On weak reverse Hölder inequalities for nondoubling harmonic measures

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ABSTRACT: In this note we point out that theorems of David - Jerison and Semmes for harmonic measure on the boundary of an NTA Ahlfors regular type domain can be extended to more general domains.

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1. Introduction.[1] Denote points in Euclidean n-space $\mathbb{R}^n$ by $x = (x_1, \ldots, x_n)$ and let $\bar{E}, \partial E$ denote the closure and boundary of $E \subset \mathbb{R}^n$, respectively. Let $|x|$ be the norm of x and let $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$. Define k dimensional Hausdorff measure, $0 < k \leq n$, in $\mathbb{R}^n$ as follows: For fixed $\delta > 0$ and $E \subseteq \mathbb{R}^n$, let $L(\delta) = \{B(x_i, r_i)\}$ be such that $E \subseteq \bigcup B(x_i, r_i)$ and $0 < r_i < \delta, \quad i = 1, 2, \ldots$ Set

$$\phi_\delta^k(E) = \inf_{L(\delta)} \left(\sum \alpha(k)\, r_i^k \right),$$

where $\alpha(k)$ denotes the volume of the unit ball in $\mathbb{R}^k$. Then

$$H^k(E) = \lim_{\delta \to 0} \phi_\delta^k(E), \quad 0 < k \leq n.$$

We write dx for dH^n.

Given a bounded domain, D (i.e a connected bounded open subset of $\mathbb{R}^n$), let diam $D = \sup\{|x-y| : x, y \in D\}$ and let $d(x)$ denote the distance from $x \in D$ to ∂D. ∂D is said to be Ahlfors regular provided there exists $1 \leq M < \infty$ such that

$$M^{-1}\, r^{n-1} \;\leq\; H^{n-1}(\partial D \cap B(x,r)) \;\leq\; M\, r^{n-1} \tag{1.1}$$

whenever $0 < r \leq$ 2 diam D and $x \in \partial D$. We say that D satisfies a uniform interior ball condition provided for some $\delta_1 > 0$ the following statement is true. If $x \in \partial D$ and $0 < r \leq 2$ diam D, there exists

$$B(\hat{x}, r') \subset B(x,r) \cap D \text{ with } r' \geq \delta_1 r. \tag{1.2}$$

D satisfies a uniform exterior plane condition provided there exists $\delta_2 > 0$ such that whenever $x \in \partial D$ and $0 < r \leq 2$ diam D, we have

$$B(\hat{x}, r') \cap P \subset B(x,r) \cap (\mathbb{R}^n \setminus \bar{D}) \text{ for some } \hat{x} \text{ in the hyperplane } P \text{ and } r' \geq \delta_2 r. \tag{1.3}$$

Next we say that D satisfies a Harnack chain condition provided for each $L > 0$ there is a constant $c(L) \geq 1$ such that if $y, z \in D$, with $|y - z| \leq L \min\{d(y), d(z)\}$, then there is a chain $\{B(z_i, d(z_i)/2)\}_1^N$ with $y \in B(z_1, d(z_1)/2)$, $z \in B(z_N, d(z_N)/2)$, and $B(z_i, d(z_i)) \subset D$ for $1 \leq i \leq N$. Moreover

$$B(z_i, d(z_i)/2) \cap B(z_{i+1}, d(z_{i+1})/2) \neq \emptyset, \text{ for } 1 \leq i \leq N-1 \text{ while } N \leq c(L). \tag{1.4}$$

Finally we shall say that ∂D satisfies a uniform interior big pieces of Lipschitz graphs condition provided there exists $a_1, a_2 \geq 1$ such that whenever $x \in D$ we can find a domain Ω with

(a) $\quad \Omega \subset B(x, 20\, d(x)) \cap D,$

(b) $\quad$ If $w' = (w_1, \ldots, w_{n-1}), x = (x', x_n)$, then after a rotation of coordinates if necessary

$$\Omega = \{w = (w', w_n) : \psi(w') < w_n < x_n + d(x)/2,\ |x' - w'| < d(x)/a_1\},$$

(c) $\quad \|\psi\|_\infty = \sup_{z' \in \mathbb{R}^{n-1}} |\psi(z')| \leq x_n - d(x)/a_1,$

$$\tag{1.5}$$

[1]Dedicated to the memory of Matts Essén and his contributions to the study of harmonic measure.

for some $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$ with

$$|\psi(y') - \psi(z')| \leq a_1 \, |y' - z'| \text{ for all } y', z' \in \mathbb{R}^{n-1}. \tag{1.6}$$

Also

$$H^{n-1}(\{(w', \psi(w')) : |x' - w'| \leq d(x)/a_1\} \cap \partial D) \geq d(x)^{n-1}/a_2. \tag{1.7}$$

Given a bounded domain $D \subset \mathbb{R}^n, n \geq 2$, and $w \in D$, let $\omega(\cdot, w)$ denote harmonic measure for ∂D with respect to w, defined as usual by way of the Riesz representation theorem for continuous functions on ∂D and a method of Perron-Wiener-Brelot (see [H]). We extend $\omega(\cdot, w)$ to a Borel measure on $\mathbb{R}^n$ in the usual way by putting $\omega(E, w) = \omega(E \cap \partial D, w)$ whenever $E \subset \mathbb{R}^n$ is Borel. $\omega(\cdot, w)$ is said to be a doubling measure provided there exists $L < \infty$ such that

$$\omega(B(x, r), w) \leq \; L \, \omega(B(x, r/2), w) \tag{1.8}$$

whenever $x \in \partial D$ and $0 < r \leq 2$ diam D. Given $x \in \partial D, r > 0$, we say that $\omega(\cdot, w)$ is in a reverse Hölder class with respect to $H^{n-1}|_{B(x,r) \cap \partial D}$ provided there is a Borel measurable function h defined on $B(x, r) \cap \partial D$ with $d\omega(\cdot, w) = h \, dH^{n-1}$ and $1 \leq L < \infty, \lambda > 0$, such that

$$\int_{B(y,\rho) \cap \partial D} h^{1+\lambda} dH^{n-1} \; \leq \; L \, \rho^{(1-n)\lambda} \left(\int_{B(y,\rho) \cap \partial D} h \, dH^{n-1} \right)^{1+\lambda} = L \, \rho^{(1-n)\lambda} \, \omega[B(y, \rho)]^{1+\lambda} \tag{1.9}$$

whenever $y \in \partial D$ and $B(y, \rho) \subset B(x, r)$. We now state an equivalent version of Theorem 2 in [DJ].

Theorem A. *Let D be a bounded domain satisfying (1.1)-(1.4). Then $\omega(\cdot, w)$ is a doubling measure and $\omega(\cdot, w)$ is in a reverse Hölder class with respect to $H^{n-1}|_{B(x,r) \cap \partial D}$ whenever $x \in \partial D, 0 < r$, and $w \in D \setminus B(x, 4r)$.*

We note that the conclusion of Theorem A is also obtained in [S] when (1.3) is replaced by a uniform exterior ball condition and $\mathbb{R}^n \setminus \partial D$ contains exactly two components. To outline the proof of Theorem A (see the paragraph after (10) in [DJ]), the above authors first show

Theorem B. *If D is a bounded domain satisfying (1.1) - (1.3), then ∂D satisfies a uniform interior big pieces of Lipschitz graphs condition. a_1, a_2 depend only on M, δ_1, δ_2 and n.*

Using Theorem B, a theorem in [D] for harmonic measure in a Lipschitz domain, and the maximum principle for Laplace's equation, they then obtain the validity of the following statement. There exists θ_1, θ_2, $0 < \theta_1, \theta_2 < 1$, depending only on M, δ_1, δ_2, n, such that if $x \in D$ and $E \subset B(x, 20d(x)) \cap \partial D$ is Borel with

$$\frac{H^{n-1}(E)}{H^{n-1}[B(x, 20d(x)) \cap \partial D]} \geq 1 - \theta_1, \tag{1.10}$$

then

$$\omega(E, x) \; \geq \; \theta_2. \tag{1.11}$$

Using (1.10) implies (1.11) and comparison results for harmonic measure from [JK], one can conclude as in [DJ] that $\omega(\cdot, w)$ satisfies an A_∞ type condition. That is, there exists $\phi_1, \phi_2, 0 < \phi_1, \phi_2 < 1$ such that whenever $z \in \partial D, 0 < t, w \in D \setminus B(z, 2t)$, and $E \subset B(z,t) \cap \partial D$ is Borel with

$$\frac{H^{n-1}(E)}{H^{n-1}[B(z,t) \cap \partial D]} \geq 1 - \phi_1, \tag{1.12}$$

then

$$\frac{\omega(E, w)}{\omega[B(z,t) \cap \partial D, w]} \geq \phi_2. \tag{1.13}$$

It is well known that this A_∞ type condition and the fact that $\omega(\cdot, w)$ is a doubling measure implies the conclusion of Theorem A (see for example [CF]). We remark that Theorem B does not require the chain assumption (1.4). Moreover (1.4) is used in an essential way in Theorem A. It is needed for example to insure the doubling propery of $\omega(\cdot, w)$ and also for comparison estimates between certain harmonic measures. In this paper we show that if (1.4) is not assumed in Theorem A, then $\omega(\cdot, w)$ still satisfies a weak reverse Hölder inequality. More specifically we prove

Theorem 1. *Let D be a bounded domain satisfying (1.1), (1.2) and a uniform interior big pieces of Lipshitz graphs condition (see (1.5) - (1.7)). There exists $1 \leq L < \infty, \lambda > 0$ (depending only on M, δ_1, δ_2, n) such that if $x \in \partial D, 0 < r$, and $w \in D \setminus B(x, 4r)$, then $d\omega(\cdot, w)/dH^{n-1} = h$ on $\partial D \cap B(x,r)$ with*

$$\int_{B(y,\rho)\cap\partial D} h^{1+\lambda} dH^{n-1} \leq L\, \rho^{(1-n)\lambda} \left(\int_{B(y,2\rho)\cap\partial D} h dH^{n-1} \right)^{1+\lambda} = L\, \rho^{(1-n)\lambda}\, \omega[B(y, 2\rho)]^{1+\lambda} \tag{1.14}$$

whenever $y \in \partial D$ and $B(y, 2\rho) \subset B(x,r)$.

We note that $B(y, 2\rho)$ occurs on the righthand side of (1.14) (instead of $B(y,\rho)$) which is the reason this inequality is called a weak reverse Hölder inequality. Also observe that Theorem 1 implies the conclusion of Theorem A when ω is a doubling measure. To outline our proof we use (1.10), (1.11), but instead of showing that (1.12) implies (1.13) we prove an inequality in Lemma 2.2 which we can then use in Lemma 3.1 to get Theorem 1. Finally we remark that given Theorem B, our approach seems easier and more natural to us, as it essentially just uses the maximum principle and Hölder continuity results for positive harmonic functions in D vanishing on a portion of ∂D. Similar arguments for nondoubling parabolic measures on the boundary of a halfspace can be found in [HL, ch 2]. Finally in section 4 we give an example of a domain D satisfying (1.1) - (1.3) for which harmonic measure (with respect to any point in D) is not doubling and make some remarks concerning results related to Theorem 1.

2. Proof of the Main Lemma. In the sequel, unless otherwise specified, we let $c = c(\alpha, \beta, \gamma, \delta) \geq 1$ be a constant which may depend only on $\alpha, \beta, \gamma, \delta$, not necessarily the same at each occurrence. We shall need the following lemma.

Lemma 2.1. *Let D, ω be as in Theorem 1. If $z \in \partial D, r > 0$, then there exists $1 \leq c_1 < \infty$,* *depending only on n, M, such that*

$$(\alpha) \quad c_1\,\omega(B(z,r),x) \geq 1 \text{ whenever } x \in B(z,r/2)\cap D$$

$$(\beta) \quad \lim_{x\longrightarrow y} \omega(B(z,r),x) = 1 \text{ whenever } y \in B(z,r)\cap \partial D$$

$$(\gamma) \quad \lim_{x\longrightarrow y} \omega(B(z,r),x) = 0 \text{ whenever } y \in \partial D \setminus \bar{B}(z,r).$$

Proof: Using (1.1) it is easily shown that $B(z,r)\cap\partial D$ and $B(z,r)$ have comparable Newtonian capacities (logarithmic capacities when $n=2$). This fact and well known Wiener type estimates for subharmonic functions (see [H], [L]) imply Lemma 2.1. □

Next we prove our main lemma.

Lemma 2.2. *Let D be as in Theorem 1. Given $\epsilon > 0$, there exists $\eta = \eta(\epsilon, M, \delta_1, \delta_2, n), 0 < \eta < 1$, and $c(\epsilon) = c(\epsilon, M, \delta_1, \delta_2, n) \geq 1$, such that if $z \in \partial D, 0 < r$, and $F \subset \partial D \cap B(z, 2r)$ is Borel with $H^{n-1}(F) \geq (1-\eta)H^{n-1}(\partial D\cap B(z,2r))$, then*

$$\omega(B(z,r),w) \leq \epsilon\,\omega(B(z,2r),w) \,+\, c(\epsilon)\,\omega(F,w) \tag{2.3}$$

whenever $w \in D\setminus B(z,4r)$.

Proof: Clearly it suffices to prove Lemma 2.2 for $\epsilon > 0$ small, say $0 < \epsilon \leq \epsilon_0$. We shall use the fact that (1.10) implies (1.11). For completeness we show how this implication follows from (1.5) - (1.7). Indeed, given $x \in D$, there exists a Lipschitz domain Ω satisfying (1.5)-(1.7) for an appropriate Lipschitz ψ. If $\theta_1 > 0$ is small enough in (1.10), then from (1.7) we see that

$$H^{n-1}(\{(w',\psi(w')) : |x'-w'| \leq d(x)/a_1\}\cap E) \geq d(x)^{n-1}/(2a_2). \tag{2.4}$$

If ω_Ω denotes harmonic measure for Ω, then it follows from (2.4), (1.5), (1.6), and [D] that for some $c = c(a_1, a_2, n) \geq 1$,

$$c\,\omega_\Omega(E\cap\partial\Omega, x) \geq 1.$$

Using the maximum principle for harmonic functions we deduce first that

$$\omega_\Omega(E\cap\partial\Omega,\cdot) \leq \omega(E\cap\partial\Omega,\cdot)$$

and second from the above inequality that (1.10) implies (1.11). To continue the proof of Lemma 2.2, let A, $1 \leq A \leq \epsilon_0^{-1/2}$, be a large positive constant to be chosen later and let j be the greatest integer $\leq A/\epsilon$. Put

$$U_k = \{y : (\tfrac{5}{4}+\tfrac{k}{4j})r < |y-z| < (\tfrac{5}{4}+\tfrac{k+1}{4j})r\} \text{ and } S_k = \{y\in D : |y-z| = (\tfrac{5}{4}+\tfrac{k+1/2}{4j})r\} \tag{2.5}$$

for $1 \leq k \leq j-1$. Let $\epsilon' = \epsilon/A$ and first suppose that there exists $x \in S_k$ with $d(x) = (\epsilon'/100)r$. In this case we note that for $\eta(\epsilon) > 0$ small enough, we have

$$H^{n-1}[F\cap B(x,20d(x))] \,\geq (1-\theta_1)\,H^{n-1}[\partial D\cap B(x,20d(x))]$$

where θ_1 is as in (1.10). Applying (1.11) we conclude for some $\hat{c}(\epsilon) \geq 1$ (with the same dependence as $c(\epsilon)$ in Lemma 2.2) that

$$\hat{c}(\epsilon)\, \omega(F, x) \geq 1.$$

Using Harnack's inequality for positive harmonic functions it follows from this inequality that for some $\tilde{c}(\epsilon) \geq 1$ (with the same dependence as $\hat{c}(\epsilon)$)

$$\tilde{c}(\epsilon)\, \omega(F, \cdot) \geq 1 \text{ on } S_k \cap \{w : d(w) \geq \epsilon' r/100\}. \tag{2.6}$$

Now if $x \in S_k \cap \{w : d(w) < \epsilon' r/100\}$, then from Lemma 2.1 (α) we get

$$c_1\, \omega(U_k\,, x) \geq 1. \tag{2.7}$$

Combining (2.6) and (2.7) we find from the maximum principle for subharmonic functions and Lemma 2.1 $(\beta), (\gamma)$ that for some $\bar{c}(\epsilon) = \bar{c}(\epsilon, M, \delta_1, \delta_2, n) \geq 1$,

$$1 \leq\, c_1\, \omega(U_k\,, \cdot) + \bar{c}(\epsilon)\omega(F, \cdot) \text{ on } S_k. \tag{2.8}$$

If $S_k \cap \{w : d(w) = \epsilon' r/100\} = \emptyset$, then by continuity of d, either $S_k \subset \{w : d(w) < \epsilon' r/100\}$ or $S_k \subset \{w : d(w) > \epsilon' r/100\}$. In the first case (2.7) holds for $x \in S_k$ so (2.8) remains valid. Otherwise using continuity of d it follows that there exists $\rho > 0$ with

$$\{y : \rho \leq |y - z| \leq (\tfrac{5}{4} + \tfrac{k+1/2}{4j})r\} \subset \{w : d(w) \geq \epsilon' r/100\}$$

and $d(x) = \epsilon' r/100$ for some $x \in \partial B(z, \rho)$. Applying the same analysis as previously we find first that $\tilde{c}(\epsilon)\, \omega(F, \cdot) \geq 1$ on $\partial B(z, \rho)$ and thereupon from Harnack's inequality that (2.8) is still valid for suitably large $\bar{c}(\epsilon)$. Thus (2.8) is true in all cases.

From (2.8) and once again the maximum principle for subharmonic functions, as well as Lemma 2.1 $(\beta), (\gamma)$, we find for $1 \leq k \leq j - 1$ and $w \in D \setminus B(z, 4r)$ that

$$\omega(B(z, r), w) \leq c_1\, \omega(U_k\,, w) + \bar{c}(\epsilon)\, \omega(F, w). \tag{2.9}$$

Summing (2.9) over $1 \leq k \leq j - 1$ we get

$$(j-1)\omega(B(z, r), w) \leq\, c_1\, \omega(B(z, 2r), w) + (j-1)\bar{c}(\epsilon)\, \omega(F, w). \tag{2.10}$$

Dividing (2.10) by $j - 1$ and choosing A large enough we conclude that Lemma 2.2 is true. □

3. Proof of Theorem 1. In this section we first prove the following measure theoretic lemma.

Lemma 3.1 *Let G be a bounded Ahlfors regular domain with constant M (as in (1.1) with D replaced by G). Let μ be a positive Borel measure on $\mathbb{R}^n$ with support contained in ∂G and $\mu(\partial G) = 1$. Suppose for some $x \in \partial G, r > 0$, that there exists $\epsilon_1, \tau > 0, c_2 \geq 1$, such that*

$$\mu(B(y, \rho)) \leq \epsilon_1\, \mu(B(y, 2\rho)) \,+\, c_2\, \mu(P) \tag{3.2}$$

whenever $P \subset B(y, 2\rho) \cap \partial G$ is a Borel set with

$$\frac{H^{n-1}(P)}{H^{n-1}[B(y, 2\rho) \cap \partial G]} \geq 1 - \tau \tag{3.3}$$

and $y \in \partial G$ with $B(y,2\rho) \subset B(x,r)$. If $\epsilon_1 = \epsilon_1(M,n) > 0$ is small enough, then there exists $C, 1 \leq C < \infty, \gamma > 0$ (depending only on M, c_2, τ, n) and a Borel measurable function g such that $d\mu/dH^{n-1} = g$ on $\partial G \cap B(x,r)$ while

$$\int_{B(y,\rho)\cap\partial G} g^{1+\gamma}\, dH^{n-1} \leq C\,\rho^{(1-n)\gamma}\left(\int_{B(y,2\rho)\cap\partial G} g\, dH^{n-1}\right)^{1+\gamma} = C\,\rho^{(1-n)\gamma}\,\mu[B(y,2\rho)]^{1+\gamma}. \quad (3.4)$$

Proof: We first show that $\mu|_{B(x,r)\cap\partial G}$ is absolutely continuous with respect to $H^{n-1}|_{B(x,r)\cap\partial G}$. To this end we claim for any fixed $K > 2^{n-1}$ and μ almost every $y \in$ supp $\mu \subset \partial G$ that

$$\liminf_{\rho\longrightarrow 0} \frac{\mu[B(y,2\rho)]}{\mu[B(y,\rho)]} < K \quad (3.5)$$

Indeed if the above inequality were false then it would follow from (3.5) and iteration that

$$\lim_{\rho\longrightarrow 0} \frac{\mu[B(y,\rho)]}{\rho^{n-1}} = 0$$

for y in a Borel measurable set E with $\mu(E) > 0$. To obtain a contradiction let $\epsilon > 0$ and use a well known covering lemma (see [M]) to get $\{B(y_i, 5\rho_i)\}$ such that

$$\begin{aligned}
&(a) \quad E \subset \textstyle\bigcup_i B(y_i, 5\rho_i) \\
&(b) \quad B(y_i,\rho_i)\cap B(y_j,\rho_j) = \emptyset,\ i \neq j, \\
&(c) \quad \mu[B(y_i,5\rho_i)] \leq \epsilon\rho_i^{n-1} \text{ for all } i.
\end{aligned} \quad (3.6)$$

Using (3.6) and Ahlfors regularity of ∂G (see (1.1)), we get

$$\begin{aligned}
\mu(E) &\leq \sum_i \mu[B(y_i,5\rho_i)] \leq \epsilon \sum_i \rho_i^{n-1} \\
&\leq M\epsilon \sum_i H^{n-1}[B(y_i,\rho_i)\cap\partial G] \leq M\epsilon\, H^{n-1}(\partial G).
\end{aligned} \quad (3.7)$$

Letting $\epsilon \longrightarrow 0$ we obtain a contradiction to our assumption that E has positive μ measure. Thus (3.5) is true. Next observe from a standard argument, using the Besicovitch covering lemma (see [M, Cor 2.14]), that for μ almost every $y \in F$ Borel $\subset B(x,r)\cap\partial G$ we have

$$\lim_{\rho\longrightarrow 0} \frac{\mu[B(y,\rho)\setminus F]}{\mu[B(y,\rho)]} = 0. \quad (3.8)$$

Now if μ were not absolutely continuous with respect to H^{n-1} measure on $B(x,r)\cap\partial G$, then for some F Borel $\subset B(x,r)\cap\partial G$ we would have $H^{n-1}(F) = 0$ and $\mu(F) > 0$. Choose $y \in F$ so that (3.5) holds when $K = 2^n$ and also (3.8). To get a contradiction we note from (3.2) with P replaced by $B(y,2\rho)\setminus F$ and (3.5), that for some small $\rho > 0$

$$\begin{aligned}
\mu[B(y,\rho)] &\leq \epsilon_1\,\mu[B(y,2\rho)] + c_2\,\mu[B(y,2\rho)\setminus F] \\
&\leq 2^n\,\epsilon_1\,\mu[B(y,\rho)] + c_2\,\mu[B(y,2\rho)\setminus F].
\end{aligned}$$

Dividing this inequality by $\mu[B(y,\rho)]$ and letting $\rho \to 0$ through a sequence so that (3.5), (3.8) hold, we get a contradiction provided $\epsilon_1 < 2^{-n}$. Thus from the Radon - Nikodym theorem, $d\mu = g\, dH^{n-1}$ on $B(x,r) \cap \partial G$.

To continue the proof of Lemma 3.1 we use an argument which first appeared in [G] (see also [Gi], [CF]). We assume, as we may, that $\mu[B(y,2\rho)] \neq 0$, since otherwise (3.4) is trivial. Let $\delta(z) = 2\rho - |y-z|$ denote the distance from $z \in B(y,2\rho)$ to $\partial B(y,2\rho)$. Put

$$f(z) = \left(\frac{\delta(z)}{2\rho}\right)^{n-1} g(z) \text{ for } z \in B(y,2\rho) \cap \partial G.$$

Clearly,

$$2^{n-1} f(z) \geq g(z) \text{ when } z \in B(y,\rho) \cap \partial G \text{ and } f(z) \leq g(z) \text{ on } B(y,2\rho) \cap \partial G. \tag{3.9}$$

If $z \in \partial G,\ B(z,t) \subset B(y,2\rho)$, let

$$⨍_{B(z,t)\cap\partial G} f\, dH^{n-1} = \left\{H^{n-1}[B(z,t)\cap \partial G]\right\}^{-1} \int_{B(z,t)\cap\partial G} f\, dH^{n-1}$$

and let

$$\lambda \geq 2^{n-1} M\, \rho^{1-n} \int_{B(y,2\rho)\cap\partial G} g\, dH^{n-1} = \lambda_0. \tag{3.10}$$

We note for any $z \in B(y,2\rho) \cap \partial G$ that

$$\begin{aligned} ⨍_{B(z,\delta(z)/2)\cap\partial G} f\, dH^{n-1} &\leq 2^{n-1} M \delta(z)^{1-n} \int_{B(z,\delta(z)/2)\cap\partial G} \left[\frac{\delta(z)}{\rho}\right]^{n-1} g\, dH^{n-1} \\ &\leq 2^{n-1} M\, \rho^{1-n} \int_{B(y,2\rho)\cap\partial G} g\, dH^{n-1} = \lambda_0. \end{aligned} \tag{3.11}$$

Also from differentiation theory (see [M, ch 2]) we see that

$$\lim_{t\to 0} ⨍_{B(z,t)} f\, dH^{n-1} = f(z) \tag{3.12}$$

for H^{n-1} almost every $z \in B(y,2\rho)\cap\partial G$. Next put

$$F(\lambda) = \{z \in B(y,2\rho)\cap\partial G : f(z) > \lambda\} \text{ for } \lambda > 0.$$

Let $z \in F(\lambda), \lambda \geq \lambda_0$, be a point where (3.12) holds. Then from Ahfors regularity of ∂G and (3.11) we see there exists, $t = t(z)$, $0 < 5t \leq \delta(z)/2$, such that

$$\begin{aligned} &(i) \quad ⨍_{B(z,5t)\cap\partial G} f\, dH^{n-1} \leq M^2 \lambda, \\ &(ii) \quad ⨍_{B(z,t')\cap\partial G} f\, dH^{n-1} \geq \lambda \text{ for } 0 < t' \leq 5t\,. \end{aligned} \tag{3.13}$$

We observe from (3.13), the definition of f, g and Ahlfors regularity of ∂G that

$$(100)^{n-1} M^4 \mu(B(z,t/2)) \geq \mu(B(z,5t)) \geq \mu(B(z,t)). \tag{3.14}$$

Set

$$E(\lambda) = \{w \in B(y,2\rho) \cap \partial G : f(w) \leq \lambda\}$$

for $\lambda > 0$ and suppose for given small $\delta > 0$ that

$$H^{n-1}[E(\delta\lambda) \cap B(z,t)] \geq (1-\tau) H^{n-1}[B(z,t) \cap \partial G],$$

where τ is as in (3.3). Then from (3.14) and (3.2) with $y, 2\rho, P$, replaced by $z, t, E(\delta\lambda) \cap B(z,t)$, we see that if $\lambda \geq \lambda_0$ and $0 < \epsilon_1 \leq 200^{-n} M^{-4}$, then

$$\mu(B(z,t/2)) \leq 2c_2 \mu(E(\delta\lambda) \cap B(z,t)).$$

Multiplying this inequality by $(\delta(z)/2\rho)^{n-1} \{H^{n-1}[B(z,t) \cap \partial G]\}^{-1}$ and using the definition of f, g, (1.1), (3.13) (ii) to make estimates we get (after cancelling out λ) that

$$1 \leq 20^n c_2 M^4 \delta.$$

Thus if $\delta_0 = [40^n c_2 M^4]^{-1}$, then

$$H^{n-1}[F(\delta_0\lambda) \cap B(z,t)] \geq \tau H^{n-1}[B(z,t) \cap \partial G]. \tag{3.15}$$

Using once again a well known covering argument, we find a sequence $\{B(z_i, 5t_i)\}$ for which (3.13), (3.15) hold with (z,t) replaced by (z_i, t_i) and also

$$\begin{aligned} &(+) \quad H^{n-1}[F(\lambda) \setminus \textstyle\bigcup_i B(z_i, 5t_i)] = 0 \\ &(++) \quad B(z_i, t_i) \cap B(z_j, t_j) = \emptyset, \ i \neq j, \text{ for all } i, j. \end{aligned} \tag{3.16}$$

From (3.13), (1.1), (3.15), and (3.16) we deduce

$$\begin{aligned} \int_{F(\lambda)} f \, dH^{n-1} &\leq \sum_i \int_{B(z_i,5t_i)\cap\partial G} f \, dH^{n-1} \leq 5^{n-1} M^4 \lambda \sum_i H^{n-1}[B(z_i,t_i) \cap \partial G] \\ &\leq \tau^{-1} 5^{n-1} M^4 \lambda \sum_i H^{n-1}[F(\delta_0\lambda) \cap B(z_i,t_i)] \leq \tau^{-1} 5^n M^4 \lambda H^{n-1}[F(\delta_0\lambda)]. \end{aligned} \tag{3.17}$$

Let $A = \tau^{-1} 5^{n-1} M^4$ and for a given large positive integer m set

$$\tilde{f} = \min\{f, m\}$$

$$\tilde{F}(\lambda) = \{z \in B(y,2\rho) \cap G : \tilde{f}(z) > \lambda\}.$$

Then from (3.17) we see for $\lambda \geq \lambda_0$ that

$$\int_{\tilde{F}(\lambda)} \tilde{f} \, dH^{n-1} \leq A \lambda H^{n-1}[\tilde{F}(\delta_0\lambda)]. \tag{3.18}$$

Lemma 3.2 now follows from (3.18) and a standard argument. For completeness we give the proof. From Fubini's theorem, and integration by parts we get

$$
\begin{aligned}
&\int_{\tilde F(\lambda_0)} \tilde f^{1+\gamma}\, dH^{n-1} \\
&= \gamma \int_{\lambda_0}^{\infty} \lambda^{\gamma-1} \left(\int_{\tilde F(\lambda)} \tilde f \, dH^{n-1} \right) d\lambda \,+\, \lambda_0^{\gamma} \int_{\tilde F(\lambda_0)} \tilde f dH^{n-1} \\
&\le A\,\gamma \int_{\lambda_0}^{\infty} \lambda^{\gamma}\, H^{n-1}(\tilde F(\delta_0 \lambda))\, d\lambda \,+\, 2^{n-1}\, A\, M\, \lambda_0^{1+\gamma}\, \rho^{n-1} \\
&\le\; A\gamma(1+\gamma)^{-1}\, \delta_0^{-1-\gamma} \int_{B(y,2\rho)\cap\partial G} \tilde f^{1+\gamma}\, dH^{n-1} \,+\, 2^{n-1}\, A\, M\, \lambda_0^{1+\gamma}\, \rho^{n-1}.
\end{aligned}
\tag{3.19}
$$

Choose γ so that

$$
\frac{2A\gamma}{(1+\gamma)\delta_0^{1+\gamma}} \;\le\; 1.
$$

Then from (3.19) and simple estimates we see for c suitably large that

$$
\int_{B(y,2\rho)\cap\partial G} \tilde f^{1+\gamma}\, dH^{n-1} \;\le\; (1/2) \int_{B(y,2\rho)\cap\partial G} \tilde f^{1+\gamma}\, dH^{n-1} \,+\, c\, \lambda_0^{1+\gamma}\, \rho^{n-1}
$$

and thereupon for some large $c' = c'(M, \tau, c_2, n)$ that

$$
\int_{B(y,2\rho)} \tilde f^{1+\gamma}\, dH^{n-1} \le\; c'\, \rho^{-(n-1)\gamma} \left(\int_{B(y,2\rho)} \tilde f\, dH^{n-1} \right)^{1+\gamma}.
$$

Letting $m \to \infty$ and using the monotone convergence theorem, we conclude first that the above inequality holds with $\tilde f$ replaced by f and second from (3.9) that Lemma 3.1 is true. □

Theorem 1 follows easily from Lemma 2.2 and Lemma 3.1. Indeed from Lemma 2.2 we see that $\mu = \omega(\cdot, w)$ satisfies the hypotheses of Lemma 3.1 with $c_2 = c(\epsilon_1)$, $\tau = \eta(\epsilon_1)$. Applying Lemma 3.1 we get Theorem 1 with $\lambda = \gamma$. The proof of Theorem 1 is now complete. □

4. Concluding Remarks. Let $Q(z,\rho) = \{x \in \mathbb{R}^n : |x_i - z_i| < \rho \text{ for } 1 \le i \le n\}$ be the cube with center z and side length ρ. Let $e_1 = (1, 0, \ldots, 0)$ and put

$$
K_m = \bar Q(2^{-m}\, e_1,\, (1+m^{-1})2^{-m-3}) \setminus \left\{ Q(2^{-m}\, e_1,\, 2^{-m-3}) \bigcup Q([2^{-m} + 2^{-m-3}]e_1, m^{-1}2^{-m-2}) \right\}
$$

$$
D = Q(0,1) \setminus \left(\bigcup_{m=100}^{\infty} [\, K_m \,\cup Q(2^{-m}\, e_1,\, 2^{-m-4}) \cup \{0\}] \right)
$$

It is easily checked that (1.1) - (1.3) are valid for some M, δ_1, δ_2. Also making estimates similar to those in Lemma 2.1 (α) and iterating (see [L]) we deduce that if

$$
y_m = (2^{-m} + 2^{-m-4})e_1,\; r_m = 2^{-m-4} \text{ for } m = 100, \ldots,
$$

and $x = (1/2, 0, \ldots, 0)$, then $y_m \in \partial D$ and

$$\omega[B(y_m, r_m), x] \;\leq\; c\, m^{-\alpha}\, \omega[B(y_m, 2r_m), x]$$

for some $c, 1 \leq c < \infty$, and $\alpha, 0 < \alpha \leq 1$. Letting $m \to \infty$ we find that $\omega(\cdot, x)$ is not a doubling measure. From Harnack's inequality it follows that $\omega(\cdot, z)$ is a nondoubling measure for any $z \in D$.

The domains considered by [DJ] in Theorem B are a subclass of Ahlfors regular domains whose boundaries have big projections in the sense of [DS], which in turn are a subclass of domains whose boundaries are locally uniformly rectifiable, again in the terminology of [DS]. An example in [BJ, section 8] shows that harmonic measure on a locally uniformly rectifiable boundary need not be absolutely continuous with respect to H^{n-1} measure. However if certain bad closed balls are extracted from such a domain, then harmonic measure on the boundary of the resulting domain satisfies a weak reverse Hölder inequality as in Theorem 1. This latter statement can be proved as in [LV, sections 5,6]. Look for caloric analogues of these statements and Theorem 1 in [LN].

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