ON THE SINGULAR SET IN THE NAVIER-STOKES EQUATIONS

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ABSTRACT: We consider suitably weak solutions (u, p) to the incompressible Navier-Stokes equations and under various assumptions on u obtain estimates for the size of its singular set. One of our results improves a well known theorem of Caffarelli, Kohn and Nirenberg.

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1. Introduction. In this paper we investigate suitable weak solutions of the incompressible Navier-Stokes equations with viscosity $\nu = 1$,

$$u_t^i - \Delta u^i + u \cdot \nabla u^i + \frac{\partial p}{\partial x_i} = f_i,\ 1 \le i \le 3, \quad \nabla \cdot u = 0 \tag{1.1}$$

in $D = \Omega \times (0,T) \subset \mathbb{R}^3 \times \mathbb{R}$ where $f = (f_1, f_2, f_3)$ is an external force with $\nabla \cdot f = 0$. In recent years this equation has received much attention. Of particular interest has been the initial boundary value problem on a bounded smooth domain $\Omega \subset \mathbb{R}^3$:

$$\begin{aligned} &(i) \quad u(x,t) = 0, (x,t) \in \partial\Omega \times (0,T), \\ &(ii) \quad u(x,0) = u_0(x),\ \nabla \cdot u_0(x) = 0,\ x \in \Omega, \end{aligned} \tag{1.2}$$

where the boundary conditions are understood in an appropriate sense (see [T, ch 3]). It was originally shown by Leray[L] and Hopf[H] using essentially what now is known as the Faedo-Galerkin method that for $f \in L^2(D)$ and $u_0 \in L^2(\Omega)$, there exists a solution u to (1.1), (1.2) in the sense that

$$\begin{aligned} &(a) \quad u \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega)). \\ &(b) \quad \int_D [-u \cdot \phi_t + \nabla u \cdot \nabla \phi + (u \cdot \nabla u - f) \cdot \phi]\, dxdt = 0 \text{ and } \int_D u \cdot \nabla\theta\, dxdt = 0 \text{ for} \\ &\qquad \theta, \phi = (\phi_1, \phi_2, \phi_3) \text{ infinitely differentiable in } D \text{ with compact support and } \nabla \cdot \phi \equiv 0. \\ &(c) \quad \int_\Omega |u|^2(\cdot,t)dx + 2\int_0^t \int_\Omega |\nabla u|^2\, dx\, dt \le \int_\Omega |u_0|^2\, dx + 2\int_0^t \int_\Omega f \cdot u\, dx\, dt,\ 0 < t < T. \\ &(d) \quad \lim_{t \to 0} u(\cdot,t) = u_0(\cdot) \text{ in the norm of } L^2(\Omega). \end{aligned} \tag{1.3}$$

We note that (b) implies the existence of p as a distribution. Also (c) is known as the energy inequality. This concept of a solution appears too weak to guarantee that u satisfies Caccioppoli and local energy type inequalities or that p is an integrable function. Thus for weak solutions as above, it is difficult to attack such fundamental questions as: *Can a weak solution to the incompressible Navier Stokes equations develop singularities and is it unique?* To obtain partial regularity results, [CKN] define a suitable weak solution (u,p) to be a weak solution (as above) with $p \in L^{5/4}(D)$ and for which u satisfies Caccioppoli-local energy inequalities. Also $f \in L^q(D)$ for some $q > 5/2$. An exact definition of suitable weak solutions will be given in section 2. These authors define the singular set, $\mathcal{S} \subset D$, of a suitable weak solution u as follows: $(x,t) \in \mathcal{S}$ if and only if u is not essentially bounded in any neighborhood of (x,t). They prove

Theorem A $\Lambda_1(\mathcal{S}) = 0$.

In Theorem A, Λ_1 denotes Hausdorff one measure using parabolic cylinders for coverings (as defined above (1.4)).

In this paper we show that Theorem A can be improved on. To be more specific we shall need some notation. Given $r > 0, (x,t) \in I\!R^3 \times I\!R = I\!R^4$, let

$$B_r(x) = \{y \in I\!R^3 : |y-x| < r\}, \quad Q_r(x,t) = B_r(x) \times (t-r^2, t+r^2), \; t \in I\!R.$$

We call $Q_r(x,t)$ a parabolic cylinder with center at (x,t). Let h be an increasing continuous function on (0,1] with $\lim\limits_{t\to 0} h(t) = 0$, and $h(1) = 1$. Define Hausdorff measure with respect to h and parabolic cylinders in the following way. For fixed $\delta > 0$ and $E \subset I\!R^3 \times I\!R$ let $L(\delta)$ be the family of all coverings $\{Q_{r_i}(x_i,t_i)\}$ of E with sidelength $\le \delta$. That is $E \subset \bigcup Q_{r_i}(x_i,t_i)$ and $0 < r_i \le \delta$. Put

$$\Psi_\delta(E,h) = \inf_{L(\delta)} \left(\sum_i h(r_i) \right)$$

and set

$$\Lambda(E,h) = \lim_{\delta\to 0} \Psi_\delta(E,h).$$

If $h(t) = t^k, k$ a real number, we write Λ_k for $\Lambda(\cdot,h)$ and call Λ_k, Hausdorff k measure defined with respect to parabolic rectangles. We note that Λ_k is normalized differently than the usual Hausdorff k measure. Next if $Q_r(x,t) \subset D$ and $r > 0$ put

$$\begin{aligned} A_1(r) &= A_1(r,x,t) = r^{-1} \int_{Q_r(x,t)} |\nabla u|^2 \, dx \, dt. \\ A_2(r) &= A_2(r,x,t) = r^{-1} \sup_{|s-t|<r^2} \int_{B_r(x)} |u|^2(\cdot,s) \, dx \,. \\ A_3(r) &= A_3(r,x,t) = r^{-2} \int_{Q_r(x,t)} |u|^3 \, dx \, dt. \end{aligned} \tag{1.4}$$

$$F_i(M) = \{ (x,t) \in D : \limsup_{r\to 0} A_i(r) \le M \}, \; 1 \le i \le 3, \; M < \infty.$$

We first prove in section 4.

Theorem 1 *Let u, p, f, q be as in the definition of a suitable weak solution. There exists $\epsilon > 0$ depending only on M such that $\Lambda_{1-\epsilon}(\mathcal{S} \cap F_i(M)) = 0, \; 1 \le i \le 3$. In fact for some absolute constant $\tilde{c}$ one can take*

$$\epsilon = \max_{b\in[9/5,2]} \left[(2-b)\,\tilde{c}^{-1} \begin{cases} (M+1)^{(3b-56)/6} & \text{when } i = 3, \\ (M+1)^{-(3b+68)/6} & \text{when } i = 2, \\ (M+1)^{(9b-106)/6} & \text{when } i = 1. \end{cases} \right]$$

Next we prove in section 5.

Theorem 2 *Let u, p, f, q be as in the definition of a suitable weak solution. If $h(t) = t\,(\ln e/t)^\sigma$, then $\Lambda(\mathcal{S}, h) = 0$ whenever $0 \le \sigma < 3/44$.*

As for the proof of Theorem 1, originally we tried to use the Caccioppoli inequality for a suitable weak solution and interpolation inequalities to argue as in [C], [KL] in order to conclude first a reverse Hölder inequality for $|\nabla u|$ and then that $|\nabla u|^{2+\epsilon}$ is locally integrable, which would imply Theorem 1. Unfortunately for i fixed, $1 \leq i \leq 3$, we can prove such an inequality only for parabolic rectangles with centers in $\mathcal{S} \cap F_i(M)$, with constants which depend on M. Thus we are not able to prove higher integrability of $|\nabla u|$. However we are able to show that $d^{-\epsilon}\,|\nabla u|^2$ is integrable locally which easily implies Theorem 1. Here $d(y,s)$ is essentially the parabolic distance from (y,s) to $\mathcal{S} \cap F_i(M)$. Theorem 2 is proved similarly although the arguments are more technical and we do not quite get a reverse Hölder inequality for u on $\mathcal{S}$. Still though we are able to show that $(\log 1/d)\,|\nabla u|^2$ is locally integrable on a certain subset of $\mathcal{S}$ which is enough to prove Theorem 2. A key ingredient in both proofs is Corollary 1 in [CKN].

As for the plan of this paper, in section 2 we give the definition of a suitable weak solution and state the Caccioppoli inequality which follows readily from the definition. We also state a representation formula for the pressure, Corollary 1 in [CKN], and interpolation inequalities. In section 3 we prove Lemmas 1 and 2 which together show that $A_1(r,x,t) \approx$ constant for small r when $(x,t) \in \mathcal{S} \cap F_i(M)$, $1 \leq i \leq 3$. In section 4 we use Lemmas 1,2 and our Caccioppoli inequality to prove Theorem 1 in the way outlined above. In section 5 we prove Lemmas 3 and 4 which are analogues of Lemmas 1 and 2. We then use these lemmas to prove Theorem 2 as outlined above. Throughout the proof of Theorems 1 and 2 we shall refer often to results in [CKN], so the reader is advised to have this paper at hand.

Finally part of this research was carried out while the second author was a visitor at KAIST. He would like to thank KAIST for the gracious hospitality extended him while he was a visitor.

2. Preliminary Reductions. Following [CKN, section 2] we say that (u,p) are a suitable weak solution to (1.1) if u, p, f are Lebesgue measurable functions on D and

$$\begin{aligned} &(\alpha) \quad p \in L^{5/4}(D) \text{ while } f \in L^q(D) \text{ for some } q > 5/2 \text{ with } \nabla \cdot f = 0, \text{ weakly.} \\ &(\beta) \quad (1.3)(a),(b) \text{ are valid.} \\ &(\gamma) \quad 2\int |\nabla u|^2\,\theta\,dxdt \,\leq\, \int [\,|u|^2\,(\theta_t + \Delta\theta) \,+\, (\,|u|^2 + 2p\,)\,u\cdot\nabla\theta \,+\, 2\,(u\cdot f)\,\theta\,]\,dx\,dt \\ &\qquad \text{whenever } \theta \in C_0^\infty(D). \end{aligned} \tag{2.1}$$

Choosing θ appropriately one deduces from (2.1) the Caccioppoli inequality (see [CKN, (2.17)])

$$\begin{aligned} \int_{Q_r(x,t)} |\nabla u|^2\,dxdt \,+\, \operatorname{ess}\sup_{|s-t|\leq r^2} \int_{B_r(x)} |u|^2(\cdot,s)\,dx \,\leq\, c\,r^{-2}\int_{Q_{2r}(x,t)} |u|^2\,dxdt \\ +\, c\,r^{-1}\int_{Q_{2r}(x,t)} [\,|u|^3(\cdot,s) \,+\, |p|\,|u|\,]\,dxdt \,+\, c\int_{Q_{2r}(x,t)} |f|\,|u|\,dxdt. \end{aligned} \tag{2.2}$$

In this equation as in the sequel c denotes an absolute contant ≥ 1. In general, $c(a_1, a_2, \ldots)$ will denote a positive constant ≥ 1 depending only on $a_1, a_2, \ldots$, not necessarily the same at each occurrence. From (2.1) it also follows for $\psi \in C_0^\infty(\Omega)$ and fixed $\tau \in (0,T)$ that at $(z,\tau) \in D$ (see

[CKN, (2.15)])

$$p\,\psi \;=\; p_1 + p_2 + p_3 \text{ where}$$

$$p_1 = \frac{3}{4\pi}\int \frac{\partial^2}{\partial y_i \partial y_j}\left(\frac{1}{|z-y|}\right)(\psi\, u^i\, u^j)\,(y,\tau)\,dy$$

$$p_2 = \frac{3}{4\pi}\int \left[\frac{2\,(z_i-y_i)}{|z-y|^3}\,(\frac{\partial\psi}{\partial y_j}\,u^i\,u^j)(y,\tau) \;+\; \frac{1}{|z-y|}\,(\frac{\partial^2\psi}{\partial y_i y_j}\,u^i\,u^j)(y,\tau)\right]\,dy \tag{2.3}$$

$$p_3 = \frac{3}{4\pi}\int \left[\frac{1}{|z-y|}\,(\Delta\psi\,p)(y,\tau) + \frac{2\,(z_i-y_i)}{|z-y|^3}\,(\frac{\partial\psi}{\partial y_i}\,p)(y,\tau)\right]dy.$$

Here repeated indices denote summation from 1 to 3 and the integral involving p_1 is understood as a prinicpal value. We now state some interpolation inequalities. Let $Q_\rho(x,t)\subset D, s\in(t-\rho^2,t+\rho^2)$, and let $a_\rho(s,u^i)$ denote the average of $u^i(\cdot,s)$ on $B_\rho(x)$. Put $a_\rho(s,u)=(a_\rho(u^1,s),a_\rho(u^2,s),a_\rho(u^3,s))$. From Hölder's inequality we have for $1<b\le 2$,

$$\begin{aligned}&\int_{B_\rho(x)}|u(\cdot,s)-a_\rho(s,u)|^3\,dy\\ &\quad\le\; c\left(\int_{B_\rho(x)}|u(\cdot,s)-a_\rho(s,u)|^{2b/(b-1)}\right)^{(b-1)/2}\cdot\left(\int_{B_\rho(x)}|u|^2\,(\cdot,s)\,dy\right)^{(3-b)/2}.\end{aligned} \tag{2.4}$$

Using (2.4), Sobolev's Theorem, and integrating in time one obtains for $9/5\le b\le 2$

$$\begin{aligned}&\int_{Q_\rho(x,t)}|u-a_\rho(s,u)|^3\,dyds\\ &\quad\le\; c\rho^{(5b-9)/2}\left(\int_{Q_\rho(x,t)}|\nabla u|^b\,dyds\right)\cdot\left(\sup_{|s-t|\le\rho^2}\int_{B_\rho(x)}|u|^2\,(\cdot,s)\,dy\right)^{(3-b)/2}\end{aligned} \tag{2.5 a}$$

and

$$\begin{aligned}&\int_{Q_\rho(x,t)}|u-a_\rho(s,u)|^3\,dyds\\ &\quad\le\; c\,\rho^{1/2}\left(\int_{Q_\rho(x,t)}|\nabla u|^2\,dyds\right)^{3/4}\cdot\left(\sup_{|s-t|\le\rho^2}\int_{B_\rho(x)}|u|^2\,(\cdot,s)\,dy\right)^{3/4}.\end{aligned} \tag{2.5 b}$$

We note that (2.5 b) was chosen from the many possible inequalities because it gives the largest exponent on the righthand side (3/4) for the integrals involved in the supremum. Next we list some inequalities that are easily derived from (2.5). The case $b=2$ of (2.5 a) will play an important role in our proofs. Thus we list it seperately in (2.6 a). If $Q_\rho(x,t)\subset D$ and $0<r\le\rho$, then

$$A_3(r)\;\le\;c\,(\rho/r)^2\,A_1(\rho)\;A_2(\rho)^{1/2}\;+\;c\,\min\{(r/\rho)^3\,A_2(\rho)^{3/2},\;(r/\rho)A_3(\rho)\,\}. \tag{2.6 a}$$

Also from (2.5 b) we have

$$A_3(r) \le c(\rho/r)^2 A_1(\rho)^{3/4} A_2(\rho)^{3/4} + c \min\{(r/\rho)^3 A_2(\rho)^{3/2}, (r/\rho)A_3(\rho)\}. \tag{2.6 b}$$

Finally for $9/5 \le b < 2$ we get from (2.5 a)

$$\begin{aligned} A_3(r) \le\, & c r^{-2} \rho^{2b-3} A_2(\rho)^{(3-b)/2} \textstyle\int_{Q_\rho(x,t)} |\nabla u|^b dxdt \\ & + c \min\{(r/\rho)^3 A_2(\rho)^{3/2}, (r/\rho)A_3(\rho)\}. \end{aligned} \tag{2.6 c}$$

For example (2.6 c) follows from

$$\begin{aligned} A_3(r) &\le c r^{-2} \int_{Q_\rho(x,t)} |u - a_\rho(s,u)|^3 \, dy \, ds + c r \int_{t-r^2}^{t+r^2} |a_\rho(s,u)|^3 \, ds \\ &\le c r^{-2} \rho^{(5b-9)/2} \left(\int_{Q_\rho(x,t)} |\nabla u|^b \, dyds \right) \cdot \left(\sup_{|s-t|\le \rho^2} \int_{B_\rho(x)} |u|^2 (\cdot, s) \, dy \right)^{(3-b)/2} \\ &\quad + c \min\{(r/\rho)^3 A_2(\rho)^{3/2}, (r/\rho)A_3(\rho)\} \\ &\le c r^{-2} \rho^{2b-3} A_2(\rho)^{(3-b)/2} \int_{Q_\rho(x,t)} |\nabla u|^b \, dxdt + c \min\{(r/\rho)^3 A_2(\rho)^{3/2}, (r/\rho)A_3(\rho)\}. \end{aligned}$$

In addition to the quantities defined in (1.4) we shall need

$$\begin{aligned} A_4(r) &= A_4(r,x,t) = r^{-2} \int_{Q_r(x,t)} |u||p| \, dxdt \\ L(r) &= L(r,x,t) = r^{-13/4} \int_{t-r^2}^{t+r^2} \left(\int_{B_r(x)} |p| \, dx \right)^{5/4} dt \\ G(r) &= G(r,x,t) = r^{3q-5} \int_{Q_r(x,t)} |f|^q \, dx \, dt. \end{aligned} \tag{2.7}$$

From Hölder's inequality we see that

$$r^{-1} \int_{Q_r(x,t)} |f||u| dxdt \le c\, G(r)^{1/q} \min\{A_2(r)^{1/2}, A_3(r)^{1/3}\}. \tag{2.8}$$

Next suppose that $\psi \in C_0^\infty(B_{2r_0}(x))$ with $|\nabla \psi| \le 1000\, r_0^{-1}$ and $\psi \equiv 1$ on $B_{3r_0/2}(x)$. If $2r \le \rho \le r_0$ and $B_{2r_0}(x) \subset \Omega$, then from (2.3) we deduce with $\tau \in (t-r^2, t+r^2)$ and $z = x$ that

$$\begin{aligned} r^{-2} \int_{B_r(x)} |p||u|(\cdot,\tau) \, dy &\le r^{-2} \int_{B_r(x)} (\,|p_1| + |p_2| + |p_3|\,)|u|(\cdot,\tau) dy \\ &\le r^{-2} \int_{B_r(x)} |p_1||u|(\cdot,\tau) \, dy + c\, A_2(r)^{1/2} \, r_0^{-3} \left(\int_{B_{2r_0}(x)} (|p| + |u|^2)(\cdot,\tau) \, dy \right). \end{aligned}$$

Integrating in time one gets

$$
\begin{aligned}
A_4(r) \;&=\; r^{-2}\int_{Q_r(x,t)} |p||u|\,dyd\tau \\
&\le\; c\,r^{-2}\int_{Q_r(x,t)} |p_1|\,|u|dyd\tau \;+\; \beta(r)A_2(r)^{1/2}
\end{aligned}
\tag{2.9}
$$

where

$$
\beta(r) = c\,r^{2/5}\,r_0^{-12/5}\;\left(\int_{t-r^2}^{t+r^2}\int_{B_{2r_0}(x)} (|p|+|u|^2)^{5/4}\,dxdt\right)^{4/5}.
$$

We write $p_1 = p_{11} + p_{12}$ where

$$
p_{11} = \frac{3}{4\pi}\int \frac{\partial^2}{\partial y_i \partial y_j}\left(\frac{1}{|z-y|}\right)(\chi\,\psi\,u^i\,u^j)(\cdot,\tau)\,dy
$$

and χ is the characteristic function of $B_\rho(x)$. From Calderón-Zygmund theory and Hölder's inequality we deduce that

$$
\begin{aligned}
\int_{B_r(x)} |p_{11}||u|(\cdot,\tau)\,dy \;&\le\; \left(\int_{B_r(x)} |p_{11}|^{3/2}(\cdot,\tau)\,dy\right)^{2/3}\cdot\left(\int_{B_r(x)}|u|^3(\cdot,\tau)\,dy\right)^{1/3} \\
&\le\; c\,\left(\int_{B_\rho(x)} |u|^3(\cdot,\tau)\,dy\right)^{2/3}\cdot\left(\int_{B_r(x)}|u|^3(\cdot,\tau)\,dy\right)^{1/3}.
\end{aligned}
$$

Integrating in time and using Hölder's inequality it follows that

$$
r^{-2}\int_{Q_r(x,t)} |p_{11}||u|\,dyd\tau \;\le\; c(\rho/r)^{4/3}\,A_3(\rho)^{2/3}A_3(r)^{1/3}. \tag{2.10}
$$

Next observe for $z \in B_r(x)$ that

$$
|p_{12}(z,\tau)| \;\le\; c|p_{12}(x,\tau)| \le\; c\int_\rho^{2r_0} \sigma^{-4}\,(\int_{B_\sigma(x)} |u|^2\,(\cdot,\tau)\,dy)\,d\sigma\,.
$$

Thus

$$
r^{-2}\int_{Q_r(x,t)} |p_{12}||u|\,(\cdot,\tau)\,dyd\tau \;\le\; c\,(r/\rho)^2\,A_3(r)^{1/3}\sup_{\sigma\in[\rho,2r_0]} A_2(\sigma). \tag{2.11 a}
$$

Also from the above observation, Hölder's inequality and Minkowski's inequality for integrals (see [S]) we deduce

$$
\begin{aligned}
r^{-2}\int_{Q_r(x,t)} |p_{12}||u|(\cdot,\tau)\,dy\,d\tau &\le r^{-2}\left(\int_{Q_r(x,t)}|u|^3(\cdot,\tau)\,dyd\tau\right)^{1/3}\cdot\left(\int_{Q_r(x,t)} |p_{12}|^{3/2}(\cdot,\tau)\,dyd\tau\right)^{2/3} \\
&\le\; c\,r^{2/3}\,A_3(r)^{1/3}\left(\int_\rho^{2r_0}\sigma^{-3}\,(\int_{t-r^2}^{t+r^2}\int_{B_\sigma(x)} |u|^3\,dx\,d\tau)^{2/3}\,d\sigma\right) \\
&\le\; c(r/\rho)^{2/3}A_3(r)^{1/3}\sup_{\sigma\in[\rho,2r_0]} A_3(\sigma)^{2/3}.
\end{aligned}
\tag{2.11 b}
$$

Combining (2.10), (2.9), (2.11) we conclude that

$$\begin{aligned} A_4(r) \le\ & c\,(\rho/r)^{4/3}\, A_3(\rho)^{2/3} A_3(r)^{1/3} + \beta(r)\ A_2(r)^{1/2} \\ & + cA_3(r)^{1/3} \min\{(r/\rho)^{2/3} \sup_{\sigma\in[\rho,2r_0]} A_3(\sigma)^{2/3},\ (r/\rho)^2 \sup_{\sigma\in[\rho,2r_0]} A_2(\sigma)\}. \end{aligned} \tag{2.12}$$

Next we estimate L. From the arguments used in proving (2.10), (2.11) we see that

$$\begin{aligned} & r^{-13/4} \int_{t-r^2}^{t+r^2} \left(\int_{B_r(x)} |p_1|(\cdot,\tau) dx \right)^{5/4} dt \le cr^{-5/3} \left(\int_{Q_r(x,t)} (|p_{11}| + |p_{12}|)^{3/2}(\cdot,\tau)\, dx dt \right)^{5/6} \\ & \le c\,(\rho/r)^{5/3}\, A_3(\rho)^{5/6} + c\min\{(r/\rho)^{5/6} \sup_{\sigma\in[\rho,2r_0]} A_3(\sigma)^{5/6},\ (r/\rho)^{5/2} \sup_{\sigma\in[\rho,2r_0]} A_2(\sigma)^{5/4}\}. \end{aligned} \tag{2.13}$$

Also as in (2.9) we get

$$\begin{aligned} & r^{-13/4} \int_{t-r^2}^{t+r^2} \left(\int_{B_r(x)} (|p_2| + |p_3|)(\cdot,\tau) dx \right)^{5/4} dt \\ & \le cr^{1/2}\, r_0^{-3} \int_{t-r^2}^{t+r^2} \int_{B_{2r_0}(x)} (|p| + |u|^2)^{5/4}(\cdot,\tau) dx\, dt = \gamma(r). \end{aligned} \tag{2.14}$$

Combining (2.13), (2.14), we have

$$\begin{aligned} L(r) = r^{-13/4} \int_{t-r^2}^{t+r^2} \left(\int_{B_r(x)} |p|(\cdot,\tau) dx \right)^{5/4} dt \le\ & c\,(\rho/r)^{5/3}\, A_3(\rho)^{5/6} + \gamma(r) \\ + c\min\{(r/\rho)^{5/6} \sup_{\sigma\in[\rho,2r_0]} A_3(\sigma)^{5/6},\ (r/\rho)^{5/2} \sup_{\sigma\in[\rho,2r_0]} A_2(\sigma)^{5/4}\}. & \end{aligned} \tag{2.15}$$

Next we state Corollary 1 in [CKN] (see also [Li]).

Theorem B *There are absolute constants ϵ_1 and $\hat{c} > 0$, and a constant $\epsilon_2 > 0$ depending only on q with the following property. If u, p, f are as in the definition of a suitable weak solution, $Q_r(x,t) \subset D$ and*

$$A_3(r) + A_4(r) + L(r) \le \epsilon_1, \quad G(r) \le \epsilon_2,$$

then $|u| \le \hat{c}\, r^{-1}$ almost everywhere in $Q_{r/2}(x,t)$.

We note that $(x,t) \notin \mathcal{S}$ under the scenario of Theorem B. Also [CKN] state this result for parabolic cylinders with different centers but their result clearly implies Theorem B.

Finally in this section we note that if g is a real valued function on $(0, 2r_0]$ which is bounded on any closed subinterval of $(0, 2r_0]$ and if $\limsup_{r\to 0} g(r) = \infty$, then there exists a decreasing sequence of points $(s_k)_1^\infty$ in $(0, r_0]$ with

$$\begin{aligned} (*) \quad & s_k \to 0, g(s_k) \to \infty \text{ as } k \to \infty. \\ (**) \quad & 2g(s_k) \ge \sup_{s\in[s_k,2r_0]} g(s). \end{aligned} \tag{2.16}$$

(2.16) follows easily from the definition of supremum. In the applications we shall often take $g = A_2$.

3. Useful Lemmas. In this section we first prove

Lemma 1 *Fix $i, 1 \le i \le 3$, and suppose that $(x,t) \in F_i(M)$. There exists an absolute constant c_1 such that for $1 \le j \le 4$,*

$$\eta_j = \limsup_{r \to 0} A_j(r) \le c_1 \begin{cases} M+1 \text{ when } i=3 \\ M^3+1 \text{ when } i=2 \\ M^2+1 \text{ when } i=1. \end{cases}$$

Proof: We assume as we may that $M \ge 1$. Let $(x,t) \in F_i(M)$ and choose r_0 so small that

$$A_i(r) \le 2M \text{ for } 0 < r \le 2r_0. \tag{3.1}$$

We also suppose r_0 is so small that

$$G(r) \le 1, 0 < r \le r_0, \tag{3.2}$$

which is possible as we see from (2.7) and $q > 5/2$. We first prove Lemma 1 for $i = 3$. Observe from (1.4), (2.7), (2.2), (2.8), (3.2) and Hölder's inequality that for $0 < r \le r_0$,

$$A_1(r) + A_2(r) \le c(A_3(2r) + A_4(2r) + \min\{A_2(2r), A_3(2r)^{2/3}\} + 1). \tag{3.3}$$

whenever $Q_{2r}(x,t) \subset D$. Using (3.1), (3.3) and (2.12) with r, ρ replaced by $2r, 4r$ we get

$$A_1(r) + A_2(r) \le cM + \beta(2r)\, A_2(2r)^{1/2}. \tag{3.4}$$

We note for fixed r_0 that $\beta(2r) \to 0$ as $r \to 0$. Thus we may choose $r_1 \le r_0$ so that $\beta(2r) \le 1/4$ for $0 < r \le r_1$. We claim that (3.4) implies

$$\eta_2 = \limsup_{r \to 0} A_2(r) < \infty. \tag{3.5}$$

Indeed otherwise we can apply (2.16) with $g = A_2$. Let $(s_n)_1^\infty$ be as in (2.16). Using (3.4) with $r = s_n$, and (2.16) $(**)$, we see for large n that

$$A_2(s_n) \le cM + (1/2)A_2(s_n)$$

so that $A_2(s_n) \le 2cM$. Letting $n \to \infty$ we get a contradiction. Thus (3.5) is valid. Using (3.5) and letting $r \to 0$ through a properly chosen sequence in (3.4) we obtain $\eta_2 \le cM$. Using this conclusion in (2.12) it follows that Lemma 1 holds for $i = 3, j = 4$. Finally from (3.3) we conclude first that Lemma 1 is valid for $i = 3, j = 1$ and second that it is true for $i = 3, 1 \le j \le 4$.

Next we prove Lemma 1 for $i = 2$. Let r_0, r_1 be as above. From (2.6 b),(3.1) with r, ρ replaced by r^* we have

$$A_3(r^*) \le cM^{3/4} \max\{A_1(r^*), M\}^{3/4}. \tag{3.6}$$

Replace r^* by $2r, 4r$ in (3.6) and recall that $\beta(2r) \le 1/4$. Using these inequalities and (3.1) in (2.12) with r, ρ replaced by $2r, 4r$, we get

$$A_4(2r) \le cM^{3/4} \max\{A_1(2r), A_1(4r), M\}^{3/4}. \tag{3.7}$$

Putting (3.6), (3.7) in (3.3) we find that

$$A_1(r) \le cM^{3/4} \max\{A_1(2r), A_1(4r), M\}^{3/4}. \tag{3.8}$$

We conclude from (3.8) as in the proof of (3.5) that first $\eta_1 < \infty$ and thereupon that $\eta_1 \le cM^3$. Using this inequality in (3.6), (3.7) we obtain Lemma 1 for $i = 2$, $1 \le j \le 4$.

Finally we prove Lemma 1 for $i = 1$. We first show that (3.5) is valid. Again the proof of (3.5) is by contradiction using (2.16) with $g = A_2$. Let $(s_k)_1^\infty$ be as in (2.16). Note from (2.6 a) that if $\lambda = c^{-1}$ is small enough, $r/\rho = \lambda$, and $s_n \le r \le \rho \le r_1$, then

$$A_3(r) \le c\, MA_2(s_n)^{1/2} + (1/2)\, A_3(\rho). \tag{3.9}$$

Iterating this inequality we see for large n that

$$A_3(r) \le c\, MA_2(s_n)^{1/2} \tag{3.10}$$

whenever $s_n \le r \le \lambda^{-1} r_1$. Using this inequality in (2.12) with r, ρ replaced by $2s_n, 4s_n$ we conclude for large n that

$$A_4(2s_n) \le c\, MA_2(s_n)^{1/2} \tag{3.11}$$

Putting (3.11), (3.10) in (3.3) we obtain

$$A_2(s_n) \le c\, MA_2(s_n)^{1/2}$$

which clearly contradicts (2.16) when n is large. Thus (3.5) is true. Using this fact and arguing as in (3.9) we see that

$$\eta_3 \le c\, M\eta_2^{1/2} + (1/2)\eta_3$$

which implies $\eta_3 \le cM\eta_2^{1/2}$. Using this inequality in (2.12) we conclude that also $\eta_4 \le cM\eta_2^{1/2}$. Finally letting $r \to 0$ through a certain sequence in (3.3) we find from these estimates that $\eta_2 \le cM\eta_2^{1/2}$ which implies $\eta_2 \le cM^2$. Retracing our steps we see that this estimate also can be used for η_3, η_4. The proof of Lemma 1 is now complete. □

For ease of notation we write

$$N = N(i) = \begin{cases} c_1(M+1) \text{ when } i = 2, 3, \\ c_1(M^2+1) \text{ when } i = 1. \end{cases}$$

Then from Lemma 1 we have $\eta_2 \le N$. If $9/5 \le b < 2$, we also put

$$J(r) = J(r, x, t, b) = r^{2b-5} \int_{Q_r(x,t)} |\nabla u|^b dx dt \text{ whenever } Q_r(x,t) \subset D.$$

Next we prove

Lemma 2 *Fix $i, 1 \le i \le 3$, and suppose that $(x,t) \in F_i(M) \cap \mathcal{S}$. There exists $\delta = \delta(N, b) > 0$ so that $\liminf_{r \to 0} J(r) \ge \delta$. In fact for some absolute c_2 one can take*

$$c_2\, \delta = N^{(3b-25)/6}.$$

Proof. We shall use Theorem B to show for fixed $(x,t) \in F_i(M) \cap \mathcal{S}$ and ρ_1 small that if $\nu = J(\rho_1)$, then ν cannot be too small as otherwise Theorem B would imply (x,t) is not in $\mathcal{S}$. To begin we suppose ρ_1 is so small that $A_2(r) \le 2N$ for $r \le \rho_1$. and also that

$$G(r) \;\le\; \epsilon_2 \text{ and } \beta(r) + \gamma(r) \;\le\; \tfrac{\epsilon_1}{8\,(N+1)} \text{ for } 0 < r \le \rho_1. \tag{3.12}$$

Here c is the constant in (2.8) and ϵ_1, ϵ_2 are the constants in Theorem B. We first note that (2.6 c) can be written as

$$A_3(r) \;\le\; c(\rho/r)^2\, A_2(\rho)^{(3-b)/2}\, J(\rho) \;+\; c\,\min\{\,(r/\rho)^3 A_2(\rho)^{3/2}, (r/\rho)A_3(\rho)\,\}.$$

From the above inequalities we deduce for $r^* < \rho_1$ that

$$A_3(r^*) \;\le\; c\,(\rho_1/r^*)^2\,\nu N^{(3-b)/2} \;+\; c\,(r^*/\rho_1)^3\, N^{3/2}. \tag{3.13}$$

From (2.12), (2.15) we also have for $2r \le \rho \le \rho_1$,

$$A_4(r) \le c(\rho/r)^{4/3}\, A_3(\rho)^{2/3} A_3(r)^{1/3} \;+\; c(r/\rho)^2 N A_3(r)^{1/3} \;+\; \epsilon_1/4, \tag{3.14}$$

$$L(r) \;\;\le c(\rho/r)^{5/3}\, A_3(\rho)^{5/6} \;+\; + c(r/\rho)^{5/2} N^{5/4} \;+\; \epsilon_1/4. \tag{3.15}$$

We use (3.13)-(3.15) to obtain an estimate for the infimum of $A_3(r) + A_4(r)$ on $(0, \rho_1]$. Indeed for fixed r one sees that the minimum of (3.14) is essentially obtained when $r = N^{-3/10}\,\rho\, A_3(\rho)^{1/5}$. Using this expression for r in (3.13) and requiring that the right hand side for $A_3(r)$ be a constant multiple of the righthand side for $A_4(r)$ and vice versa, we get that the infimum of the sum of the righthand sides in these inequalities is essentially obtained when $r = \nu^{8/25} N^{-4b/25} \rho_1$ and $\rho = \nu^{1/5}\, N^{-b/10} \rho_1$ provided ν is small enough. For these values of r, ρ one computes

$$A_3(r) + A_4(r) \;\le\; c\, N^{(75-9b)/50}\, \nu^{9/25} + \epsilon_1/4$$

Also

$$L(r) \;\le\; c\, N^{(25-3b)/20}\, \nu^{3/10} \;+\; \epsilon_1/4.$$

Thus from Theorem B we must have

$$\epsilon_1^{6/5} \;\le\; c\, N^{(75-9b)/50}\, \nu^{9/25}. \tag{3.16}$$

Clearly (3.16) implies Lemma 2. □

4. **Proof of Theorem 1.** Let

$$\hat{N} \;=\; \hat{N}(i) \;=\; \begin{cases} c_1(M+1), i = 1, 3, \\ c_1(M^3+1), i = 2 \end{cases} \tag{4.1}$$

and observe from Lemma 1 that $\eta_1 \le \hat{N}$. Let $\delta = \delta(N, b), J = J(\cdot, b), 9/5 \le b < 2$, be as in Lemma 2 and for fixed $i, 1 \le i \le 3$, let H_k, $k = 1, 2, \ldots$ be the set of all $(x,t) \in F_i(M) \cap \mathcal{S}$ such that for $0 < r \le k^{-1}$,

$$\delta/2 \;\le J(r) \text{ and } A_1(r) \;\le\; 2\hat{N}. \tag{4.2}$$

Put $r_0 = k^{-1}$. From Lemmas 1 and 2 we see that $F_i(M) \cap \mathcal{S} = \bigcup H_k$ so by the countable subadditivity of Hausdorff measure it suffices to prove Theorem 1 for H_k. If $(x,t) \in H_k$, then from (4.2) we have

$$A_1(r_1) \leq 4\hat{N}\delta^{-1} J(r_2) \text{ whenever } 0 < r_1, r_2 \leq r_0. \tag{4.3}$$

Also if $\kappa = (3\delta/32\pi)^{1/b}$ and

$$K_1 = K_1(\rho, x, t, b) = \{(y,s) \in Q_\rho(x,t) : |\nabla u| \leq \kappa\rho^{-2}\},$$

$$K_2 = K_2(\rho, x, t, b) = Q_\rho(x,t) \setminus K_1,$$

then for $0 < \rho \leq r_0$, we have

$$\delta\,\rho^{5-2b}/2 \leq \int_{Q_\rho(x,t)} |\nabla u|^b \, dxdt = \int_{K_1} \ldots dxdt + \int_{K_2} \ldots dxdt \leq \delta\rho^{5-2b}/4 + \int_{K_2} \ldots dxdt.$$

Also from Hölder's inequality and (4.2) we see that

$$J(\rho) \leq (8\pi/3)^{(2-b)/2} A_1(\rho)^{b/2} \leq 3\hat{N}^{b/2}.$$

Combining these two inequalities we get

$$J(\rho) \leq 12\hat{N}^{b/2}\delta^{-1}\rho^{2b-5} \int_{K_2} |\nabla u|^b \, dxdt. \tag{4.4}$$

Next let

$$d(x,t) = \inf\{|s-t|^{1/2} + |x-y| : (y,s) \in H_k\}$$

$$E(r) = \{(x,t) \in \mathbb{R}^4 : d(x,t) < 2r\}.$$

Given $r, 0 < r \leq r_0$, we use a well known covering argument (see [S, ch 1]) to get $\{Q_{5r}(x_i,t_i)\}$ with

$$(x_i, t_i) \in H_k$$

$$H_k \subset \textstyle\bigcup_i Q_{5r}(x_i,t_i) \tag{4.5}$$

$$Q_r(x_m,t_m) \cap Q_r(x_n,t_n) = \emptyset, \; m \neq n.$$

We note from the definition of d and (4.5) that

$$E(r) \subset \cup_i Q_{10r}(x_i,t_i). \tag{4.6}$$

Using (4.3)- (4.6) we see that if $E_1(r) = \{(x,t) : |\nabla u|(x,t) > \kappa r^{-2}\} \cap E(r)$, then

$$\begin{aligned}\int_{E(r)} |\nabla u|^2 \, dxdt &\leq \sum_i \int_{Q_{10r}(x_i,t_i)} |\nabla u|^2 \, dxdt \leq 40\hat{N}\delta^{-1}r^{2b-4}\sum_i \int_{Q_r(x_i,t_i)} |\nabla u|^b \, dxdt \\ &\leq 480\hat{N}^{(2+b)/2}\delta^{-2}r^{2b-4}\sum_i \int_{K_2(r,x_i,t_i,b)} |\nabla u|^b \, dxdt \leq 480\hat{N}^{(2+b)/2}\delta^{-2}r^{2b-4}\int_{E_1(r)} |\nabla u|^b \, dxdt.\end{aligned} \tag{4.7}$$

We multiply each side of (4.7) by $r^{-(1+\epsilon)}$ and integrate from n^{-1} to r_0 where $n > k$. If $d_n(x,t) = \max\{d(x,t), n^{-1}\}$ we obtain upon interchanging the order of integration (permissible by the Tonelli theorem) that

$$\int_{n^{-1}}^{r_0} r^{-(1+\epsilon)} \left(\int_{E(r)} |\nabla u|^2 \, dxdt \right) dr \;=\; \int_{E(r_0)} \epsilon^{-1}[d_n(x,t)^{-\epsilon} \;-\; r_0^{-\epsilon}] \, |\nabla u|^2 \, dxdt$$

$$\leq\; 480 \hat{N}^{(2+b)/2} \delta^{-2} \int_{n^{-1}}^{r_0} r^{2b-\epsilon-5} \left(\int_{E_1(r)} |\nabla u|^b \, dxdt \right) dr \tag{4.8}$$

$$\leq\; \frac{480 \hat{N}^{(2+b)/2}}{\delta^2(4+\epsilon-2b)} \int_{E(r_0)} (\min\{|\nabla u|/\kappa,\, d_n^{-2}\})^{(4-2b+\epsilon)/2} \;\; |\nabla u|^b \, dx \, dt.$$

Now

$$d_n^{-\epsilon} \, |\nabla u|^2 \;\geq\; \kappa^{2-b} \left(\min\{ \, |\nabla u|/\kappa, d_n^{-2} \, \} \right)^{(4-2b+\epsilon)/2} \; |\nabla u|^b.$$

Hence if

$$\epsilon = \frac{\kappa^{2-b} \, (4-2b) \, \delta^2}{1000 \hat{N}^{(2+b)/2}} \tag{4.9}$$

we can multiply (4.8) by ϵ and absorb the resulting righthand side into the lefthand side. We get

$$\int_{E(r_0)} d_n(x,t)^{-\epsilon} \, |\nabla u|^2 \, dxdt \;\leq\; r_0^{-\epsilon} \int_{E(r_0)} |\nabla u|^2 \, dxdt.$$

Letting $n \to \infty$ and using the monotone convergence theorem it follows that

$$\int_{E(r_0)} d(x,t)^{-\epsilon} \, |\nabla u|^2 \, dxdt \;\leq\; r_0^{-\epsilon} \int_{E(r_0)} |\nabla u|^2 \, dxdt. \tag{4.10}$$

To get Theorem 1 observe from (4.2) with $b = 2$, (4.5), (4.6), and (4.10) that for $h(t) = t^{1-\epsilon}$ and $\Psi_{5r} = \Psi_{5r}(\cdot, h)$ defined as in section 1 that

$$\Psi_{5r}(H_k) \;\leq\; 100\delta^{-1} \sum_i r^{-\epsilon} \int_{Q_r(x_i,t_i)} |\nabla u|^2 \, dxdt \;\leq\; 100\delta^{-1} \int_{E(r_0)} d^{-\epsilon} \, |\nabla u|^2 \, dxdt. \tag{4.11}$$

Letting first $r \to 0$ and then $r_0 \to 0$, we get $\Lambda_{1-\epsilon}(H_k) = 0$. Using (4.9), the definition of N, δ in Lemma 2 and the definition of $\hat{N}$ in (4.1) we get the estimate for ϵ in Theorem 1. The proof of Theorem 1 is now complete. $\Box$.

5. Several Lemmas and Theorem 2. In the proof of Theorem 2 we shall need two lemmas analogous to Lemmas 1 and 2. We begin with

Lemma 3 *Let $0 < \sigma \leq 1$, and set*

$$\hat{F}(\sigma) = \{(x,t) : \limsup_{r \to 0} \frac{A_1(r)}{(\log 1/r)^\sigma} \;\leq\; 1\}.$$

If $(x,t) \in \hat{F}(\sigma)$, then for some absolute c_3 we have

$$\limsup_{r \to 0} \frac{A_2(r)}{(\log 1/r)^{2\sigma}} \leq c_3.$$

Proof: We proceed as in the case $i = 1$ of Lemma 1. Again the proof is by contradiction. If Lemma 3 is false, let $r_1 < r_0 \le 1/2$ be as in this lemma and suppose $A_1(r) \le 2(\log 1/r)^\sigma$ for $0 < r \le \rho_1$. Put $g(r) = (\log 1/r)^{-2\sigma} A_2(r), 0 < r \le r_0$ and let $(s_n)_1^\infty$ be as in (2.16). From (2.6a) we deduce that if $c = \lambda^{-1}$ is small enough, $r/\rho = \lambda$, and $s_n \le r \le \rho \le r_1$, then

$$A_3(r) \le c(\log 1/s_n)^\sigma A_2^{1/2}(s_n) + A_3(\rho)/2. \tag{5.1}$$

Iterating this inequality we see for large n that

$$A_3(r) \le c(\log 1/s_n)^\sigma A_2(s_n)^{1/2} \tag{5.2}$$

Inserting (5.2) in (2.12) with r replaced by $2s_n, 4s_n$ we conclude for large n that

$$A_4(2s_n) \le c(\log 1/s_n)^\sigma A_2(s_n)^{1/2}. \tag{5.3}$$

Putting (5.2), (5.3) into (3.3) we obtain

$$A_2(s_n) \le c(\log 1/s_n)^\sigma A_2(s_n)^{1/2}$$

which implies $g(s_n) \le c$, a contradiction. Thus

$$\limsup_{r \to 0} \frac{A_2(r)}{(\log 1/r)^{2\sigma}} = \alpha < \infty.$$

Using this inequality and arguing as in (5.1) we deduce first that

$$\limsup_{r \to 0} \frac{A_3(r)}{(\log 1/r)^{2\sigma}} \le c\alpha^{1/2}$$

and second from (2.12) as in (5.3) that also

$$\limsup_{r \to 0} \frac{A_4(r)}{(\log 1/r)^{2\sigma}} \le c\alpha^{1/2}.$$

Using (3.3) to estimate A_2 in terms of A_3, A_4 we see from the above two inequalities that

$$\alpha \le c\,\alpha^{1/2}.$$

The proof of Lemma 3 is now complete. □

Remark. We could have proved a more general version of Lemma 3 with $\hat{F}_i(\sigma)$ replacing $\hat{F}(\sigma)$ in the spirit of Lemma 1. However the above is all we need for the proof Theorem 2. Next we prove an analogue of Lemma 2.

Lemma 4 *Let $(x,t) \in \hat{F}(\sigma) \cap \mathcal{S}$ and $9/5 \le b < 2$. Define J relative to b as above Lemma 2. Then for some absolute c_4, we have*

$$\liminf_{r \to 0} [(\log 1/r)^{(25-3b)\sigma/3} J(r)] \ge c_4^{-1}.$$

Proof: Let ρ_1, ν be as in Lemma 2. We essentially just copy the proof of Lemma 2 with N replaced by $(\log 1/\rho_1)^{2\sigma}$. We assume as we may that $A_2(r) \leq c(\log 1/r)^{2\sigma}$ for $0 < r \leq \rho_1 \leq r_1/4$, thanks to Lemma 3. In place of (3.12) we assume that

$$G(r) \leq \epsilon_2 \text{ and } \gamma(r) + \beta(r)\, A_2(r)^{1/2} \leq \epsilon_1/4 \quad \text{for } 0 < r \leq \rho_1,$$

which is permissible since $G(r), \gamma(r)$, and $r^{-2/5}\,\beta(r) \to 0$, as $r \to 0$. Using these inequalities in the display above (3.13) with $\rho = \rho_1$ we see that

$$A_3(r^*) \leq c\,(\rho_1/r^*)^2 \nu (\log 1/\rho_1)^{(3-b)\sigma} + c\,(r^*/\rho_1)^3\,(\log 1/\rho_1)^{3\sigma}. \tag{5.4}$$

Moreover for $\rho_1^2 \leq 2r \leq \rho \leq \rho_1$ we can write as in (3.14), (3.15).

$$A_4(r) \leq c(\rho/r)^{4/3}\, A_3(\rho)^{2/3} A_3(r)^{1/3} + c(r/\rho)^2 (\log 1/\rho_1)^{2\sigma} A_3(r)^{1/3} + \epsilon_1/4, \tag{5.5}$$

$$L(r) \leq c(\rho/r)^{5/3}\, A_3(\rho)^{5/6} + c(r/\rho)^{5/2} (\log 1/\rho_1)^{5\sigma/2} + \epsilon_1/4. \tag{5.6}$$

Arguing as following (3.15) we see that if

$$r/\rho_1 = \nu^{8/25} (\log 1/\rho_1)^{-8b\sigma/25}$$

$$\rho/\rho_1 = \nu^{1/5} (\log 1/\rho_1)^{-b\sigma/5} \rho_1$$

then for these values of r, ρ one computes in (5.4) - (5.6),

$$A_3(r) + A_4(r) \leq c(\log 1/\rho_1)^{(75-9b)\sigma/25} \nu^{9/25} + \epsilon_1/4.$$

Similarly

$$L(r) \leq c\,(\log 1/\rho_1)^{(75-9b)\sigma/30}\, \nu^{3/10} + \epsilon_1/4.$$

Using these inequalities and Theorem B we conclude that $A_1(\rho_1) \geq c^{-1} (\log 1/r)^{(3b-25)\sigma/3}$ which implies Lemma 4. $\square$

Proof of Theorem 2. Armed with Lemmas 3 and 4 we can now use the same game plan as in the proof of Theorem 1. To begin let G_k, $k = 1, 2, \ldots$ be the set of all $(x,t) \in \hat{F}(\sigma) \cap \mathcal{S}$ such that for $0 < r \leq k^{-1}$,

$$c_4^{-1}/2 \leq (\log 1/r)^{(25-3b)\sigma/3}\, J(r) \text{ and } A_1(r) \leq 2\,(\log 1/r)^{\sigma}. \tag{5.7}$$

From Lemma 4 we see that $\hat{F}(\sigma) \cap \mathcal{S} = \bigcup G_k$. Put $r_0 = k^{-1}$ and observe from (5.7) that for $(x,t) \in G_k$,

$$A_1(r_1) \leq c\,(\log 1/r_1)^{\sigma}\, (\log 1/r_2)^{(25-3b)\sigma/3}\, J(r_2) \text{ whenever } 0 < r_1, r_2 \leq r_0. \tag{5.8}$$

Also if c_5 is large enough and

$$\hat{K}_1 = \hat{K}_1(\rho, x, t, b) = \{(y,s) \in Q_\rho(x,t) : |\nabla u| \leq (\log 1/\rho)^{(3b-25)\sigma/3b}\, (c_5 \rho)^{-2}\},$$

$$\hat{K}_2 = \hat{K}_2(\rho, x, t, b) = Q_\rho(x,t) \setminus \hat{K}_1,$$

then for $0<\rho\le r_0$, we have

$$\rho^{5-2b}(\log 1/\rho)^{(3b-25)\sigma/3} \;\le\; c\int_{Q_\rho(x,t)} |\nabla u|^b\,dxdt \;=\; c\int_{\hat K_1}\dots dxdt \;+\; c\int_{\hat K_2}\dots dxdt$$

$$\le\; (1/2)\rho^{5-2b}\,(\log 1/\rho)^{(3b-25)\sigma/3} \;+\; c\int_{\hat K_2}\dots dxdt.$$

Moreover from (5.7) and Hölder's inequality we see that

$$J(\rho)\;\le\; c\,A_1(\rho)^{b/2}\;\le\; c\,(\log 1/\rho)^{b\sigma/2}\;. \tag{5.9}$$

Combining these two inequalities we get

$$J(\rho)\le\, c\,\rho^{2b-5}\,(\log 1/\rho)^{(50-3b)\sigma/6}\int_{\hat K_2}|\nabla u|^b\,dxdt. \tag{5.10}$$

Let

$$\hat d(x,t)\;=\;\inf\{|s-t|^{1/2}\;+\;|x-y|\;:(y,s)\in G_k\,\}$$

$$\hat E(r)\;=\;\{(x,t)\in I\!\!R^4 : \hat d(x,t)<2r\}$$

and choose $\{Q_{5r}(x_i,t_i)\}$ satisfying (4.5), (4,6) with H_k, d, E replaced by $G_k, \hat d, \hat E$. Using (4.5), (4.6), (5.8) and (5.10) it follows that if

$$\hat E_1(r)\;=\;\{(y,s)\in D:\;(\log 1/r)^{(3b-25)\sigma/3b}\,(c_5 r)^{-2}<\;|\nabla u|(y,s)\}\cap\hat E(r)$$

then

$$\int_{\hat E(r)}|\nabla u|^2\,dxdt\;\le\;\sum_i\int_{Q_{10r}(x_i,t_i)}|\nabla u|^2\,dxdt\;\le\;c\,r^{2b-4}\,(\log 1/r)^{(28-3b)\sigma/3}\sum_i\int_{Q_r(x_i,t_i)}|\nabla u|^b\,dxdt$$

$$\le\;c\,r^{2b-4}\,(\log 1/r)^{(106-9b)\sigma/6}\sum_i\int_{\hat K_2(r,x_i,t_i,b)}|\nabla u|^b\,dxdt$$

$$\le\;c\,r^{2b-4}(\log 1/r)^{(106-9b)\sigma/6}\int_{\hat E_1(r)}|\nabla u|^b\,dxdt. \tag{5.11}$$

We multiply each side of (5.11) by r^{-1} and integrate from n^{-1} to r_0 where $n>k$. If $\hat d_n(x,t)=\max\{\hat d(x,t),n^{-1}\}$ we obtain upon interchanging the order of integration that for small enough σ

$$\int_{n^{-1}}^{r_0} r^{-1}\left(\int_{\hat E(r)}|\nabla u|^2\,dxdt\right)dr\;=\;\int_{E(r_0)}\log(r_0/\hat d_n)\,|\nabla u|^2\,dxdt$$

$$\le\;c\int_{n^{-1}}^{r_0} r^{2b-5}\,(\log 1/r)^{(106-9b)\sigma/6}\left(\int_{\hat E_1(r)}|\nabla u|^b\,dxdt\right)dr$$

$$\le\;c(2-b)^{-1}\int_{E(r_0)}\Big(\min\{|\nabla u|(\log|\nabla u|)^\alpha\,,\;\hat d_n^{-2}(\log 1/\hat d_n)^{(106-9b)\sigma/[6(2-b)]}\}\Big)^{(2-b)}\;|\nabla u|^b\,dx\,dt \tag{5.12}$$

where

$$\alpha = \sigma \left(\frac{25 - 3b}{3b} + \frac{106 - 9b}{6(2-b)} \right).$$

Assume σ is so small that

$$\alpha(2-b) < 1. \tag{5.13}$$

Then we can essentially absorb the right hand side of (5.12) into the left hand side. To see this note that if $\hat{d}_n^{-2} \leq |\nabla u|$, then

$$\left(\min\{|\nabla u|(\log|\nabla u|)^{\alpha}, \ \hat{d}_n^{-2}(\log 1/\hat{d}_n)^{(106-9b)\sigma/[6(2-b)]}\}\right)^{(2-b)} \ |\nabla u|^b \ \leq \ c\,|\nabla u|^2(\log 1/\hat{d}_n)^{(106-9b)\sigma/6}$$

while if $|\nabla u| < \hat{d}_n^{-2}$, then

$$\left(\min\{|\nabla u|(\log|\nabla u|)^{\alpha}, \ \hat{d}_n^{-2}(\log 1/\hat{d}_n)^{(106-9b)\sigma/[6(2-b)]}\}\right)^{(2-b)} \ |\nabla u|^b \ \leq \ c\,|\nabla u|^2\,(\log 1/\hat{d}_n)^{\alpha(2-b)}.$$

In either case we see that there is a constant $c^*(\alpha, b, r_0)$ such that if $|\nabla u|(y,s) > c^*$, then the integrand on the righthand side of (5.12) evaluated at (y,s) is at most one half of the integrand on the lefthand side evaluated at (y,s). Thus if (5.13) holds then

$$\int_{E(r_0)} (\log 1/\hat{d}_n)\,|\nabla u|^2\,dxdt \ \leq \ c(\alpha, b, r_0) \ < \ \infty.$$

Letting $n \to \infty$ and using the monotone convergence theorem it follows that

$$\int_{E(r_0)} (\log 1/\hat{d})\,|\nabla u|^2\,dxdt \ < \ \infty. \tag{5.14}$$

Next let $a = (3-19\sigma)/3$ and put $h_1(t) = t\,(\log e/t)^a$ for $0 < t \leq 1$. Now from (5.7) with $b = 2$, (4.5), (4.6), and (5.14) we see that if $\Psi_{5r} = \Psi_{5r}(\cdot, h_1)$ is defined as in section 1 then for $0 < r \leq r_0 \leq 1/2$,

$$\Psi_{5r}(G_k) \ \leq \ c\,(\log 1/r) \sum_i \int_{Q_r(x_i,t_i)} |\nabla u|^2\,dxdt \ \leq \ \int_{E(r_0)} (\log 1/\hat{d})\,|\nabla u|^2\,dxdt < \infty. \tag{5.15}$$

Letting $r_0 \to 0$ we conclude that $\Lambda(G_k, h_1) = 0$ when (5.13) holds. To get Theorem 2 we note that if $h(t) = t\,(\log e/t)^{\sigma}$, then it is easily seen that $\Lambda(\mathcal{S} \setminus \hat{F}(\sigma), h) = 0$. Indeed in this case one can estimate $\Psi_{\delta}(\mathcal{S} \setminus \hat{F}(\sigma))$ by the integral of $|\nabla u|^2$ over a certain set in such a way that the measure of the set tends to zero as $\delta \to 0$. We conclude from the above note and the countable subadditivity of Hausdorff measure that if (5.13) holds and $a \geq \sigma$, then

$$\Lambda(\mathcal{S}, h) = 0. \tag{5.16}$$

Now the equation $\alpha(2-b) = 1$ defines σ as a function of b on $[9/5\ ,\ 2)$. Moreover this function is easily seen to be increasing and $a - \sigma$ is bounded below by a positive constant on $[9/5,\ 2)$. Evaluating this function as $b \to 2$, we see by continuity that (5.13) holds whenever $\sigma < 3/44$. In view of (5.16), the proof of Theorem 2 is now complete. $\Box$

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