (see [KT]) to show that there exists $\epsilon_0 > 0, r_0 = r_0(\epsilon)$ defined for $\epsilon \in (0, \epsilon_0)$, such that whenever $y \in B(w,r) \cap \partial D$ and $0 < s < r_0(\epsilon)$, then

$$\fint_{B(y,s)\cap\partial D} |\nabla u|^p\, dH^{n-1} \;\leq\; (1+\epsilon) \left(\fint_{B(y,s)\cap\partial D} |\nabla u|^{p-1}\, dH^{n-1} \right)^{p/(p-1)}. \tag{5.23}$$

Here

$$\fint_E f\, dH^{n-1} = [H^{n-1}(E)]^{-1} \int_E f dH^{n-1}$$

whenever $E \subset \partial D$ is Borel measurable with finite positive H^{n-1} measure and f is an H^{n-1} integrable function on E. The proof of (5.23) is by contradiction. Indeed, otherwise, there exists sequences $(s_m)_1^\infty, (y_m)_1^\infty$ satisfying $y_m \in \partial D \cap B(w,r), m = 1, 2, \dots, \lim_{m \to \infty} s_m = 0$, and for which (5.23) is false with y, s replaced by $y_m, s_m, m = 1, 2, \ldots$ To obtain a contradiction we note from C^1 smoothness of $\partial D \cap B(w, 8r)$ that $\partial D \cap B(w, 4r)$ is Reifenberg flat with vanishing constant (see [KT]) in the following sense. Given $\delta > 0$, small, there is an $a = a(\delta) < 10^{-6} r$, such that whenever $y \in \partial D \cap B(w, 4r)$ and $0 < s \le a$, then

$$\begin{aligned} \{z + t\nu \in B(y, s/\delta), z \in P, t > \delta s\} \subset D \\ \{z - t\nu \in B(y, s/\delta), z \in P, t > \delta s\} \subset \mathbb{R}^n \setminus \bar{D}. \end{aligned} \tag{5.24}$$

Here $P = P(y)$ is the tangent plane to $\partial D \cap B(w, 4r)$ at y and $\nu = \nu(y)$ is the corresponding unit normal pointing into D. For fixed $A > 10^6$ we can also choose $\delta_0 = \delta_0(A) > 0$ so small that if $y' = y + As\nu(y)$, then the domain $\Omega(y')$, obtained by drawing all line segments from points in $B(y', \frac{As}{4})$ to points in $\partial D \cap B(y, As)$, is starlike Lipschitz with respect to y'. We assume, as we may, that $s_m \le a(\delta_0)$, for $m = 1, 2, \ldots$. Let y'_m be the point corresponding to y_m as above and put $D_m = \Omega(y'_m) \setminus \bar{B}(y'_m, As_m/8)$. From C^1 smoothness of $\partial D \cap B(w, 8r)$, we see that $D_m, m = 1, 2, \ldots,$ has Lipschitz constant $\le c_0$ where c_0 is an absolute constant. Let u_m be the p capacitary function for $D_m, m = 1, 2, \ldots$. From Theorem 2 with $r, w, \hat{u}_1, \hat{u}_2$ replaced by $\frac{As_m}{100}, y_m, u, u_m$, we deduce that if $w_1, w_2 \in B(y_m, 2s_m) \cap D, m = 1, 2, \ldots,$ then

$$\left| \log \left(\frac{u_m(w_1)}{u(w_1)} \right) - \log \left(\frac{u_m(w_2)}{u(w_2)} \right) \right| \le cA^{-\alpha},$$

where c, α are the constants in Theorem 2, so are independent of m. Letting $w_1, w_2 \to z_1, z_2 \in \partial D \cap B(y_m, 2s_m)$, and using Theorem 3, we get H^{n-1} almost everywhere that

$$\left| \log \left(\frac{|\nabla u_m(z_1)|}{|\nabla u(z_1)|} \right) - \log \left(\frac{|\nabla u_m(z_2)|}{|\nabla u(z_2)|} \right) \right| \le cA^{-\alpha}.$$

Exponentiating this inequality we have for c large enough,

$$(1 - \tilde{c}\, A^{-\alpha}) \frac{|\nabla u_m(z_1)|}{|\nabla u_m(z_2)|} \le \frac{|\nabla u(z_1)|}{|\nabla u(z_2)|} \le (1 + \tilde{c}\, A^{-\alpha}) \frac{|\nabla u_m(z_1)|}{|\nabla u_m(z_2)|},$$

where $\tilde{c}$ depends only on p, n, and the Lipschitz constant for D. Using this inequality and the fact that (5.23) is false, we obtain

$$\begin{aligned} &\frac{\displaystyle ⨍_{B(y_m, s_m) \cap \partial D} |\nabla u_m|^p \, dH^{n-1}}{\left(\displaystyle ⨍_{B(y_m, s_m) \cap \partial D} |\nabla u_m|^{p-1} \, dH^{n-1} \right)^{p/(p-1)}} \\ &\qquad \ge (1 - cA^{-\alpha}) \frac{\displaystyle ⨍_{B(y_m, s_m) \cap \partial D} |\nabla u|^p \, dH^{n-1}}{\left(\displaystyle ⨍_{B(y_m, s_m) \cap \partial D} |\nabla u|^{p-1} \, dH^{n-1} \right)^{p/(p-1)}} \ge (1 - cA^{-\alpha})(1 + \epsilon). \end{aligned} \tag{5.25}$$

Let T_m be a conformal affine mapping of $\mathbb{R}^n$ which maps the origin, e_n, onto y_m, y'_m respectively and $W = \{x \in \mathbb{R}^n : x_n = 0\}$ onto $P(y_m)$ for $m = 1, 2, \ldots$ It is easily seen that T_m is the composition of a translation, rotation, dilation. Let D'_m, u'_m be such that $T_m(D'_m) = D_m$ and $u_m(T_m x) = u'_m(x)$ whenever $x \in D'_m$. Since the p Laplace equation is invariant under translations, rotations, and dilations, we see that u'_m is the p capacitary function for D'_m. From (5.25) we also get

$$\begin{aligned}
&\frac{\displaystyle\fint_{B(0,1/A)\cap\partial D'_m} |\nabla u'_m|^p \, dH^{n-1}}{\left(\displaystyle\fint_{B(0,1/A)\cap\partial D'_m} |\nabla u'_m|^{p-1} \, dH^{n-1}\right)^{p/(p-1)}} \\
&\qquad = \frac{\displaystyle\fint_{B(y_m,s_m)\cap\partial D} |\nabla u_m|^p \, dH^{n-1}}{\left(\displaystyle\fint_{B(y_m,s_m)\cap\partial D} |\nabla u_m|^{p-1} \, dH^{n-1}\right)^{p/(p-1)}} \geq (1 - cA^{-\alpha})(1+\epsilon).
\end{aligned} \tag{5.26}$$

Letting $m \to \infty$ we see from Lemmas 2.1, 2.2 that u'_m converges uniformly on $\mathbb{R}^n$ to u' where u' is the p capacitary function for the starlike Lipschitz ring domain, $D' = \hat{\Omega} \setminus B(e_n, 1/8)$. Also $\hat{\Omega}$ is obtained by drawing all line segments connecting points in $B(0,1) \cap W$ to points in $B(e_n, 1/4)$. To get a contradiction we shall show that

$$\begin{aligned}
&\limsup_{m\to\infty} \left[\frac{\displaystyle\fint_{B(0,1/A)\cap\partial D'_m} |\nabla u'_m|^p \, dH^{n-1}}{\left(\displaystyle\fint_{B(0,1/A)\cap\partial D'_m} |\nabla u'_m|^{p-1} \, dH^{n-1}\right)^{p/(p-1)}} \right] \\
&\qquad \leq \frac{\displaystyle\fint_{B(0,1/A)\cap W} |\nabla u'|^p \, dH^{n-1}}{\left(\displaystyle\fint_{B(0,1/A)\cap W} |\nabla u'|^{p-1} \, dH^{n-1}\right)^{p/(p-1)}}.
\end{aligned} \tag{5.27}$$

To see how (5.27) leads to a contradiction, note from Schwarz reflection that u' has a p harmonic extension to $B(0,1/8)$ with $u' \equiv 0$ on $W \cap B(0,1/8)$. From barrier estimates we have $c^{-1} \leq |\nabla u'| \leq c$ on $B(0,1/16)$ where c depends only on p, n, and from Lemma 2.3 we find that $|\nabla u'|$ is Hölder continuous with exponent σ on $\bar{B}(0,1/16) \cap W$ (one could take $\sigma = 1$ here). Using these facts we conclude first that for some c

$$(1 - cA^{-\sigma})\, |\nabla u'(0)| \;\leq\; |\nabla u'(z)| \;\leq\; (1 + cA^{-\sigma})\, |\nabla u'(0)|$$

when $z \in B(0,1/A)$ and second from (5.26), (5.27) that

$$(1 + cA^{-\sigma}) \geq \frac{\displaystyle\fint_{B(0,1/A)\cap W} |\nabla u'|^p \, dH^{n-1}}{\left(\displaystyle\fint_{B(0,1/A)\cap W} |\nabla u'|^{p-1} \, dH^{n-1}\right)^{p/(p-1)}} \geq (1 - cA^{-\alpha})(1+\epsilon).$$

Clearly this inequality cannot hold if $A = e^{1/\epsilon}$ and ϵ_0 is sufficiently small. From this contradiction we conclude that in order to prove Theorem 4 it suffices to prove (5.27). To prove (5.27) we shall need the following lemma.

Lemma 5.28. *Given $\eta > 0$ small, there exists c, depending only on p, n, and the Lipschitz constant for D and $m_0 = m_0(\eta)$, such that if $m \geq m_0, x \in \partial D'_m \cap B(0, 1/100)$, and $0 \leq s \leq 1/100$, then $u'_m(a_s(x)) \leq c\, s^{1-\eta}$.*

Proof: Let ν'_m be the outer unit normal to $\partial D'_m$. We note for large m that since ∂D is C^1, there is a truncated cone $L \subset \mathbb{R}^n \setminus D'_m$ with vertex at x, axis parallel to $\nu'_m(x)$, angle opening $\theta > \pi/2$, and height $1/50$, where θ can be arbitrarily near $\pi/2$. let T be the cone with vertex at the origin, axis parallel to e_n, and of angle opening θ. Arguing as in [Kr] we see there exists a positive solution $f = f(\cdot, \theta)$ to the p Laplacian in T of the form

$$f(z) = |z|^{\gamma}\, \psi(\tfrac{\langle z, e_n \rangle}{|z|}),\ z \in T, \tag{5.29}$$

where $\psi \in C^{\infty}[\cos\theta, 1]$ and $\psi(1) = 1, \psi(\cos\theta) = 0$. Using the maximum principle for p harmonic functions, rotation and translation invariance of the p Laplacian, as well as Theorem 1, it follows that the conclusion of Lemma 5.28 holds with $1-\eta$ replaced by $\gamma = \gamma(\theta)$. Thus to complete the proof of Lemma 5.28 it suffices to show that $\gamma(\theta) \rightarrow 1$ continuously as $\theta \rightarrow \pi/2$. This can be easily verified using a compactness argument, since any convergence subsequence say $\{f(\cdot, \theta_j)\}$ with $\theta_j \rightarrow \pi/2$ is a positive solution to the p Laplacian on $\{x : x_n > 0\}$ which vanishes on W. This solution also has the form (5.29) with $\gamma = \lim_{j \rightarrow \infty} \gamma(\theta_j)$. Using Theorem 1 and the fact that x_n is also a positive solution to the p Laplacian on $\{x : x_n > 0\}$ which vanishes on W, it follows that $\gamma = 1$.

Another proof of Lemma 5.28 which uses only the fact that $\partial D \cap B(w, 4r)$ is Reifenberg flat with vanishing constant (see (5.24)) and an iteration argument, can be given as in [L, (2.37)]. The proof in [L] gives a lower bound for $u'_m(a_s(x))$ and is only in $\mathbb{R}^2$. However thanks to Theorem 1, essentially the same iterative argument can be used to get Lemma 5.28. □

Finally we use Lemma 5.28 to prove (5.27). Let $\mu_m = \mu_m^+ - \mu_m^-$, be the measure associated with u'_m for $m = 1, 2, \ldots$ From (5.20) we see that

$$d|\mu_m| = |\nabla u'_m|^{p-1}\, dH^{n-1} \text{ on } \partial D'_m. \tag{5.30}$$

From Lemmas 2.1, 2.2, and 2.35 we see that $\{|\mu_m|\}$ converges weakly to $|\mu_{\prime}|$ where $\mu_{\prime} = \mu_{\prime}^+ - \mu_{\prime}^-$ is the measure associated with u'. Also (5.30) holds for $|\mu_{\prime}|, u'$. From the definition of weak convergence and (5.30) we deduce

$$\int_{B(0,1/A) \cap W} |\nabla u'|^{p-1}\, dH^{n-1} = \lim_{m \longrightarrow \infty} \int_{B(0,1/A) \cap \partial D'_m} |\nabla u'_m|^{p-1}\, dH^{n-1}. \tag{5.31}$$

Let $C = \{x = (x', x_n) \in \mathbb{R}^n : |x'| < 1/A, |x_n| < 1/A\}$ be the symmetric cylinder of radius $1/A$, height $2/A$ with center at the origin and axis parallel to e_n. Let ν^* be the outer unit normal to C. To handle the top integral in (5.27) we argue as in the proof of Lemma 2.39 leading to (2.41). As noted after (5.22), this inequality holds with $u, D(t), B(w, s)$, replaced by u'_m, D'_m, C, and with $\nabla u'_m$ as the normal, H^{n-1} almost everywhere, to $\partial D'_m$. We get

$$L_m = (p-1) \int_{\partial D'_m \cap C} \langle e_n - x, \nabla u'_m \rangle\, |\nabla u'_m|^{p-1} dH^{n-1} = J_m + K_m. \tag{5.32}$$

where

$$J_m = (p-n)\int_{D'_m \cap C} |\nabla u'_m|^p \, dx$$

and

$$K_m = \int_{D'_m \cap \partial C} \langle\, x - e_n,\, \nu^* \,\rangle \, |\nabla u'_m|^p \, dH^{n-1}$$

$$- \, p \int_{D'_m \cap \partial C} \langle\, \nabla u'_m,\, \nu^* \,\rangle \, \langle \nabla u'_m, x - e_n \rangle |\nabla u'_m|^{p-2} \, dH^{n-1}.$$

Now $u'_m, \nabla u'_m$ converge uniformly on compact subsets of D' to $u', \nabla u'$. Replace u by u'_m in Theorem 3. From the proof of Theorem 3 (a) (see (5.16)) we deduce that the L^q norm $(q > p)$ of $N(|\nabla u'_m|)$ on $\partial D'_m \cap B(0, 1/8)$ is uniformly bounded for $m = 1, 2, \ldots$ Using these facts and standard convergence theorems we see that

$$\lim_{m \longrightarrow \infty} J_m = (p-n)\int_{D' \cap C} |\nabla u'|^p \, dx. \tag{5.33}$$

The same argument can be applied to K_m to get

$$\begin{aligned} \lim_{m \longrightarrow \infty} K_m = & \int_{D' \cap \partial C} \langle\, x - e_n,\, \nu^* \,\rangle \, |\nabla u'|^p \, dH^{n-1} \\ & - \, p \int_{D' \cap \partial C} \langle\, \nabla u',\, \nu^* \,\rangle \, \langle \nabla u', x - e_n \rangle |\nabla u'|^{p-2} \, dH^{n-1} \end{aligned} \tag{5.34}$$

provided we first show that

$$\int_{D'_m \cap \partial C} |\nabla u'_m|^{p+1} \, dH^{n-1} \, \leq \, M < \infty \tag{5.35}$$

for sufficiently large m, say $m \geq m_0$. To prove (5.35) note from our construction that $\partial D'_m \cap \partial C$ is a Lipschitz image of $\{x' \in \mathbb{R}^{n-1} : |x'| = 1/A\}$, with Lipschitz constant independent of m. Let

$$\Lambda_k = \{x \in D'_m \cap \partial C : A^{-1}\, 2^{-k-1+n} < d(x, \partial D'_m) < A^{-1} 2^{-k-n}\} \text{ for } k = 3, \ldots$$

From Lemma 5.28 and Lemma 2.5 (ii) for u'_m, we see that if m_0 is large enough, then for $m > m_o$,

$$|\nabla u'_m|^{p+1} \leq cA2^{k/2} \tag{5.36}$$

on Λ_k where c is independent of m and k. If $|x'| = 1/A$, and l denotes the line through x' parallel to e_n, then from (5.36) and the above note we deduce that

$$\int_{l \cap \Lambda_k} |\nabla u'_m|^{p+1}(x', \cdot) \, dH^1 \, \leq \, c2^{-k/2}.$$

Integrating with respect to H^{n-2} measure over the sphere of radius $1/A$ in $\mathbb{R}^{n-1}$, it follows that

$$\int_{\Lambda_k} |\nabla u'_m|^{p+1} dH^{n-1} \, \leq \, c2^{-k/2} A^{2-n}. \tag{5.37}$$

Let $\Lambda = \bigcup_{k=1}^{\infty} \Lambda_k$. Summing (5.37) we get

$$\int_{\Lambda} |\nabla u'_m|^{p+1} dH^{n-1} \leq c\, A^{2-n}. \tag{5.38}$$

Now from (5.24) we see for m_0 large enough that $c\, d(C \setminus \Lambda, \partial D'_m) \geq A^{-1}$ where c is independent of m. From this fact,(5.38), and uniform convergence of $\nabla u'_m$ to $\nabla u'$ on compact subsets of D' we find that (5.35) is valid. Thus (5.34) is also valid. Now, (5.32) also holds with u'_m replaced by u' and ∂D_m by $\partial D'$. In view of (5.33), (5.34) and (5.32) for u', u'_m we deduce that

$$\begin{aligned} \lim_{m \to \infty} L_m &= (p-1) \int_{W \cap B(0,1/A)} \langle e_n - x, \nabla u' \rangle\, |\nabla u'|^{p-1} dH^{n-1} \\ &= (p-1) \int_{W \cap B(0,1/A)} |\nabla u'|^{p} dH^{n-1}. \end{aligned} \tag{5.39}$$

Since $\nabla u'_m(x), x \in \partial D'_m$, is normal H^{n-1} almost everywhere to the tangent plane through x, we deduce from C^1 smoothness of $\partial D \cap B(w, 8r)$ and our construction that

$$||\nabla u'_m| + \langle x - e_n, \nabla u'_m \rangle| \leq b_m |\nabla u'_m| \text{ on } \partial D'_m \cap C$$

where $b_m \to 0$ as $m \to \infty$. Thus,

$$\lim_{m \to \infty} L_m = (p-1) \lim_{m \to \infty} \int_{\partial D'_m \cap C} |\nabla u'_m|^{p} dH^{n-1}. \tag{5.40}$$

Next we note from C^1 smoothness of $\partial D \cap B(w, 8r)$ that

$$H^{n-1}[\partial D'_m \cap B(0, 1/A)] \to H^{n-1}[W \cap B(0, 1/A)] \to \text{ as } m \to \infty. \tag{5.41}$$

Using (5.40) in (5.39), the fact that $B(0, 1/A) \subset C$, (5.41), and (5.31), we conclude that (5.27) is true. From our earlier remarks, we now get Theorem 4. □

REFERENCES

[AS] H. Aikawa and N. Shanmugalingam, *Carleson type estimates for p harmonic functions and the conformal Martin boundary of John domains in metric measure spaces,* Michigan Math. J. **53** (2005), no. 1, 165-188.

[AKSZ] H. Aikawa, T. Kilpeläinen, N. Shanamuligan, and X. Zhong, *Boundary Harnack principle for p harmonic functions in smooth Euclidean domains,* submitted.

[A] A. Ancona, *Principe de Harnack à la frontière et théorème de Fatou pour un opérateur elliptique dans un domain lipschitzien,* Ann. Inst. Fourier (Grenoble) **28** (1978), no. 4, 169-213.

[BL] B. Bennewitz and J. Lewis, *On the dimension of p harmonic measure,* Ann. Acad. Sci. Fenn. **30** (2005), 459-505.

[BL1] B. Bennewitz and J. Lewis, *On weak reverse Hölder inequalities for nondoubling harmonic measures,* Complex Variables **49** (2004), no. 7 - 9, 571-582.

[C1] L. Caffarelli, *A Harnack inequality approach to the regularity of free boundaries. Part I, Lipschitz free boundaries are* $C^{1,\alpha}$, Revista Math. Iberoamericana **3** (1987), 139-162.

[C2] L. Caffarelli, *A Harnack inequality approach to the regularity of free boundaries. II. Flat free boundaries are Lipschitz,* Comm. Pure Appl. Math. **42** (1989), no. 1, 55–78.

[C3] L. Caffarelli, *A Harnack inequality approach to the regularity of free boundaries. III. Existence theory, compactness, and dependence on* X, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **15** (1988), no. 4, 583–602 (1989).

[CFMS] L. Caffarelli, E. Fabes, S. Mortola, S. Salsa *Boundary behavior of nonnegative solutions of elliptic operators in divergence form,* Indiana J. Math. **30** (4) (1981) 621-640.

[CF] R. Coifmann and C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals,* Studia math. **51** (1974), 241-250.

[D] B. Dahlberg, *On estimates of harmonic measure,* Arch Rational Mech. Anal. **65** *(1977), 275-288.*

[Di] E. DiBenedetto, $C^{1+\alpha}$ *local regularity of weak solutions of degenerate elliptic equations,* Nonlinear Anal. **7** (1983), 827-850.

[EL] A. Eremenko and J. Lewis, *Uniform limits of certain A-harmonic functions with applications to quasiregular mappings,* Ann. Acad. Sci. Fenn. AI , Math. **16** (1991), 361-375.

[FKS] E. Fabes, C. Kenig, and R. Serapioni, *The local regularity of solutions to degenerate elliptic equations,* Comm. Partial Differential Equations, **7** (1982), no. 1, 77 - 116.

[FJK] E. Fabes, D. Jerison, and C. Kenig, *The Wiener test for degenerate elliptic equations,* Ann. Inst. Fourier (Grenoble) **32** (1982), no. 3, 151-182.

[FJK1] E. Fabes, D. Jerison, and C. Kenig, *Boundary behavior of solutions to degenerate elliptic equations.* Conference on harmonic analysis in honor of Antonio Zygmund, Vol I, II Chicago, Ill, 1981, 577-589, Wadsworth Math. Ser, Wadsworth Belmont CA, 1983.

[G] F. Gehring, *On the* L^p *integrability of the derivatives of a quasiconformal mapping,* Acta Math. **130** (1973), 265-277.

[GZ] R. Gariepy and W.P. Ziemer, *A regularity condition at the boundary for solutions of quasilinear elliptic equations,* Arch. Rat. Mech. Anal. **67** (1977), 25-39.

[GT] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order,* second edition, Springer-Verlag, 1983.

[HKM] J. Heinonen, T. Kilpeläinen, and O. Martio, *Nonlinear potential theory of degenerate elliptic equations,* Oxford University Press, 1993.

[HL] S. Hofmann and J. Lewis, *The Dirichlet problem for parabolic operators with singular drift term,* Mem. Amer. Math. Soc. **151** (2001), no. 719, 1-113.

[JK] D. Jerison and C. Kenig, *Boundary behavior of harmonic functions in nontangentially accessible domains,* Advances in Math. **46** (1982), 80-147.

[JK1] D. Jerison and C. Kenig, *The logarithm of the Poisson kernel of a C^1 domain has vanishing mean oscillation,* Trans. Amer. Math. Soc. **273** (1984), 781-794.

[K] J. Kemper, *A boundary Harnack inequality for lipschitz domains and the principle of positive singularities,* Comm. Pure Appl. Math. **25** (1972), 247-255.

[KT] C. Kenig and T. Toro, *Harmonic measure on locally flat domains,* Duke Math J. **87** (1997), 501-551.

[KT1] C. Kenig and T. Toro, *Free boundary regularity for harmonic measure and Poisson kernels,* Ann. of Math. **150** (1999), 369-454.

[KT2] C. Kenig and T. Toro, *Poisson kernel characterization of Reifenberg flat chord arc domains,* Ann. Sci. Ecole Norm. Sup. (4) **36** (2003), no. 3, 323 - 401.

[KT3] C. Kenig and T. Toro, *Free boundary regularity below the continuous threshold: 2-phase problems,* submitted.

[KP] C. Kenig and J. Pipher, *The Dirichlet problem for elliptic operators with drift term,* Publ. Mat. **45** (2001), no. 1, 199-217.

[KZ] T. Kilpeläinen and X. Zhong, *Growth of entire A - subharmonic functions,* Ann. Acad. Sci. Fenn. AI , Math. **28** (2003), 181-192.

[Kr] I. N. Krol', *On the behavior of the solutions of a quasilinear equation near null salient points of the boundary,* Proc. Steklov Inst. Math. **125** (1973) 130-136.

[LV] J. Lewis and A. Vogel, *Uniqueness in a free boundary problem,* to appear in Comm. in PDE.

[LV1] J. Lewis and A. Vogel, *Symmetry problems and uniform rectifiability,* submitted.

[L] J. Lewis, *Note on p harmonic measure,* to appear in Computational Methods in Function Theory.

[L1] J. Lewis, *Regularity of the derivatives of solutions to certain degenerate elliptic equations,* Indiana University Mathematics Journal **32** (6) (1983), 849-858

[Li] G.M. Lieberman, *Boundary regularity for solutions of degenerate elliptic equations,* Nonlinear Anal. **12** (1988), no. 11, 1203-1219.

[LSW] W. Littman, G. Stampacchia, and H. F. Weinberger, *Regular points for elliptic equations with discontinuous coefficients,* Ann. Scuola Norm. Sup. Pisa (3) **17** (1963), 43 - 77.

[Ma] P. Mattila, *Geometry of sets and measures in Euclidean spaces,* Cambridge University Press, 1995.

[S] J. Serrin, *Local behavior of solutions of quasilinear elliptic equations,* Acta Math. **111** (1964), 247-302.

[St] E.M. Stein, *Singular integrals and differentiability properties of functions,* Princeton University Press, Princeton, NJ, 1970.

[T] P. Tolksdorff, *Everywhere regularity for some quasi-linear systems with lack of ellipticity,* Ann. Math. Pura Appl. **134** (1984), no 4, 241-266.

[W] J. M. Wu, *Comparisons of kernel functions, boundary Harnack principle and relative Fatou theorem on Lipschitz domains,* Ann. Inst. Fourier (Grenoble) **28** (1978), no. 4, 147-167.