Symmetry Theorems and Uniform Rectifiability

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Abstract: We study overdetermined boundary conditions for positive solutions to some elliptic partial differential equations of p Laplacian type in a bounded domain D. We show these conditions imply uniform rectifiability of ∂D and also that they yield the solution to certain symmetry problems.

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1. Introduction. Denote points in Euclidean n - space, $\mathbb{R}^n$, by $x = (x_1, ..., x_n)$ and let $\bar{E}$, ∂E, denote the closure and boundary of $E \subseteq \mathbb{R}^n$, respectively. Let $\langle x, y\rangle$ denote the standard inner product in $\mathbb{R}^n$, $|x| = \langle x, x\rangle^{1/2}$, and set $B(x, r) = \{y \in \mathbb{R}^n : \ |y - x| < r\}$ whenever $x \in \mathbb{R}^n, r > 0$. Define k dimensional Hausdorff measure, $1 \leq k \leq n$, in $\mathbb{R}^n$ as follows: For fixed $\delta > 0$ and $E \subseteq \mathbb{R}^n$, let $L(\delta) = \{B(x_i, r_i)\}$ be such that $E \subseteq \cup B(x_i, r_i)$ and $0 < r_i < \delta, \ \ i = 1, 2, ...$ Set

$$\phi_\delta^k(E) = \inf_{L(\delta)} \left(\sum \alpha(k) r_i^k \right)$$

where $\alpha(k)$ denotes the volume of the unit ball in $\mathbb{R}^k$. Then

$$H^k(E) = \lim_{\delta \to 0} \phi_\delta^k(E), \quad 1 \leq k \leq n.$$

If $O \subset \mathbb{R}^n$ is open and $1 \leq q \leq \infty$, let $W^{1,q}(O)$ be the space of equivalence classes of functions f with distributional gradient $\nabla f = (f_{x_1}, \dots, f_{x_n})$, both of which are q th power integrable on O. Let

$$\|f\|_{1,q} = \|f\|_q + \|\nabla f\|_q$$

be the norm in $W^{1,q}(O)$ where $\| \cdot \|_q$ denotes the usual Lebesgue q norm in O. Let $C_0^\infty(O)$ be the infinitely differentiable functions with compact support in O and let $W_0^{1,q}(O)$ be the closure of $C_0^\infty(O)$ in the norm of $W^{1,q}(O)$. Next for fixed $p, 1 < p < \infty$, and constants $c_1, c_2, 0 < c_1 < 1 < c_2 < \infty$, suppose that $A(s,t)$ is a positive continuous function on $(0,\infty) \times (0, \infty)$ with continuous first partials in t and

$$\begin{aligned}
&(a) \qquad c_1\, t^{p/2} \leq tA(s,t) \leq c_2\, t^{p/2} \\
&(b) \qquad c_1 \ \leq t \tfrac{\partial}{\partial t} \log[tA(s, t^2)] \ \leq c_2 \\
&(c) \qquad |A(s_1, t) - A(s_2, t)| \leq c_2 |s_1 - s_2| (1 + t)^{p/2 - 1}
\end{aligned} \tag{1.1}$$

whenever $s_1, s_2, t \in (0, \infty)$. We note for later use that from (1.1) $(a), (b)$ it follows for fixed s and any $\eta, \xi \in \mathbb{R}^n \setminus 0$ that

$$c \,\langle\, A(s, |\eta|^2)\, \eta \ - \ A(s, |\xi|^2)\, \xi \ , \, \eta - \xi \,\rangle \ \geq \ (|\eta| + |\xi|)^{p-2}\, |\eta - \xi|^2\,. \tag{1.2}$$

In (1.2), $c \geq 1$ denotes a positive constant depending on p, c_1, c_2, n. We consider positive weak solutions u to

$$\nabla \cdot \big[A(u, |\nabla u|^2) \nabla u\big] + C(u, |\nabla u|^2) = 0$$

in $D \cap N$ where D is a bounded domain and $N \supset \partial D$ is an open neighborhood of ∂D. Here $C : (0, \infty) \times (0, \infty) {\to} [0, \infty)$ with

$$|C(s,t)| \ \leq c_2 \ < \infty, \ (s,t) \in (0, \infty) \times (0, \infty). \tag{1.3}$$

Moreover $u \in W^{1,p}(D \cap N)$ with

$$\int_{D \cap N} [\, \langle A(u, |\nabla u|^2) \nabla u, \nabla \theta \rangle - C(u, |\nabla u|^2)\, \theta \,]\, dx = 0, \tag{1.4}$$

where $\theta \in W_0^{1,p}(D \cap N)$ and dx denotes H^n measure. If $A(u, |\nabla u|^2) = |\nabla u|^{p-2}$, $C \equiv 0$ in (1.4), we say that u is a weak solution to the p Laplacian partial differential equation in $N \cap D$. To simplify matters, we shall always assume that

$$u(x) \to 0 \text{ as } x \to \partial D. \tag{1.5}$$

Put $u \equiv 0$ in $N \setminus D$ and note that $u \in W^{1,p}(N)$. In section 2 we point out that there exists a unique finite positive Borel measure μ such that

$$\int_{D \cap N} [-\langle A(u, |\nabla u|^2) \nabla u, \nabla \phi \rangle + C(u, |\nabla u|^2)\phi] dx = \int \phi \, d\mu \tag{1.6}$$

whenever $\phi \in C_0^\infty(N)$. Finally we assume for some β, $0 < \beta < \infty$, that

$$\mu(B(y,r) \cap \partial D) \leq \beta \, r^{n-1} \tag{1.7}$$

for $0 < r \leq r_0$ and all $y \in \partial D$. Here r_0 is so small that $\cup_{y \in \partial D} \bar{B}(y, r_0) \subset N$. Under these assumptions we prove in section 2, the following important square function estimate.

Theorem 1.8. *Fix p, δ_0, with $0 < \delta_0 \leq 1 < p < \infty$, and suppose that u, D, μ satisfy (1.1)-(1.7). There exists $\hat{r}_0, 0 < \hat{r}_0 \leq r_0$, and k_0 a positive integer (depending on c_1, c_2) , such that if $z \in \partial D$ and $0 < r \leq \hat{r}_0$, then for $k \geq k_0$,*

$$\int_{D \cap B(z,r)} u \, \max(|\nabla u| - \delta_0, 0)^k \sum_{i,j=1}^n u_{x_i x_j}^2 \, dx \leq c r^{n-1} \tag{1.9}$$

where $c, \hat{r}_0$ depend on $n, p, k, c_1, c_2, \delta_0, \beta$ but not on $z \in \partial D$.

Armed with Theorem 1.8 we shall prove the following theorem in section 3.

Theorem 1.10. *Let u, D, p, μ be as in Theorem 1.8 and suppose also that for some $\gamma, 0 < \gamma < \infty$,*

$$\gamma \, r^{n-1} \leq \mu(B(z,r)) \text{ whenever } z \in \partial D \text{ and } 0 < r \leq r_0. \tag{1.11}$$

If k_0 is as in Theorem 1.8 then for $k \geq k_0$ and some $\tilde{r}_0 > 0$,

$$\int_{D \cap B(z,r)} u \, |\nabla u|^k \sum_{i,j=1}^n u_{x_i x_j}^2 \, dx \leq c r^{n-1}, \; 0 < r \leq \tilde{r}_0,$$

where $c, \tilde{r}_0$ depend on $n, p, k, c_1, c_2, \beta, \gamma$. Moreover ∂D is locally uniformly rectifiable in the sense of David - Semmes.

By local uniform rectifiability of ∂D, we mean that $P \cup \partial D$ is uniformly rectifiable where P is any $n-1$ dimensional plane whose distance from ∂D is $\approx$ equal to the diameter of D. For numerous equivalent definitions of uniform rectifiability we refer the reader to [DS], [DS1]. In section 4 we begin the study of some overdetermined boundary value problems. As motivation for these problems we note that in [S, Theorem 2], Serrin proved

Theorem 1.12. *Suppose that the bounded region D has a C^2 boundary. If there is a positive solution $u \in C^2(\bar{D})$ to the uniformly elliptic equation*

$$\Delta u + k(u, |\nabla u|^2) \sum_{i,j=1}^{n} u_{x_i}\, u_{x_j}\, u_{x_i x_j} \;=\; l(u, |\nabla u|^2) \tag{1.13}$$

where k, l are continuously differentiable everywhere with respect to their arguments and if u satisfies the boundary conditions

$$u = 0 \text{ and } \tfrac{\partial u}{\partial n} = a = \text{ constant on } \partial D, \tag{1.14}$$

then D is a ball and u is radially symmetric about the center of D.

In (1.14), $\frac{\partial}{\partial n}$ denotes the inner normal derivative of u at a point in ∂D. In this paper we continue a project (see [V, LV, LV1, LV3]) whose goal is to obtain the conclusion of Serrin's theorem under minimal regularity assumptions on ∂D and the boundary values of $|\nabla u|$. To begin we note that uniform ellipticity in (1.13) means for all $q \in \mathbb{R}^n \setminus \{0\}, \xi \in \mathbb{R}^n$ with $|\xi| = 1$, and $s > 0$ that

$$\infty > \Lambda \geq 1 \,+\, k(s, |q|^2)\langle q, \xi\rangle^2 \;\geq\; \lambda > 0. \tag{1.15}$$

Next observe that (1.13) can be written in divergence form as

$$\nabla \cdot \big[A^*(u, |\nabla u|^2)\, \nabla u \,\big] + C^*(u, |\nabla u|^2) \;=\; 0, \tag{1.16}$$

where

$$\begin{aligned} \log A^*(s,t) &= \tfrac{1}{2} \int_0^t k(s,\tau) d\tau \\ C^*(s,t) &= -A^*(s,t)\ \big[\, l(s,t) \,+\, t\, \tfrac{\partial}{\partial s} \log A^*(s,t) \,\big]\,. \end{aligned} \tag{1.17}$$

Uniform ellipticity of A^* and smoothness properties of A^*, C^* can be garnered from (1.15) and smoothness of k, l. We note that if ∂D is smooth enough then

$$d\mu^* = A^*(0, |\nabla u|^2)\, |\nabla u|\, dH^{n-1}, \tag{1.18}$$

where μ^* is defined as in (1.6) relative to A^*, C^*. Thus a weak formulation of (1.14) is (1.5) and

$$\mu^* = a\, A(0, a^2)\, H^{n-1}|_{\partial D}. \tag{1.19}$$

A natural first question is whether Theorem 1.12 remains true when (1.14) is replaced by (1.19), (1.5) and no assumption is made on ∂D. We note that the answer to this question is no for related problems when $p = 2$ (see [LV2]) and for $n = 2, 1 < p < \infty$ (see [B]). Moreover, at least for some A^*, C^* we believe the techniques in [LV2] for $p = 2$ and [B] for $n = 2, 1 < p < \infty$, could be used to construct examples of functions u satisfying (1.16) in $D \neq$ ball and also the overdetermined boundary condition (1.19), (1.5). The examples in [LV2], [B], have the property that $|\nabla u|(x) \to \infty$ as $x \to \partial D$ through a certain sequence. Also, in proving Theorem 1.8 we show that (1.7) is equivalent to the assumption that u has a bounded Lipschitz extension to a neighborhood of ∂D. Thus, a second question (which rules out known counterexamples) is whether Theorem 1.12 remains true when (1.14) is replaced by (1.5), (1.7), (1.19), under appropriate structure - smoothness assumptions on A^*, C^*. As evidence for a yes answer we discuss recent work in [LV3]. To do so, consider the following free boundary problem. Given $F \subset \mathbb{R}^n$ a compact convex set, $a > 0, 1 < p < \infty$, find $\hat{u}$ and a bounded domain $\Omega = \Omega(a,p)$ with $F \subset \Omega, \hat{u} \in W_0^{1,p}(\Omega)$, and

$$
\begin{aligned}
&(*) \qquad \nabla \cdot (|\nabla \hat{u}|^{p-2} \nabla \hat{u}) = 0 \text{ weakly in } \Omega \setminus F. \\
&(**) \qquad \hat{u}(x) = 1 \text{ continuously on } F \text{ and } \hat{u}(x) \to 0 \text{ as } x \to y \in \partial\Omega. \\
&(***) \quad |\nabla \hat{u}(x)| \to a \text{ whenever } x \to y \in \partial\Omega.
\end{aligned}
\tag{1.20}
$$

This problem was solved in [HS] (see also [HS1], [HS2] for related problems). They proved

Theorem 1.21. *If F has positive p capacity, then there exists a unique $\hat{u}, \Omega$ satisfying (1.20). Moreover Ω is convex with a smooth (C^∞) boundary.*

We remark that the above authors assume F has nonempty interior. However their theorem can easily be extended to more general F (see [LV3]). In [LV3] we proved

Theorem 1.22. *Let D, u, p, a be as in (1.20) $(*), (**)$ with $\hat{u}, \Omega$ replaced by u, D, and let μ be the measure corresponding to u as in (1.6) relative to $A(u, |\nabla u|^2) = |\nabla u|^{p-2}$. If μ satisfies (1.7), (1.19) (for this A and with $\mu = \mu^*$), then $D = \Omega(a,p)$.*

Note from Theorems 1.22, 1.21 that if F is a ball, then necessarily D is a ball since in this case radial solutions satisfying the overdetermined boundary conditions always exist. To outline the proof of Theorem 1.22, the key step is to show that

$$
\limsup_{x \longrightarrow \partial D} |\nabla u(x)| \leq a. \tag{1.23}
$$

Theorem 1.22 then follows from Theorem 1.21, the minimizing property of a p capacitary function for the ‘ Dirichlet ’ integral, and the fact that the nearest point projection onto a convex set is Lipschitz with norm ≤ 1. Our proof in [LV3] uses the square function estimate in Theorem 1.8 but also makes important use of the fact that u, u_{x_k} are solutions to the same divergence form equation.

We would like to prove an inequality similar to (1.23) when u, a weak solution to (1.4), satisfies (1.5) while (1.7), (1.19) hold for μ. Unfortunately, however, the p Laplace partial differential equation seems to be essentially the only divergence form partial differential equation of the form (1.1) with the property that a solution, u, and its partial derivatives, $u_{x_i}, 1 \leq i \leq n$, both satisfy the same divergence form partial differential equation To see why, suppose $A(u, |\nabla u|^2) = A(|\nabla u|^2)$ and $C \equiv 0$ in (1.4). Suppose that u is a strong smooth solution to the new version of (1.4) at $x \in D, \nabla u(x) \neq 0$, and $A \in C^\infty[(0,\infty)]$. Differentiating $\nabla \cdot [A(|\nabla u|^2) \nabla u] = 0$, we deduce for $\zeta = \langle \nabla u, \eta \rangle$ that at x,

$$
\tilde{L}\zeta = \nabla \cdot [2A'(|\nabla u|^2) \langle \nabla u, \nabla \zeta \rangle \, \nabla u \, + \, A(|\nabla u|^2) \nabla \zeta] = 0.
$$

Clearly,

$$
\tilde{L}u = \nabla \cdot [2A'(|\nabla u|^2) |\nabla u|^2 \, \nabla u]
$$

at x and this equation is only obviously zero if $A(t) = at^\lambda$ for some real a, λ. Without such an equation for $u, |\nabla u|^2$, we are not able to use u to make estimates as in [LV3]. Instead, in order to carry through the argument in [LV3], it appears that one is forced to consider some rather delicate estimates concerning the absolute continuity of elliptic measure with respect to H^{n-1} measure on ∂D. To outline our attempts to prove an analogue of (1.23) for a general A, C as in (1.1)-(1.3), we note for sufficiently large k, that $|\nabla u|^k$ is a subsolution to (see section 4)

$$\hat{L}w = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(b_{ij} w_{x_j}) = 0, \tag{1.24}$$

where thanks to Theorem 1.10,

$$\int_{B(z,r)\cap D} u \sum_{i,j=1}^{n} \left(\frac{\partial b_{ij}}{\partial x_j}\right)^2 dx \le c r^{n-1} \text{ whenever } z \in \partial D \text{ and } 0 < r \le \hat{r}_0. \tag{1.25}$$

Moreover, the extra assumption (1.11) allows us to conclude in Theorem 1.10 that ∂D is locally uniformly rectifiable.

At one time we believed that local uniform rectifiability of ∂D would imply elliptic measure is absolutely continuous with respect to H^{n-1} measure on ∂D. Here the desired elliptic measure is defined relative to a point in D and a certain elliptic operator which agrees with $\hat{L}$ on $\{x \in D : |\nabla u(x)| \ge \delta_0\}$. However we found an illuminating example in [BJ, section 8] which shows that harmonic measure in $\mathbb{R}^2$ for the complement of a compact locally uniformly recifiable set, need not be absolutely continuous with respect to H^1 measure on this set. Thus we first assumed that D satisfied a Carleson measure type analogue of the following chain condition:
There exists $1 \le c_3 < \infty$ such that if $z \in \partial D, 0 < r \le r_0, |z-x| + |z-y| \le r$, and x, y, lie in the same component P of $B(z, r_0) \cap D$, with $\min\{d(x, \partial P), d(y, \partial P)\} \ge r/100$, then there is a chain, $\{B(w_i, d(w_i, \partial P)/2)\}_1^k$, connecting x to y with the properties:

$$\begin{aligned} &(a) \quad x \in B(w_1, d(w_1, \partial P)/2), y \in B(w_k, d(w_k, \partial P)/2) \text{ and } \bigcup_{i=1}^{k} B(w_i, d(w_i, \partial P)) \subset P, \\ &(b) \quad B(w_i, d(w_i, \partial P)/2) \cap B(w_{i+1}, d(w_{i+1}, \partial P)/2) \neq \emptyset \text{ for } 1 \le i \le k-1, \\ &(c) \quad k \le c_3. \end{aligned} \tag{1.26}$$

Here, as in the sequel, $d(E, F)$ denotes the Euclidean distance between the sets E and F. Later we observed that in order to obtain the desired analogue of (1.23) it suffices to prove absolute continuity with respect to H^{n-1} of an elliptic measure concentrated on the boundary of a certain subdomain $D_1 \subset D$. Here ∂D_1 is locally uniformly rectifiable and D_1 is constructed by removing from D certain balls on which $|\nabla u|$ is 'small'. With this intuition we finally were able to make the required estimates and thus obtain the following theorem.

Theorem 1.27. *Let $A, p, D, u, \mu, \beta, \gamma$ be as in Theorem 1.10. Suppose also that A has continuous second partials and C has continuous first partials on $(0,\infty) \times (0,\infty)$ each of which extends continuously to $[0,\infty) \times (0,\infty)$. If*

$$\mu(B(z,r) \cap \partial D) \le \beta_1 H^{n-1}(B(z,r) \cap \partial D) \textit{ for } 0 < r \le r_0 \textit{ and all } z \in \partial D, \tag{1.28}$$

then

$$\limsup_{x \to z} |\nabla u|(x)\, A(u(x), |\nabla u|^2(x)) \le \beta_1 \textit{ for each } z \in \partial D.$$

Our proof of Theorem 1.27 does not require any specific knowledge of uniform rectifiability although the arguments are certainly inspired by [DS], [DS1] and the reader who is not well versed in these arguments may have trouble following our rather complicated but complete argument. In section 4 we first prove Theorem 1.27 under the additional assumption that D satisfies a Carleson measure type version of (1.26). This assumption allows us to argue as in [DJ] and use a theorem of [KP] to conclude that elliptic measure associated with a certain partial differential equation of the form (1.24), (1.25) is absolutely continuous with respect to $H^{n-1}|_{\partial D}$ and in fact that the corresponding Radon Nikodym derivative satisfies a weak reverse Hölder inequality on $B(x,r) \cap \partial D$ whenever $x \in \partial D$ and $0 < r \le r_0$. We can then use essentially the argument in [LV3] to get Theorem 1.27. In section 5 we construct $D_1 \subset D$ (as mentioned above) and using our work in section 4 reduce the proof of Theorem 1.27 to proving an inequality for a certain elliptic measure on ∂D_1. In section 6 we prove this inequality by a rather involved stopping time argument and thus finally obtain Theorem 1.27 without the chain assumption (1.26). We note that Theorem 1.10 implies ∂D is contained in a surface for which H^{n-1} almost every point has a tangent plane (see [DS]). Using this fact, Lemma 2.10, and blowup type arguments one can show that the conclusion of Theorem 1.27 is valid 'nontangentially' for H^{n-1} almost every $z \in \partial D$. Thus the arguments in sections 4-6 are to show that the '$\limsup'$ in Theorem 1.27 must occur nontangentially on a set of positive H^{n-1} measure $\subset \partial D$.

The main difficulty in proving more general symmetry theorems under assumptions similar to those in Theorem 1.27 is that one is forced to use more sophisticated boundary maximum principles (such as the Alexandroff moving plane argument) in a domain whose boundary is not apriori smooth. We can overcome this difficulty by making further assumptions on ∂D. To this end we say that ∂D is δ Reifenberg flat if whenever $z \in \partial D$ and $0 < r \le r_0$, there exists a plane $P = P(z,r)$ containing z with unit normal n such that

$$\begin{aligned} \{y + \rho n \in B(z,r) : y \in P, \rho > \delta r\} \subset D \\ \{y - \rho n \in B(z,r) : y \in P, \rho > \delta r\} \subset \mathbb{R}^n \setminus D. \end{aligned} \tag{1.29}$$

As our final theorem we prove in section 7,

Theorem 1.30. *Let u, p, A, C, D, be as in Theorem 1.27, except that now u is a weak solution to the PDE above (1.3) in all of D. Also assume that equality holds in (1.28) whenever $z \in \partial D$ and $0 < r \le r_0$. If ∂D is $\delta > 0$ Reifenberg flat and δ is sufficiently small, then D is a ball.*

To prove Theorem 1.30 we first show that Theorem 1.27 and work of [ACF] imply that ∂D is $C^{2,\alpha}$ for some $\alpha > 0$. Second we use the ' moving plane argument ' as in [V] to conclude that D is a ball. Finally at the end of section 7 we make some remarks concerning possible generalizations of our theorems.

2. Proof of Theorem 1.8. We state here some lemmas that will be used throughout this paper. In these lemmas, $c \ge 1$, denotes a positive constant depending only on n, p, c_1, c_2, not necessarily the same at each occurrence. We say that c depends on the ' data. ' In general, $c(a_1, \dots, a_m) \ge 1$ depends only on $a_1, \dots, a_m$ and the data. Also $a \approx b$ means $c^{-1} a \le b \le ca$ for some $c \ge 1$ depending only on the data

Lemma 2.1. *Let u, A, p, D, N be as in (1.1) - (1.5). If $\bar B(z, 2r) \subset N$ and $\hat u(x) = \max[u, r^{p/(p-1)}]$, then*

$$r^{p-n} \int_{B(z,r/2)} |\nabla u|^p dx \,\leq\, c \max_{B(z,r)} \hat u^p \,\leq\, c^2\, r^{-n} \int_{B(z,2r)} \hat u^p \, dx \tag{2.2}$$

while if $B(z, 2r) \subset D \cap N$, then

$$\max_{B(z,r)} \hat u \,\leq\, c \min_{B(z,r)} \hat u. \tag{2.3}$$

Proof: (2.2) is a standard subsolution type estimate while (2.3) is a standard weak Harnack inequality (see [S1]). □

Lemma 2.4. *Let u, A, p, D, N be as in (1.1) - (1.5). Then ∇u is locally Hölder continuous in $D \cap N$ for some $\sigma \in (0, 1)$ with*

$$|\nabla u(x) - \nabla u(y)| \leq c\,(|x-y|/r)^\sigma \left[\max_{B(z,r)} |\nabla u| \,+\, r^\sigma \right] \,\leq c\,(|x-y|/r)^\sigma \left[r^{-1} \max_{B(z,2r)} u \,+\, r^\sigma \right] \tag{2.5}$$

whenever $\bar B(z, 2r) \subset N \cap D$ and $x, y \in B(z, r/2)$. Also u has distributional second partials on $\{x : |\nabla u(x)| > 0\} \cap D \cap N$ and there is a positive integer k_0 (depending on the data) such that if $k \geq k_0$,

$$\int_{B(z,r/2)} \sum_{i,j=1}^{n} |\nabla u|^k \, u^2_{x_i x_j} \, dx \,\leq\, c(k) r^{n-2} \max_{B(z,r)} (1 + |\nabla u|^{k+2}) \tag{2.6}$$

whenever $B(z, 2r) \subset D \cap N$.

Proof: For a proof of (2.5) when A has no dependence on u and $C = 0$ see [GL]. The proof in the general case follows from this special case and Campanato type estimates (see for example [M], [C]). Given (2.5), (2.6) follows in a standard way. One can for example use difference quotients and make Sobolev type estimates or first show that $|\nabla u|^k$ is essentially a weak subsolution to a uniformly elliptic divergence form partial differential equation on $\{x : |\nabla u|(x) > 0\}$ and then use $|\nabla u|^2$ times a smooth cutoff as a test function. □

Lemma 2.7. *If u, A, p, D, N are as in (1.1) - (1.5), then there exists a positive Borel measure μ satisfying (1.6) with support $\subset \partial D$ and $\mu(\partial D) < \infty$.*

Proof: Lemma 2.7 is given in [HKM] under slightly different structure assumptions. Here we outline for the reader's convenience another proof. We claim that it suffices to show

$$\int_{D \cap N} [-\langle\, A(u, |\nabla u|^2)\, \nabla u, \nabla \psi \,\rangle + C(u, |\nabla u|^2)\, \psi] \, dx \geq 0$$

whenever $\psi \in C_0^\infty(N)$ is nonnegative. Indeed once this claim is established, it follows from Lemma 2.1 and the same argument as in the proof of the Riesz representation theorem for positive linear functionals on the space of continuous functions that Lemma 2.7 is true. To prove our claim we note that $\phi = [(\eta + \max[u - \epsilon, 0])^{\epsilon} - \eta^{\epsilon}\,]\ \psi$ is an admissible test function in (1.4) for small $\eta > 0$, as is easily seen. We then use (1.1) to get that

$$\int_{\{u \geq \epsilon\}} [(\eta + \max[u - \epsilon, 0])^{\epsilon} - \eta^{\epsilon}]\ [\langle\ A(u, |\nabla u|^2)\,\nabla u, \nabla \psi\,\rangle\ - C(u, |\nabla u|^2)\,\psi]\,dx \leq\ 0.$$

Using dominated convergence, letting first η and then $\epsilon{\rightarrow}0$ we get our claim. Lemma 2.7 then follows from our earlier remarks. □

Next given $z \in \partial D$ let

$$W(z, r) = \int_0^r \left(\frac{\mu[B(z,t)]}{t^{n-p}}\right)^{1/(p-1)} \frac{dt}{t}, 0 < r \leq r_0.$$

Lemma 2.8. *If $z \in \partial D$, (1.1)-(1.7) hold for u, μ, and $\hat{u}$ is as in Lemma 2.1, then for some $1 \leq c_4 \leq c_5 < \infty$, depending only on the data, we have*

$$\left(\frac{\mu(B(z,r/2)}{r^{n-p}}\right)^{1/(p-1)} \leq\ c_4 \max_{B(z,r)} \hat{u}\ \leq\ c_5\,[W(z, c_5 r/2) +\ r^{p/(p-1)}]\ \textit{for}\ 0 < r \leq r_0/c_5. \tag{2.9}$$

Proof: The lefthand inequality in (2.9) is easily proved by choosing, $\phi \in C_0^{\infty}(B(z,r))$ with $\phi \equiv 1$ on $B(z, r/2)$ in (1.6) and using (1.1), (1.3), Lemma 2.1. The righthand inequality in (2.9) was proved for $C \equiv 0$ in [KZ] under slightly different structure assumptions. To adapt the proof in [KZ] to our situation we note that these authors consider two cases. One case uses results from [KM] while the other uses an argument in [EL]. The proof in [KM] requires only (1.1) (a) and thus in this case the arguments in [KM], [KZ] can be copied verbatim, if one first replaces the measure in these papers with $d\mu + |C|dx$, thanks to (1.3). The proof in [EL] uses only (1.1), (1.2). In [EL] use is made of a certain solution to (1.4) with $C = 0$. In our situation one can replace this solution by an appropriate weak supersolution to (1.4) and then the argument in [EL], [KZ], can be copied essentially verbatim. □

Lemma 2.10. *If (1.1)-(1.7) are true for u, μ, then for all $z \in \partial D$, and $0 < r \leq r_0/c_3$,*

$$\max_{B(z,r)} u\ \leq\ c\beta^{1/(p-1)} r. \tag{2.11}$$

Moreover if either $u \geq \lambda r$ or $|\nabla u| \geq \lambda$ at some x in $B(z,r) \cap D$ with $d(x, \partial D) \geq \lambda r$, then

$$r^{n-1}\ \leq c(\lambda)\mu[B(z, c_5 r)]\ \textit{ for } 0 < r \leq r(\lambda). \tag{2.12}$$

Proof: Using (1.7) in the integral defining W and integrating we see that $W(z, c_5 r) \leq c\beta^{1/(p-1)} r$. This inequality and Lemma 2.8 imply (2.11). To get (2.12) first note from Lemma 2.4 that there exists λ_1, depending only on λ and the data, such that $u \geq \lambda_1 r$ at some point in $B(z, 2r)$ whenever $0 < r \leq r(\lambda)$. Using (1.7) we see that if λ_2, having the same dependence as λ_1, is small enough, then $4c_5 W(z, \lambda_2 r) \leq \lambda_1 r$. Using this fact and Lemma 2.8 we conclude that

$$r \leq c[W(z, c_5 r) - W(z, \lambda_2 r)] \leq c(\lambda) \left(\mu(B(z, c_5 r) r^{p-n}\right)^{1/(p-1)}$$

provided $0 < r \leq r(\lambda)$. This inequality clearly implies (2.12). □

Proof of Theorem 1.8: The proof of Theorem 1.8 is similar to the proof of Lemma 2.5 in [LV3], however our more general structure assumptions force us to work harder. We note from (2.11) and (2.5) that

$$|\nabla u| \leq c\beta^{1/(p-1)} < \infty \tag{2.13}$$

in $N_1 \cap D$ for some neighborhood N_1 with $\partial D \subset N_1$. To simplify matters we first assume that

$$A \text{ and } C \text{ are infinitely differentiable on } (0, \infty) \times (0, \infty). \tag{2.14}$$

Then from Schauder type estimates we see that u is infinitely differentiable at each $x \in D$ where $|\nabla u(x)| \neq 0$. Let $\{\bar{Q}_i = \bar{Q}_i(y_i, r_i)\}$ be a Whitney cube decomposition of D with center y_i and radius r_i. We choose this sequence so that

$$\begin{aligned} &(a) \qquad Q_i \cap Q_j = \emptyset, i \neq j, \\ &(b) \qquad 10^{-5n} d(Q_i, \partial D) \leq r_i \leq 10^{-n} d(Q_i, \partial D) \\ &(c) \qquad \textstyle\bigcup_i \bar{Q}_i = D. \end{aligned} \tag{2.15}$$

Next let η_i be a partition of unity adapted to $\{Q_i\}$. That is

$$\begin{aligned} &(i) \qquad \sum_i \eta_i \equiv 1 \\ &(ii) \qquad \text{The support of } \eta_i \text{ is } \subset \textstyle\bigcup\{Q_j : \bar{Q}_j \cap \bar{Q}_i \neq \emptyset\}. \\ &(iii) \qquad \eta_i \text{ is infinitely differentiable with } \eta_i \geq c^{-1} \text{ on } Q_i \text{ and } |\nabla \eta_i| \leq c r_i^{-1}. \end{aligned} \tag{2.16}$$

Next for fixed $\xi \leq 10^{-20}, r$ small, and $z \in \partial D$, let $\Lambda = \{j : Q_j \cap B(z, 2r) \neq \emptyset$ and $r_j \geq \xi r\}$. If $\Lambda \neq \emptyset$, set $\Omega = \bigcup_{i \in \Lambda} Q_i$, and put $\sigma(|\nabla u|) = \max(|\nabla u|^2 - \delta_0^2, 0)^{k/2}$. Integrating by parts we see that

$$
\begin{aligned}
I_1 = c^{-1} \int_\Omega u\,\sigma(|\nabla u|) \sum_{i,j=1}^{n} (u_{x_i x_j})^2 \ dx \ &\leq \sum_{m\in\Lambda} \int \ u\,\sigma(|\nabla u|) \ \sum_{i,j=1}^{n} (u_{x_i x_j})^2 \, \eta_m \, dx \\
&= -\sum_{m\in\Lambda} \int \sigma(|\nabla u|) \ \sum_{i,j=1}^{n} \ u_{x_i} u_{x_j} u_{x_i x_j} \eta_m dx \\
&- \sum_{m\in\Lambda} \int k\, u\,\sigma(|\nabla u|)^{1-2/k} \sum_{j=1}^{n} \left(\sum_{q=1}^{n} u_{x_q}\, u_{x_q x_j} \right)^2 \ \eta_m dx \qquad (2.17) \\
&- \sum_{m\in\Lambda} \int u\,\sigma(|\nabla u|) \sum_{i=1}^{n} (\Delta u_{x_i}) u_{x_i} \eta_m \, dx \\
&- \sum_{m\in\Lambda} \int \ u\,\sigma(|\nabla u|) \ \sum_{i,j=1}^{n} u_{x_i x_j}\, u_{x_i}\, (\eta_m)_{x_j} \, dx \\
&= -I_2 - I_3 - I_4 - I_5.
\end{aligned}
$$

To estimate I_5, let Λ_1 be the set of all i for which there exists Q_j, Q_k with $k \notin \Lambda, j \in \Lambda$ and $\bar{Q}_i \cap \bar{Q}_j \neq \emptyset, \ \bar{Q}_k \cap \bar{Q}_i \neq \emptyset$. Then from (2.16) $(i), (ii)$, we see that

$$
|I_5| \ \leq \sum_{m\in\Lambda_1} \int_{Q_m} u\,\sigma(|\nabla u|) \sum_{i,j=1}^{n} |u_{x_i x_j}| \, |u_{x_i}| \, |(\eta_m)_{x_j}| \, dx = I_6. \qquad (2.18)
$$

To handle I_6 we divide the integers in Λ_1 into two subsets, say $\Lambda_{11}, \Lambda_{12}$, where Λ_{11} consists of all i in Λ_1 for which $\bar{Q}_i$ touches a closed cube containing points not in $B(z, 2r)$ while $\Lambda_{12} = \Lambda_1 \setminus \Lambda_{11}$ contains integers i for which $\bar{Q}_i$ touches a closed cube Q_j with $r_j \leq \xi r$. If $j \in \Lambda_{11}$ we see from (2.6), (2.11), (2.13), (2.16) (iii) and Hölder's inequality that for $m \in \Lambda_{11}$,

$$
\begin{aligned}
&\int_{Q_m} u\,\sigma(|\nabla u|) \sum_{i,j=1}^{n} |u_{x_i x_j}| \, |u_{x_i}| \, |(\eta_m)_{x_j}| \, dx \\
&\qquad \leq c r_m^{n/2} \left(\int_{Q_m} \sum_{i,j=1}^{n} |\nabla u|^{2k+2} u_{x_i x_j}^2 \, dx \right)^{1/2} \leq c(\beta, k)\, r_m^{n-1}.
\end{aligned} \qquad (2.19)
$$

Using this inequality and (2.15) $(a), (b)$ we deduce that

$$
\sum_{m\in\Lambda_{11}} \int_{Q_m} u\,\sigma(|\nabla u|) \sum_{i,j=1}^{n} |u_{x_i x_j}| \, |u_{x_i}| \, |(\eta_m)_{x_j}| \, dx \leq c(\beta, k) r^{n-1} \qquad (2.20)
$$

Observe that the integral in (2.18) is equal to zero unless $|\nabla u| \geq \delta_0$ at some point in Q_m. Otherwise if $m \in \Lambda_{12}$ we can apply (2.12) with $r = r_m$ and (2.15) (b) to conclude as in (2.19) that

$$\int_{Q_m} u\,\sigma(|\nabla u|) \sum_{i,j=1}^{n} |u_{x_i x_j}|\,|u_{x_i}|\,|(\eta_m)_{x_j}|\,dx \leq c(\beta,k) r_m^{n-1} \;\leq\; c(\beta,k,\delta_0)\,\mu[B(y_m, 10^{10n} c_5 r_m)]. \quad (2.21)$$

From (2.15) $(a),(b)$ and the definition of Λ_{12} we see for fixed $m \in \Lambda_{12}$ that the cardinality of the set of integers $l \in \Lambda_{12}$ for which $B(y_m, 10^{40n}\,c_5 r_m) \cap B(y_l, 10^{40n} c_5 r_l) \neq \emptyset$ has cardinality $P < \infty$ where P depends only on c_5 and n. Using this fact and summing in (2.21) we get in view of (1.7) that

$$\sum_{m\in\Lambda_{12}} \int_{Q_m} u\,\sigma(|\nabla u|) \sum_{i,j=1}^{n} |u_{x_i x_j}|\,|u_{x_j}|\,|(\eta_m)_{x_i}|\,dx \leq c(\beta,k,\delta_0) r^{n-1}. \quad (2.22)$$

Adding (2.20), (2.22), we deduce from (2.18) that

$$|I_5| \;\leq\; I_6 \;\leq\; c(\beta,k,\delta_0) r^{n-1}. \quad (2.23)$$

Next we use (1.4) and estimate I_2 in the following way. First if

$$h(s,t) = \int_0^t \frac{\sigma(\tau^{1/2})}{2A(s,\tau)} d\tau,$$

then

$$\begin{aligned} I_2 &= \sum_{m\in\Lambda} \int \sigma(|\nabla u|) \sum_{i,j=1}^{n} u_{x_i} u_{x_j} u_{x_i x_j} \eta_m dx \\ &= \sum_{m\in\Lambda} \int A(u,|\nabla u|^2)\,\langle \nabla u, \nabla[h(u,|\nabla u|^2)\,\eta_m]\,\rangle\,dx \\ &- \sum_{m\in\Lambda} \int A(u,|\nabla u|^2)\,|\nabla u|^2\,h_s(u,|\nabla u|^2)\,\eta_m\,dx \\ &- \sum_{m\in\Lambda} \int A(u,|\nabla u|^2)\langle \nabla u,\,\nabla\eta_m\,\rangle\,h(u,|\nabla u|^2)\,dx \\ &= I_{21} + I_{22} + I_{23}. \end{aligned} \quad (2.24)$$

I_{23} can be estimated in the same way as I_5. We obtain

$$|I_{23}| \leq c r^{n-1}. \quad (2.25)$$

From (1.4) with $\phi = h(u,|\nabla u|^2)$, (1.1), (1.3) and (2.13) we see that

$$|I_{21}| = |\sum_{m\in\Lambda} \int C(u,|\nabla u|^2)\,h(u,|\nabla u|^2)\eta_m dx| \leq c \sum_{m\in\Lambda} r_m^n \leq c r^n. \quad (2.26)$$

Likewise,

$$|I_{22}| \le c \sum_{m\in\Lambda} r_m^n \le c r^n. \tag{2.27}$$

Combining (2.25)-(2.27) we find

$$|I_2| \le c r^{n-1}. \tag{2.28}$$

Next observe from (1.4), (2.14) that if $\nabla u(x) \neq 0$, then u is a strong solution at x to the partial differential equation

$$\begin{aligned} \Delta u = & -\frac{2A_t(u,|\nabla u|^2)}{A(u,|\nabla u|^2)} \sum_{l,q=1}^{n} u_{x_l}\, u_{x_q}\, u_{x_q x_l} \\ & - \frac{|\nabla u|^2 A_s(u,|\nabla u|^2)}{A(u,|\nabla u|^2)} - \frac{C(u,|\nabla u|^2)}{A(u,|\nabla u|^2)} = \theta. \end{aligned} \tag{2.29}$$

Taking partials of this equation with respect to x_i we get an equation for Δu_{x_i} which we can then put into the expression for I_4 and integrate by parts. Indeed,

$$\begin{aligned} -I_4 = & -\sum_{m\in\Lambda} \int u\sigma(|\nabla u|) \sum_{i=1}^{n} (\Delta u_{x_i})\, u_{x_i} \eta_m dx \\ = & \sum_{m\in\Lambda} \int u\sigma(|\nabla u|)\ \theta\, (\Delta u)\, \eta_m dx \\ & + \sum_{m\in\Lambda} \int u\, \sigma(|\nabla u|)\, \theta \sum_{i=1}^{n} \langle u_{x_i}, (\eta_m)_{x_i} \rangle dx \\ & + \sum_{m\in\Lambda} k \int u\sigma(|\nabla u|)^{1-2/k}\ \theta \sum_{i,j=1}^{n} u_{x_i} u_{x_j} u_{x_i x_j}\, \eta_m dx \\ & + \sum_{m\in\Lambda} \int |\nabla u|^2\, \sigma(|\nabla u|)\, \theta\, \eta_m dx \\ = & \ I_{41} + I_{42} + I_{43} + I_{44}. \end{aligned} \tag{2.30}$$

From (2.29), (2.13), (2.6) and a use of Hölder's inequality as in (2.19) we find that

$$\begin{aligned} |I_{41}| = & \,|\sum_{m\in\Lambda} \int u\sigma(|\nabla u|)\, \theta^2\, \eta_m\, dx| \\ \le & \ c \sum_{m\in\Lambda} \int u\sigma(\nabla u) \left(\sum_{i,j=1}^{n} u_{x_i} u_{x_j}\, u_{x_i x_j} \right)^2 \eta_m\, |\nabla u|^{-4} dx + c(\beta,k) r^n, \end{aligned} \tag{2.31}$$

where c depends only on the data. Also as in the estimate of I_5 and I_{23} we obtain

$$|I_{42}| \le c(\beta, k, \delta_0)\, r^{n-1}. \tag{2.32}$$

To estimate I_{43} we use (1.3), (2.13) and (2.6) as previously and make important use of (1.1) (b) to obtain

$$
\begin{aligned}
I_{43} &\leq -\sum_{m\in\Lambda} 2k \int u\,\sigma(|\nabla u|)^{1-2/k} \frac{A_t(u,|\nabla u|^2)}{A(u,|\nabla u|^2)} \left(\sum_{i,j=1}^n u_{x_i}u_{x_j}u_{x_ix_j}\right)^2 \eta_m\,dx + c(\beta,k)r^{n-1} \\
&\leq -(c_1-1)k\sum_{m\in\Lambda}\int u\sigma(|\nabla u|)^{1-2/k}\left(\sum_{i,j=1}^n u_{x_i}u_{x_j}u_{x_ix_j}\right)^2 |\nabla u|^{-2}\,\eta_m\,dx + c(\beta,k)\,r^{n-1}
\end{aligned} \tag{2.33}
$$

Finally to handle I_{44} let

$$\psi(\hat{s},\hat{t}) = -\int_0^{\hat{t}} \frac{\tau\sigma(\tau^{1/2})A_\tau(\hat{s},\tau)}{A^2(\hat{s},\tau)}d\tau.$$

Arguing as as in the estimate for I_2 we obtain

$$I_{44} \leq \left|\sum_{m\in\Lambda}\int A(u,|\nabla u|^2)\langle\nabla u,\nabla[\psi(u,|\nabla u|^2)\eta_m]\rangle\,dx\right| + c(\beta,k,\delta_0)r^{n-1} \leq c(\beta,k,\delta_0)r^{n-1}, \tag{2.34}$$

thanks again to (1.4). Putting (2.31)-(2.34) into (2.30) we conclude

$$
\begin{aligned}
-I_4 &\leq c\sum_{m\in\Lambda}\int u\sigma(\nabla u)\left(\sum_{i,j=1}^n u_{x_i}u_{x_j}\,u_{x_ix_j}\right)^2 |\nabla u|^{-4}dx \\
&\quad + (1-c_3)k\sum_{m\in\Lambda}\int u\sigma(|\nabla u|)^{1-2/k}\left(\sum_{i,j=1}^n u_{x_i}u_{x_j}u_{x_ix_j}\right)^2 |\nabla u|^{-2}\,\eta_m\,dx + c(\beta,k,\delta_0)r^{n-1}.
\end{aligned} \tag{2.35}
$$

Combining (2.23), (2.28), (2.35), and using the definition of I_3 in (2.17) we conclude for k large enough that

$$
\begin{aligned}
I_1 &= -I_2 - I_3 - I_4 - I_5 \;\leq\; c(\beta,k,\delta_0)\,r^{n-1} \\
&\quad + (-c_1k+c)\sum_{m\in\Lambda}\int u\sigma(\nabla u)\left(\sum_{i,j=1}^n u_{x_i}u_{x_j}\,u_{x_ix_j}\right)^2 |\nabla u|^{-4}dx \;\leq c(\beta,k,\delta_0)r^{n-1}.
\end{aligned} \tag{2.36}
$$

We now remove assumption (2.14). Let $r, z, \xi, \Lambda, \Lambda_1$, be as defined earlier. Let O be an open set with smooth boundary, $\bar{O} \subset D\cap N\cap\{|\nabla u| > \delta_0/2\}$, and the property that if $y \in \bigcup_{m\,\in\,\Lambda\,\cup\,\Lambda_1}$ supp η_m with $|\nabla u(y)| \geq 3\delta_0/4$, then $y \in O$. This open set can be obtained for example by regularizing $d(\cdot,\partial D \cup \{|\nabla u| = \delta_0/2\})$ and using Sard's theorem. Let $\{A(\cdot,\epsilon)\}, \{A_t(\cdot,\epsilon)\}, \{C(\cdot,\epsilon)\}$, denote sequences of infinitely differentiable functions on $\mathbb{R}^2$ which converge uniformly on compact subsets of $(0,\infty)\times(0,\infty)$ to A, A_t, C. Let $\{u_\epsilon\}$ be a sequence of infinitely differentiable functions converging uniformly on compact subsets of O to u. Let $\tilde{u} = \tilde{u}(\cdot,\epsilon)$ be the solution to the Dirichlet problem for O with boundary values u corresponding to the partial differential equation

$$\Delta\tilde{u} + \frac{A_t(u_\epsilon,|\nabla u_\epsilon|^2,\epsilon)}{A(u_\epsilon,|\nabla u_\epsilon|^2,\epsilon)}\sum_{i,j=1}^n \tilde{u}_{x_i}\tilde{u}_{x_j}\tilde{u}_{x_ix_j} = \frac{-C(u_\epsilon,|\nabla u_\epsilon|^2,\epsilon) - |\nabla u_\epsilon|^2 A_s(u_\epsilon,|\nabla u_\epsilon|^2,\epsilon)}{A(u_\epsilon,|\nabla u_\epsilon|^2,\epsilon)}$$

Using Calderón Zygmund type estimates, Lemma 2.4, and Schauder's theorem we see for ϵ small enough that $\tilde{u}$ exists and is infinitely differentiable on O. Moreover from Lemma 2.4 and arguments similar to those in [GT, section 9.6] we deduce that $\tilde{u}, \nabla\tilde{u}$ converge uniformly on compact subsets of O to $u, \nabla u$ while

$$\int_K |\tilde{u}_{x_i x_j} - u_{x_i x_j}|^2 \, dx \to 0 \text{ as } \epsilon \to 0 \tag{2.37}$$

whenever K is a compact subset of O. Let

$$\tilde{I}_1 = c^{-1} \int_\Omega u\sigma(|\nabla u|) \sum_{i,j=1}^n (\tilde{u}_{x_i x_j})^2 dx$$

where the integrand is defined to be 0 outside of O. We repeat the integration by parts in (2.17), getting $\tilde{I}_2, \tilde{I}_3, \tilde{I}_4, \tilde{I}_5$. We can then let $\epsilon\to 0$ in the integrals defining $\tilde{I}_2, \tilde{I}_3, \tilde{I}_5$, to get I_2, I_3, I_5. The estimate for I_5 is unchanged. I_2 can also be estimated as previously using the fact that the equality for Δu in (2.29) holds H^n almost everywhere on O and is square integrable on $\Omega \cap O$. As for I_4 we repeat the integration by parts in (2.30) involving third derivatives and then let $\epsilon\to 0$ again to get $I_{41}, I_{42}, I_{43}, I_{44}$ whereupon we can once again repeat the previous argument. Thus (2.36) holds without assumption (2.14). Since none of the constants depend on ξ we can let $\xi\to 0$ to conclude that Theorem 1.8 is true. $\Box$

3. Proof of Theorem 1.10. In this section we first show that

$$\int_{D\cap B(z,r)} u\,|\nabla u|^k \sum_{i,j=1}^n u_{x_i x_j}^2 \, dx \le c(\beta, k, \gamma) r^{n-1} \tag{3.1}$$

for $0 < r \le r_0$ where u, μ, D, A, p, k_0 are as in the statement of Theorems 1.8 and 1.10. In fact, the only constants which depend on δ_0 in the proof of Theorem 1.8 were those obtained when a derivative fell on η_m in the various integration by parts. That is in the estimates for I_5, I_{23}, I_{42} and I_{44}. Moreover from (2.21) we see that the dependence of these constants on δ_0 was needed to insure that

$$r_m^{n-1} \le c(\beta, k, \delta_0)\mu[B(y_m, 10^{10n} r_m)].$$

This inequality is now trivial by (1.11). Thus if (1.11) holds, then the constants in Theorem 1.8 can be chosen independent of δ_0. Letting $\delta_0\to 0$ in (1.9) we get (3.1). Next we note that for some $1 \le M < \infty$, ∂D is $n-1$ Ahlfors regular. That is,

$$M^{-1} r^{n-1} \le H^{n-1}(B(z,r) \cap \partial D) \le M\, r^{n-1} \tag{3.2}$$

whenever $z \in \partial D$ and $0 < r \le r_0$. In fact using measure theoretic type arguments one sees that μ can be replaced by H^{n-1} in (1.7), (1.11) (see [Ma]). Also from (1.11) and (2.9), we deduce that

$$r \le c \max_{B(z,r)} u \text{ for each } z \in \partial D \text{ and } 0 < r \le r_0/c_3. \tag{3.3}$$

Choose $y^* \in B(z,r)$ with $u(y^*) \ge r/c$. Using the mean value theorem from elementary calculus we find for some $\delta_1 > 0$ (depending on β, γ, as well as the data) and y on the line segment from y^* to z that

$$|\nabla u(y)| \ge \delta_1. \tag{3.4}$$

Also, from (2.11), Lemma 2.4, and (2.13) we deduce that y can be chosen so that for some $c = c(\beta, \gamma)$

$$r \leq c\, d(y, \partial D) \text{ and } r/c \leq u(y) \leq cr \tag{3.5}$$

where we are now writing $d(y)$ for $d(y, \partial D)$. Fix $k \geq k_0$ as in (3.1). If $B(x, d(x)) \subset D \cap N$, then we shall say that $B(x, d(x))$ is a good ϵ tangent ball ($0 < \epsilon \leq 100^{-100}$) provided

(α) $|\nabla u(x)| \geq \delta_1$.

(β) If x can be joined to $\hat{x}$ by a chain of at most $1/\epsilon$ balls $\{B(y_i, d(y_i)/2)\}$ with $B(y_i, d(y_i)) \subset D \cap N$ and $\epsilon^3 d(x) \leq d(y_i) \leq \epsilon^{-3} d(x)$, then (3.6)

$$\max_{z_1, z_2 \in B(\hat{x}, (1-\epsilon^{10}) d(\hat{x}))} \left| |\nabla u|^k \nabla u(z_1) - |\nabla u|^k \nabla u(z_2) \right| \leq \epsilon^{10}.$$

Otherwise $B(x, d(x))$ is said to be a bad tangent ball. By a chain of balls we mean as in (1.26) that successive balls have nonempty intersection. Let $K \subset \mathbb{R}^{n+1}$ denote the set of all $(z, r), z \in \partial D, 0 < r \leq \epsilon^5\, r_0$, for which there is a bad tangent ball $B(y, d(y)) \subset D$ with $y \in B(z, 4r)$, $|\nabla u(y)| \geq \delta_1$, and $r\epsilon^3 \leq d(y) \leq 4r$. If $(z, r) \in K$ then from the definition of K we see there exists y as above and $\hat{y}$ with $|\hat{y} - z| \leq c\epsilon^{-4} r$, $\epsilon^3 d(y) \leq d(\hat{y}) \leq \epsilon^{-3} d(y)$. Moreover, $\hat{y}$ can be joined to y by a chain of balls as in (3.6) (β) and

$$\max_{z_1, z_2 \in B(\hat{y}, (1-\epsilon^{10}) d(\hat{y}))} \left| |\nabla u|^k \nabla u(z_1) - |\nabla u|^k \nabla u(z_2) \right| > \epsilon^{10}. \tag{3.7}$$

In this case we claim for ϵ small enough, say $0 < \epsilon \leq \epsilon_0 = \epsilon_0(\beta, \gamma, k)$, that if $\Gamma = \Gamma(z, r) = \{w \in D \cap B(z, \epsilon^{-5} r) : d(w) \geq \epsilon^{15} r\}$, then

$$r^{n-1} \leq c(\beta, \gamma, k, \epsilon) \int_\Gamma u\, |\nabla u|^{2k} \sum_{i,j=1} u^2_{x_i x_j} dx. \tag{3.8}$$

In fact (3.7) and the triangle inequality imply for some $w \in B(\hat{y}, (1 - \epsilon^{10}/2) d(\hat{y}))$ and c_* depending only on n that

$$\max_{w_1, w_2 \in B(w, \epsilon^{10} d(\hat{y})/c_*)} \left| |\nabla u|^k \nabla u(w_1) - |\nabla u|^k \nabla u(w_2) \right| \geq \epsilon^{20}/c_*^2. \tag{3.9}$$

This last inequality and (2.5) yield

$$\left| |\nabla u|^k \nabla u(\hat{w}) - |\nabla u|^k \nabla u(w^*) \right| \geq \epsilon^{20}/(c_*^3) \tag{3.10}$$

whenever

$$|\hat{w} - w_1| + |w^* - w_2| \leq c(\beta, k)^{-1}\, \epsilon^{(10\sigma + 20)/\sigma}\, d(\hat{y}) = a$$

and $c(\beta, k)$ is large enough. Finally from (2.3), the chain assumption, and (3.5) we see that if $r_1(\epsilon) = \exp(-1/\epsilon), 0 < \epsilon \leq \epsilon_0$, and ϵ_0 is small enough (depending only on the data), then

$$r \leq c(\beta, \gamma, \epsilon)\, u \text{ on } B(\hat{y}, (1 - \epsilon) d(\hat{y})). \tag{3.11}$$

Using (3.9)-(3.11) and essentially Poincáre's inequality, we deduce that

$$
\begin{aligned}
a^{n-1} \;&\leq\; c(\beta,\gamma,\epsilon) a^{-1} \int_{B(0,a)} ||\nabla u|^k \, \nabla u(w_1 + x) - |\nabla u|^k \, \nabla u(w_2 + x)|^2 \, dx \\
&\leq\; c(\beta,\gamma,k,\epsilon) \int_{B(y,d(\hat{y})/c_*)} u \, |\nabla u|^{2k} \sum_{i,j=1}^{n} u_{x_i x_j}^2 dx \qquad (3.12)\\
&\leq c(\beta,\gamma,k,\epsilon) \int_{\Gamma} u \, |\nabla u|^{2k} \sum_{i,j=1}^{n} u_{x_i x_j}^2 dx.
\end{aligned}
$$

Since $r \leq c(\beta,\gamma,k,\epsilon) a$, we conclude the validity of (3.8) from (3.12).

Let diam D denote the diameter of D. We use (3.8) to show for $\hat{z} \in \partial D$, $0 < \rho \leq 2$ diam D and $\hat{K} = K \cap [B(\hat{z},\rho) \times (0,\rho)]$ that

$$
\int_{\hat{K}} dH^{n-1} dt/t = \int H^{n-1}[\hat{K} \cap (\mathbb{R}^n \times \{t\})] dt/t \;\leq\; c(\beta,\gamma,k,\epsilon) \rho^{n-1}. \qquad (3.13)
$$

To do this if $0 < \rho \leq r_1(\epsilon)^2$, we first use a well known covering lemma, to get $\{B(x_{ml}, r_{ml}) \times (0, 2r_{ml})\}$, $(x_{ml}, r_{ml}) \in \hat{K}$, $2^{-m-1}\rho \leq r_{ml} < 2^{-m}\rho$, a covering of

$$
\hat{K}_m = \hat{K} \cap [\mathbb{R}^n \times (2^{-m-1}\rho, 2^{-m}\rho)]
$$

with the balls, $\{B(x_{ml}, r_{ml}/100)\}$, pairwise disjoint. Second from (3.8) with (z,r) replaced by (x_{ml}, r_{ml}) and (3.2), we obtain

$$
\int_{\hat{K}_m} dH^{n-1} dt/t \leq c \sum_l r_{ml}^{n-1} \;\leq\; c(\beta,\gamma,\epsilon) \sum_l \int_{\Gamma(x_{ml}, r_{ml})} u \, |\nabla u|^{2k} \sum_{i,j=1}^{n} u_{x_i x_j}^2 dx. \qquad (3.14)
$$

From pairwise disjointness of $\{B(x_{ml}, r_{ml}/100)\}$ for fixed m, the usual 'volume argument,' and the definition of $\Gamma(x_{ml}, r_{ml})$ we see for fixed (m', l') that the set of all (m, l) such that $\Gamma(x_{m'l'}, r_{m'l'}) \cap \Gamma(x_{ml}, r_{ml}) \neq \emptyset$ has cardinality at most $N(\epsilon)$. Using this fact and summing over m we see from (3.14), (3.1) that

$$
\begin{aligned}
\int_{\hat{K}} dH^{n-1} dt/t \;&\leq\; c \sum_{m,l} r_{ml}^{n-1} \;\leq c(\beta,\gamma,k,\epsilon) \sum_{m,l} \int_{\Gamma(x_{ml}, r_{ml})} u \, |\nabla u|^{2k} \sum_{i,j=1}^{n} u_{x_i x_j}^2 dx \\
&\leq c(\beta,\gamma,k,\epsilon) \int_{D \cap B(\hat{z}, 2\rho)} u \, |\nabla u|^{2k} \sum_{i,j=1}^{n} u_{x_i x_j}^2 \, dx \leq\; c(\beta,\gamma,k,\epsilon) \rho^{n-1}
\end{aligned} \qquad (3.15)
$$

for $0 < \rho \leq r_1(\epsilon)^2$. From (3.2) we see that this inequality remains valid for $0 < \rho \leq 2$ diam D, provided $c(\beta,\gamma,k,\epsilon)$ is large enough. Thus (3.13) is true. We conclude from (3.13) (in the language of [DS1]) that if χ denotes the characteristic function of K, then $\chi_K \, dx \, dt/t$ is a Carleson measure on $\partial D \times$ (0, 2 diam D).

Next suppose that $(z, r), z \in \partial D, 0 < r \le r_1(\epsilon)^2$, is not in K. Then if $w \in \partial D \cap B(z, r/2)$ and $r' \ge c\epsilon^3 r$, we see that

$$B(w, r') \text{ contains a good tangent ball of radius } r'/c, \tag{3.16}$$

thanks to (3.3) - (3.5) with r, z replaced by r', w. We shall show for $(z, r) \notin K$ that the following weak exterior convexity condition holds:

$$\begin{array}{l}\text{If } \hat{x}, \hat{z} \text{ can be joined by a curve } \sigma \subset B(z,r) \setminus \partial D, \text{ with } d(\sigma, \partial D) \ge \epsilon r, \\ \text{then the line segment}, l, \text{ from } \hat{x} \text{ to } \hat{z} \text{ lies in } B(z,r) \setminus \partial D.\end{array} \tag{3.17}$$

We remark that (3.2), (3.13), (3.17) are shown in [DS1, II 3.3] to be equivalent to uniform rectifiability. To prove (3.17) let $y \in B(z, r/2)$ be as in (3.4), (3.5), and suppose $0 \in \bar{B}(y, d(y)) \cap \partial D$. From (3.16) and the definition of a good tangent ball we see that

$$||\nabla u|^{k+1}(w) - |\nabla u|^{k+1}(y)| \le c(\beta, \gamma, k)\epsilon^{10} \text{ when } w \in B(y, (1 - \epsilon^{10})d(y)). \tag{3.18}$$

Thus

$$||\nabla u|(w) - |\nabla u|(y)| \le c(\beta, \gamma, k)\epsilon^{10}. \tag{3.19}$$

Using once again the definiton of a good tangent ball and (3.19) we get

$$|\nabla u(w) - \nabla u(y)| \le c(\beta, \gamma, k)\epsilon^{10} \text{ for } w \in B(y, (1 - \epsilon^{10})d(y)) \tag{3.20}$$

and $0 < \epsilon \le \epsilon_0$, Using (3.20), (2.13) and the mean value theorem from differential calculus, we see that

$$|u(w) - u(y) - \langle \nabla u(y), w - y\rangle| \le c(\beta, \gamma, k)\epsilon^{10} d(y) \tag{3.21}$$

whenever $w \in \bar{B}(y, d(y))$. To simplify matters suppose that $y = d(y)e_n$ where $e_n = (0, \dots, 0, 1)$. Then (3.21) implies that in $B(y, d(y))$, u is within $c(\beta, \gamma, k)\epsilon^{10} d(y)$ of a linear function. Since $u > 0$ in $B(y, d(y))$ and $u(0) = 0$, it follows that

$$|\langle \nabla u(y), e_n\rangle - |\nabla u(y)|| \le c(\beta, \gamma, k)\epsilon^{10}$$

which implies

$$||\nabla u(y)|e_n - \nabla u(y)| \le c(\beta, \gamma, k)\epsilon^5. \tag{3.22}$$

Next suppose $\hat{x}$ is as in (3.6). Then from (3.6), (3.22), and the triangle inequality we see as in (3.18) - (3.20) that

$$|\nabla u(w) - |\nabla u(y)|e_n| \le c(\beta, \gamma, k)\epsilon^5 \tag{3.23}$$

whenever $w \in B(\hat{x}, (1 - \epsilon^{10})d(\hat{x}))$. Now (3.23) holds whenever $\hat{x}$ can be connected to y by a chain of at most ϵ^{-1} balls as in (3.6). Therefore (3.23) holds with $\hat{x}$ replaced by a center of a ball in the chain. Using this fact and choosing a curve γ contained in the chain connecting w to 0 we deduce from (3.23), (2.13), that

$$|u(w) - |\nabla u(y)|\, w_n| \le c(\beta, \gamma, k)H^1(\gamma)\epsilon^5 \text{ for } w \in B(\hat{x}, d(\hat{x})). \tag{3.24}$$

Note from (3.24) that if $u(w) = 0$, then

$$|w_n| \le c(\beta, \gamma, k)H^1(\gamma)\epsilon^5. \tag{3.25}$$

We claim for $c(\beta, \gamma, k)$ large enough that every point in $O = B(z, 2r/\epsilon^2) \cap \{w : w_n \geq c(\beta, \gamma, k)\epsilon^2 r\}$ can be joined by a chain of at most $1/\epsilon$ balls as in (3.6). In fact if $t \in (\epsilon^3 d(y), (2\epsilon)^{-3} d(y))$ then from (3.25) and an iterative type argument we see that $(0, \ldots, t)$ can be joined to $(0, \ldots, d(y))$ by a chain of at most $c \log(1/\epsilon)$ balls as in (3.6). Moreover (3.25) holds with $H^1(\gamma)$ replaced by $c\epsilon^{-3} d(y)$. We can then join $(0, \ldots, (2\epsilon)^{-3} d(y))$ to $(w', (2\epsilon)^{-3} d(y))$ whenever $|w'| \leq \epsilon^{-3} d(y)$ by a chain of at most 100 balls and then join this point to $(w', \epsilon^3 d(y))$ by a chain with the desired properties. Thus our claim is true. From our claim we deduce first that

$$\partial D \cap O = \emptyset \tag{3.26}$$

and second for $c^* = c^*(\beta, \gamma, k)$ large enough, that if $w_n = 0, |w - z| \leq r/\epsilon^2$, then

$$B(w, c^* \epsilon^2 r) \cap \partial D \neq \emptyset. \tag{3.27}$$

In fact if (3.27) is false, then for c^* large enough, we have $O \cap B(w, c^* \epsilon^2 r/8) \neq \emptyset$. Using our claim, (3.23), and arguing as in (3.24) we see that

$$|u(\zeta) - |\nabla u(y)| \zeta_n| \leq c(\beta, \gamma, k)\epsilon^2 r$$

for all ζ in $B(w, c^* \epsilon^2 r/4)$. This inequality is impossible for c^* large enough since $u > 0$. Thus (3.27) is valid. From (3.27) we see for $0 < \epsilon \leq \epsilon_0(\beta, \gamma, k)$, that if $\partial D \cap B(z, r) \subset S = \{w : |w_n| \leq \epsilon r/2\}$, then every curve σ as in (3.17) must satisfy $\sigma \cap S = \emptyset$. From connectivity of σ, it then follows that $l \cap \partial D = \emptyset$.

If $\partial D \cap B(z, r)$ is not contained in S we deduce from (3.26) that there exists $v \in \partial D \cap B(z, r)$ with $v_n \leq -\epsilon r/2$. Then from (3.16) we see that $B(v, \epsilon r/4) \cap D$ contains a good tangent ball $B(\hat{v}, d(\hat{v}))$ with $d(\hat{v}) \geq r\epsilon/c$. Let $v^* \in \partial B(\hat{v}, d(\hat{v})) \cap \partial D$. We can repeat the above argument leading to (3.26). We get for some η with $|\eta| = 1$ that if $O_1 = \{w : \langle (w - v^*), \eta \rangle \geq cr\epsilon^3\} \cap B(z, r/(c\epsilon))$, then $\partial D \cap O_1 = \emptyset$ provided $c = c(\beta, \gamma, k)$ is large enough. Also, as in (3.27) we find that each point of $\partial O_1 \cap B(z, r/(c\epsilon))$ lies within $cr\epsilon^3$ of a point of ∂D. Moreover $O_1 \cap O \cap B(z, 10r) = \emptyset$, since otherwise we could easily get a contradiction to (3.24) and/or (3.25). Finally we claim that every point of $\partial D \cap B(z, 10r)$ lies within $c(\beta, \gamma, k)^{-1}\epsilon r$ of a point in $\partial O \cup \partial O_1$. Indeed otherwise we could repeat the above argument getting an open set O_2 with the same properties as O, O_1. These three open sets could then not intersect in $B(z, r/\epsilon^{1/2})$ for sufficiently small ϵ which is clearly impossible. Finally we conclude from this discussion that σ as in (3.17), must lie at least $\epsilon r/2$ away from $\partial O_1 \cup \partial O_2$ and so by convexity of $B(z, r) \setminus (O_1 \cup O_2)$, we have $l \subset B(z, r) \setminus \partial D$.

Let H be the set of all (z, r) for which (3.17) is false and put $H_1 = H \cup [\partial D \times (r_1(\epsilon)^2, 2 \text{ diam } D)]$. Then we have just shown that $H \subset K$, where K is defined above (3.13). Using this fact and (3.13), it is easily seen that $d\nu = \chi_{H_1} t^{-1} \, dH^{n-1} dt$ is a Carleson measure on $\partial D \times (0, 2 \text{ diam D})$ in the sense that

$$\nu(B(z, \rho) \cap \partial D \times (0, \rho)) \leq c(\beta, \gamma, k, r_0, \epsilon, D) \, \rho^{n-1} \tag{3.28}$$

whenever $0 < \rho \leq 2$ diam D and $z \in \partial D$. Finally if P denotes any $n-1$ dimensional plane whose distance from ∂D is $\approx$ diam D, then it is easily checked that $\partial D \cup P$ satisfies a global weak exterior convexity condition and thus in view of the remark after (3.17) is uniformly rectifiable. The proof of Theorem 1.10 is now complete. □

4. Proof of Theorem 1.27 in a Special Case. We continue with the same notation as in sections 2 and 3. In this section we prove Theorem 1.27 under the assumption that D satisfies a Carleson measure chain condition. More specifically let $T \subset \mathbb{R}^{n+1}$ be the set of all $(z,r), z \in \partial D, 0 < r \leq r_0$, for which the chain condition stated above (1.26) is false. We assume for some $1 \leq c_3 < \infty$ as in (1.26) that $\chi_T \; dH^{n-1}dt/t$ is a Carleson measure on $\partial D \times (0, r_0)$ defined as in (3.13) with K replaced by T. That is, for each $z \in \partial D, 0 < \rho \leq r_0$, and $\hat{T} = T \cap [B(z,\rho) \times (0,\rho)]$, we have

$$\int_{\hat{T}} dH^{n-1}\, dt/t = \int H^{n-1}[\hat{T} \cap (\mathbb{R}^n \times \{t\})]dt/t \;\leq\; c_6 \rho^{n-1}. \tag{4.1}$$

In this section we again let c be a positive constant depending on the data but with the understanding that the data is now interpreted as $n, p, c_1 - c_6, \beta, \gamma, D, r_0$, as well as the C^2 sup norm of A, C^1 sup norm of C on $[0, 2\bar{c}\,\beta^{1/(p-1)}] \times [\delta_1/2, 2\bar{c}\,\beta^{1/(p-1)}]$ where δ_1 is as in (3.4) and $\bar{c}$ is chosen so large that $u + |\nabla u| \leq \bar{c}\,\beta^{1/(p-1)}$ in $N_1 \cap D$ (see (2.11), (2.13)). We first prove

Lemma 4.2 *Let D be as as in Theorem 1.10 and (4.1). Fix $z' \in \partial D$ and suppose that $z \in B(z', r_0/8) \cap \partial D$. If $0 < r \leq r_0/8$, then there exists $c \geq 1000$ and points $\tilde{y}, y$ in $B(z,r)$ with $\min\{d(\tilde{y}), d(y)\} \geq r/c$ and the property that $\tilde{y}, y$ are in different components of $B(z', r_0/2) \setminus \partial D$.*

Proof: Let K be as defined above (3.7). For z', r, z as above and small $\epsilon > 0$ (to be fixed at the end of the proof), we claim there exists $y' \in B(z, r/2) \cap \partial D, c(\epsilon) \geq 1$, and $\rho, r/c(\epsilon) \leq \rho \leq r/4$, such that

$$(y', \rho) \text{ is not in } T \cup K. \tag{4.3}$$

Indeed, otherwise it would follow for some $c = c(n) \geq 1$ and M as in (3.2) that

$$c(n) M \int_{\rho}^{r/2} t^{-1}\, H^{n-1}[(T \cup K) \cap (B(z, r/2) \times \{t\})]\, dt \geq r^{n-1}\, \ln[r/(2\rho)]$$

and this inequality contradicts either (3.13) or (4.1) for $\rho = r/c(\epsilon)$ small enough. Using (4.3), we can now argue as in the discussion leading to (3.26)-(3.28) to get Lemma 4.2. In fact in this discussion we showed that only two possible alignments of ∂D are possible when $(y', \rho) \notin K$. The first possible alignment is that every point in $B(y', \rho) \cap \partial D$ lies within $\epsilon\rho$ of a point of some plane P while every point in $P \cap B(y', \rho/\epsilon)$ lies within $\epsilon\rho$ of a point of ∂D. Also all the points in one component of $B(y', \rho) \setminus P$ are contained in D. The second possible alignment of ∂D is that every point in $B(y', \rho) \cap \partial D$ lies within $\epsilon\rho$ of two planes P, P_1 and every point in $(P \cup P_1) \cap B(y', \rho/\epsilon^{1/2})$ lies within $\epsilon\rho$ of a point of ∂D. Moreover $P_1 \cap P \cap B(y', \rho/\epsilon^{1/2}) = \emptyset$. Again the points in one component of $B(y', \rho) \setminus P$ are contained in D and y' lies within $\epsilon\rho$ of P. In either alignment it is easily seen for $\epsilon > 0$ small enough, that we can choose $y, \tilde{y}$ such that $y \in D \cap B(y', \rho/2), \tilde{y} \in B(y', \rho/2)$, and $y, \tilde{y}$ are symmetric with respect to P. Moreover, $\min\{d(\tilde{y}), d(y)\} \geq \rho/100$. Using (1.26) we see for ϵ small enough that $\tilde{y}, y$ cannot lie in the same component of $B(y', r_0) \cap D$ since any chain connecting y' to $\tilde{y}$ has $\approx \ln(1/\epsilon)$ members. Thus y, y' lie in different components of $B(y', r_0) \setminus \partial D$ and so in different components of $B(z', r_0/2) \setminus \partial D$. Fix ϵ subject to the above requirements. Then ϵ depends only on the data and the proof of Lemma 4.2 is complete. $\square$

Next we shall say ∂D contains big pieces of Lipschitz graphs provided there exists $c_7, c_8 \geq 1$ such that whenever $z \in \partial D$ and $x \in D \cap B(z, r_0/16)$, with $d(x) \leq r_0/c_7$, we can find a domain Ω

with

(α) $\quad \Omega \subset B(x, 20\, d(x)) \cap D.$

(β) $\quad$ After a rotation of coordinates if necessary,

$$\Omega = \{w = (w', w_n) : \psi(w') < w_n < x_n + d(x)/2,\ |x' - w'| < d(x)/c_7\}. \tag{4.4}$$

(γ) $\quad \|\psi\|_\infty = \sup_{z' \in \mathbb{R}^{n-1}} |\psi(z')| \le x_n - d(x)/c_7$

for some $\psi : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ with

$$|\psi(y') - \psi(z')| \le c_7\, |y' - z'| \text{ for all } y', z' \in \mathbb{R}^{n-1}. \tag{4.5}$$

Moreover

$$H^{n-1}(\{(w', \psi(w')) : |x' - w'| \le d(x)/c_7\} \cap \partial D) \ge d(x)^{n-1}/c_8. \tag{4.6}$$

With this notation we prove

Lemma 4.7. *∂D contains big pieces of Lipschitz graphs. c_7, c_8 depend only on the data.*

Proof: If $z \in \partial D$ and $x \in B(z, r_0/16) \cap D$ we choose $z^* \in \bar{B}(x, d(x)) \cap \partial D$. Let $\mathcal{S} = [B(z^*, 100 d(x)) \cap \partial D] \cup \partial B(z^*, 100\, d(x))$. Then using (3.2), Lemma 4.2, it is easily checked that $\mathcal{S}$ is Ahlfors regular (for radii $\le 100 d(x)$) and in the language of [DJ], [Se], satisfies a two balls condition. That is, given $w \in \mathcal{S}$, $0 < r \le 100\, d(x)$, there exists two balls of radii approximately equal to r which lie in different components of $\mathbb{R}^n \setminus \mathcal{S}$ and whose centers are in $B(w, r)$. Therefore $B(x, d(x))$ and some ball of approximately the same size are contained in $B(z^*, 10 d(x))$ and lie in different components of $\mathbb{R}^n \setminus \mathcal{S}$. Lemma 4.7 now follows for c_7, c_8 suitably large, depending only on the data, from a clever geometric argument of David and Jerison (see the remark following (10) in [DJ]). We omit the details. $\square$

Next we note from (1.1) (c) that A extends continuously to $[0, \infty) \times (0, \infty)$ and second from (1.1) (b) that $t \rightarrow t A(0, t^2)$ is increasing on $(\ 0, \infty\ )$. Thus to prove Theorem 1.27 it suffices to show that if

$$\limsup_{x \longrightarrow \partial D} |\nabla u(x)| = b \tag{4.8}$$

then

$$b A(0, b^2) \le \beta_1. \tag{4.9}$$

Note from (3.4) that $b \ge \delta_1$. To prove (4.9) let $\tau, 0 < \tau < 10^{-4} \min\{\delta_1, 1\}$, be a small positive number and put

$$v(x) = \max[|\nabla u|^2(x) - (b - \tau)^2, 0], \text{ for } x \in D \cap N.$$

We shall need to find a suitable partial differential equation for which v is a subsolution. To this end, observe from the assumptions on A, C in Theorem 1.27, linear theory for weak solutions to divergence form partial differential equations, (2.5), and (2.13) that if $x \in N_1 \cap D$ and $|\nabla u(x)| \ge \delta_1/4$, then there exists $1 \le c_9 < \infty$ such that $|\nabla u| \ge \delta_1/8$ on $B(x, 4 d(x)/c_9)$. Moreover,

$$d(x)^{n+1} \max_{y \in B(x, d(x)/c_9)} \sum_{i,j=1}^{n} u_{y_i, y_j}^2(y) \ \le\ c_9 \left[\int_{B(x, 2d(x)/c_9)} u \sum_{i,j=1}^{n} u_{y_i, y_j}^2(y)\, dy \ +\ r^{n-1} \right] \tag{4.10}$$

when $x \in D \cap N_1$, as follows (at least in spirit) from differentiating the partial differential equation that u satisfies twice and making estimates similar to those in Lemma 2.1 for $p = 2$ (with u replaced by $u_{y_i y_j}, 1 \leq i, j \leq n$). As usual, c_9 depends only on the data. Also from straightforward differentiation and (1.4) we see that $w = u_{x_l}, 1 \leq l \leq n$, is a weak solution for $y \in B(x, d(x)/c_9)$ to the partial differential equation

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial y_i}(\hat{b}_{ij} w_{y_j}) \,+\, \langle\, \hat{d}\,,\, \nabla w \,\rangle + \hat{e}\, w = 0 \tag{4.11}$$

where

$$\begin{aligned} \hat{b}_{ij}(x) &= 2A_t(u(x), |\nabla u|^2(x)) u_{y_i}(x)\, u_{y_j}(x) + \delta_{ij}\, A(u(x), |\nabla u|^2(x)), 1 \leq i, j \leq n, \\ \hat{d}(x) &= A_s(u(x), |\nabla u|^2(x)) \nabla u(x) + 2C_t(u(x), |\nabla u|^2(x)) \nabla u(x), \\ \hat{e}(x) &= \nabla \cdot (A_s(u(x), |\nabla u|^2(x)) \nabla u(x)) + C_s(u(x), |\nabla u|^2(x)). \end{aligned} \tag{4.12}$$

In (4.12), δ_{ij} is the Kronecker δ. Let $\phi \in C_0^\infty(\mathbb{R})$, $0 \leq \phi \leq 1$, with $\phi \equiv 1$ on $[\delta_1/2, 2\bar{c}\beta^{1/(p-1)}]$ and $\phi \equiv 0$ on $(-\infty, \delta_1/4]$ where $\bar{c}$ was chosen above Lemma 4.2 so that $u + |\nabla u| \leq \bar{c}\beta^{1/(p-1)}$ in $N_1 \cap D$. For each $x \in N_1 \cap D$ put

$$b_{ij}(x) = \delta_{ij} + \phi(|\nabla u(x)|)\, (\hat{b}_{ij}(x) - \delta_{ij}),\ \ 1 \leq i, j \leq n. \tag{4.13}$$

Using (1.1), (4.10) - (4.12) and arguing as in the proof of Lemma 2.7 we deduce for $\zeta \in C_0^\infty(N_1 \cap D)$, $\zeta \geq 0$, that

$$\int_{N_1 \cap D} \left[-\sum_{i,j=1}^{n} (b_{ij} v_{y_j} \zeta_{y_i}) \,+\, \Big(\langle\, \hat{d}\,,\, \nabla v \,\rangle + 2\hat{e}\, (v + [b - \tau]^2) \Big)\, \zeta \right] dy \geq 0. \tag{4.14}$$

Put $b_{ij} = \delta_{ij}$ for $1 \leq i, j \leq n$ when $y \in D \setminus N_1$. Then v is a weak subsolution on $N_1 \cap D$ to $Lw = f$ where

$$\begin{aligned} Lw &= \sum_{i,j=1}^{n} \frac{\partial}{\partial y_i}(b_{ij} w_{y_j}) \\ f &= -\langle\, \hat{d}\,,\, \nabla v \,\rangle \,-\, 2\hat{e}\, [v - (b - \tau)^2]. \end{aligned} \tag{4.15}$$

Using (1.1) it is easily shown that L is uniformly elliptic on D. From Theorem 1.8 with $\delta_0 = \delta_1/4$, (4.10), and the fact that A has continuous second partials which extend continuously to $[0, \infty) \times (0, \infty)$ we get for $B(z, r) \subset N_1 \cap D$,

$$\int_{B(z,r)} d(x) \max_{y \in B(x, d(x)/2)} \left[\sum_{i,j=1}^{n} |\nabla b_{ij}|^2(y) \right] dx \,\leq\, c\, r^{n-1}. \tag{4.16}$$

Here we have also used that fact that $B(x, d(x)/2)$ can be covered by at most c balls of radius $d(x)/c_9$.

Let $g(\cdot, y)$ denote Green's function for D with pole at y corresponding to L in (4.15) and let $\omega(\cdot, y)$ be the corresponding elliptic measure. That is, $g(\cdot, y)$ is continuous on $\mathbb{R}^n \setminus D$ with $g \equiv 0$ on this set. Moreover $g(\cdot, y)$ has locally square integrable distributional first partials in $D \setminus \{y\}$ and if $\theta \in C_0^\infty(\mathbb{R}^n)$, we have

$$\theta(y) = \int_{\mathbb{R}^n} \sum_{i,j=1}^{n} b_{ij}\, \theta_{x_i}\, g_{x_j}(\cdot, y) dx + \int_{\partial D} \theta\, d\omega(\cdot, y). \tag{4.17}$$

We shall need some basic facts and estimates for g, ω (see [LSW]). First, g is symmetric (i.e $g(\cdot, y) = g(y, \cdot)$) since L is self adjoint. Moreover from (4.17), classical theory, and our smoothness assumptions on A, C, we see that $g(\cdot, y), g(y, \cdot)$, are weak solutions to $Lw = 0$ in $D \setminus \{y\}$ and strong solutions to this equation in $N_1 \cap D$. Thus as in Lemma 2.1 we have

$$r^{2-n} \int_{B(z,r/2)} |\nabla g|^2(x, y) dy \,\leq\, c \max_{B(z,r)} g(x, \cdot)^2 \,\leq\, c^2\, r^{-n} \int_{B(z,2r)} g(x, y)^2\, dy \tag{4.18}$$

provided $x \notin B(z, 4r)$. Also, if E is a Borel subset of ∂D, then $x \to \omega(E, x)$ is a weak solution to L in D and in fact is the bounded solution to the Dirichlet problem for L with boundary value 1 on E and 0 on $\partial D \setminus E$ in the sense of Perron-Wiener-Brelot. Consequently from the weak maximum principle, $0 \leq \omega(E, \cdot) \leq 1$. If r_0 is so small that $\bigcup_{z \in \partial D} B(z, r_0) \subset N_1$, then for some $c \geq 1, 0 < \sigma \leq 1$, all $z \in \partial D$, and $0 < r \leq r_0$

$$\begin{aligned} &(i) \quad c\,\omega(B(z, r) \cap \partial D, x) \geq 1 \text{ whenever } x \in B(z, r/2) \cap D, \\ &(ii) \;\; 1 - \omega(B(z, r) \cap \partial D, x) \leq c(d(x)/r)^\sigma \text{ whenever } x \in B(z, r/2). \end{aligned} \tag{4.19}$$

(i) follows from the fact that $B(z, r) \cap \partial D$ and $B(z, r)$ have comparable Newtonian capacities (logarithmic capacities when $n = 2$) and estimates for subsolutions to linear second order divergence form partial differential equations (see for example [GZ]). (ii) follows from the same argument as in (i) and iteration. From (i), the fact that $g(\cdot, y) \leq c d(y)^{2-n}$ in $\mathbb{R}^n \setminus B(y, d(y)/8)$ and the maximum principle for weak solutions to L we get

$$g(x, y) \leq c\, d(y)^{2-n} \omega(B(y, 4d(y)) \cap \partial D,\, x) \text{ for } x \notin B(y, d(y)/2). \tag{4.20}$$

Also, if $x, y \in D, x \neq y$, then

$$g(x, y) \leq \begin{cases} c|x - y|^{2-n} & \text{for } n > 2 \\[2ex] c \log\left(\frac{\text{diam D}}{|x-y|}\right) & \text{when } n = 2. \end{cases} \tag{4.21}$$

We use (4.19), (4.20) to show that if $w \in \partial D, x \in B(w, \rho) \cap D$, and $B(w, 10^4\rho) \subset N_1$, then

$$\int_{B(w,100\rho) \cap D \setminus B(w,4\rho)} d(y)^{-1}\, g(x, y)\, dy \,\leq\, c\rho \left(\frac{d(x)}{\rho}\right)^\sigma \tag{4.22}$$

where c depends only on the data. To prove (4.22) we let

$$I_k = \{y \in D \cap B(w, 100\rho) \setminus B(w, 4\rho) : 10^{3-k}\rho < d(y) \leq 10^{4-k}\rho\} \text{ for } k = 1, 2, \ldots$$

For fixed k, let $\{B(y_j, 100d(y_j)), y_j \in I_k\}$ be a covering of $O = \bigcup_{y \in I_k} B(y, 4d(y))$ with the balls in $\{B(y_i, d(y_i)/4)\}$ pairwise disjoint. We note that each $y \in O$ lies in at most $c = c(n)$ balls in $\{B(y_i, 1000d(y_i)), y_i \in I_k\}$, as follows from the usual volume argument using disjointness of the smaller balls and the fact that all balls in the covering have proportional radii. Using this note and (4.20), (4.19) we deduce for $k \geq 10$ that

$$\int_{I_k} d(y)^{-1}\, g(x,y)\, dy \leq \int_{I_k} d(y)^{1-n}\, \omega[B(y, 4d(y)), x]\, dy$$

$$\leq c10^{-k}\rho \left\{ \sum_i \omega[B(y_i, 1000d(y_i)) \cap \partial D,\ x] \right\} \leq c10^{-k}\rho\, \omega[B(w, 200\rho) \cap \partial D \setminus B(w, 2\rho), x]$$

$$\leq c10^{-k}\rho \left(\frac{d(x)}{\rho} \right)^{\sigma} \tag{4.23}$$

where the last inequality follows from (4.19) (ii) and the fact that

$$\omega[B(w, 200\rho) \cap \partial D \setminus B(w, 2\rho), x] \leq 1 - \omega[B(w, 2\rho) \cap \partial D, x].$$

(4.23) is also true if $1 \leq k < 10$ as follows easily from (4.20), (4.19)(ii). Summing (4.23) we get (4.22).

Next we state the theorem of [KP] mentioned after (1.28) and tailored for our situation. A somewhat different proof of this theorem is given in [HL, ch 10]. Finally we remark that a nontrivial generalization of the following theorem for the heat equation in a time varying domain appears in [N].

Theorem 4.24. *Let Ω be as in (4.4)-(4.5) and let $\omega_* = \omega_*(\cdot, x)$ denote elliptic measure defined with respect to (b_{ij}) satisfying (4.16) and uniform ellipticity conditions. Then ω_* is a doubling measure and $\omega_* \in A^\infty(H^{n-1}|_{\partial\Omega})$. Equivalently, ω_* is a doubling measure and given, $l_1, 0 < l_1 < 1$, there exists l_2, depending on l_1, the constant c in (4.16), and the uniform ellipticity constants, such that if $w \in \partial\Omega, 0 < \rho \leq$ diam Ω, and $F \subset \partial\Omega \cap B(w, \rho)$ is Borel with $H^{n-1}(F) \geq (1 - l_1) H^{n-1}(\partial\Omega \cap B(w, \rho))$, then*

$$\omega_*(F, x) \geq l_2\, \omega_*(\partial\Omega \cap B(w, \rho), x).$$

The following lemma is the cornerstone for our proof of Theorem 1.27.

Lemma 4.25. *If $z \in \partial D$, $B(z, 10r) \subset N_1$ and $\hat\epsilon > 0$ is given, then there exists $\xi = \xi(\hat\epsilon), 0 < \xi \leq 10^{-9}$, such that if $E \subset \partial D \cap B(z, 2r)$ is Borel and $H^{n-1}(E) \geq (1 - \xi) H^{n-1}(\partial D \cap B(z, 2r))$, then for $w \in D \setminus B(z, 4r)$,*

$$\omega(\partial D \cap B(z, r), w) \leq \hat\epsilon \omega(\partial D \cap B(z, 2r), w) + c(\hat\epsilon)\omega(E, w).$$

Proof: The proof of Lemma 4.25 for harmonic measure can be found in [BL, Lemma 2.2]. For completeness we give the rather short proof. Clearly it suffices to prove Lemma 4.25 for $\hat{\epsilon} > 0$ small, say $0 < \hat{\epsilon} \leq \epsilon_0$. Let c^* $(1 \leq c^* \leq \epsilon_0^{-\frac{1}{2}})$ be a large positive constant to be chosen later and let j be the greatest integer $\leq c^*/\hat{\epsilon}$. Put

$$U_k = \{y : (\tfrac{5}{4} + \tfrac{k}{4j})r < |y - z| < (\tfrac{5}{4} + \tfrac{k+1}{4j})r\} \text{ and } S_k = \{y \in D : |y - z| = (\tfrac{5}{4} + \tfrac{k+1/2}{4j})r\} \tag{4.26}$$

for $1 \leq k \leq j - 1$. Let $\epsilon' = \hat{\epsilon}/c^*$ and first suppose that there exists $x \in S_k$ with $d(x) = (\epsilon'/100)r$. In this case we see from Lemma 4.7 for $\hat{\epsilon}$ small enough that there exists a Lipschitz domain Ω satisfying (4.4) - (4.6) with

$$B(x, d(x)/c) \subset \Omega \subset B(x, 20d(x)) \cap D \subset B(z, 2r) \cap D.$$

Let $\omega_*(\cdot, x)$ be elliptic measure for Ω with respect to x and L. From (4.16) and the observation, $d(w, \partial\Omega) \leq d(w)$ when $w \in \Omega$, we deduce that L restricted to Ω satisfies the hypotheses of Theorem 4.24. Applying this theorem we see that if $\hat{c}$ is large enough (depending only on the data) and $E \subset B(z, 2r) \cap \partial D$ is Borel with

$$\frac{H^{n-1}(E)}{H^{n-1}[B(z, 2r) \cap \partial D]} \geq 1 - \hat{\epsilon}^{n-1}/\hat{c}, \tag{4.27}$$

then

$$H^{n-1}(E \cap \partial\Omega) \geq d(x)^{n-1}/(2c_8)$$

and for some $c_+ \geq 1$,

$$c_+^{-1} \leq \omega_*(E \cap \partial\Omega, x) \leq \omega(E, x) \tag{4.28}$$

where the last inequality is a consequence of the weak maximum principle for L. Using Harnack's inequality for positive weak solutions to L we conclude that

$$c(\hat{\epsilon})^{-1} \leq \omega(E, \cdot) \text{ on } S_k \cap \{w : d(w) \geq \epsilon' r/100\}. \tag{4.29}$$

Now if $x \in S_k \cap \{w : d(w) < \epsilon' r/100\}$, then from (4.19) (i) we get

$$\omega(U_k \cap \partial D, x) \geq c^{-1}. \tag{4.30}$$

Combining (4.29) and (4.30) we conclude that

$$1 \leq c\, \omega(U_k \cap \partial D, \cdot) + c(\hat{\epsilon})\omega(E, \cdot) \text{ on } S_k. \tag{4.31}$$

If $S_k \cap \{w : d(w) = \epsilon' r/100\} = \emptyset$, then by continuity of d, either $S_k \subset \{w : d(w) < \epsilon' r/100\}$ or $S_k \subset \{w : d(w) > \epsilon' r/100\}$. In the first case (4.30) holds on S_k so (4.31) remains valid. Actually this case cannot occur as we see from (2.11), (3.3), and the weak maximum principle for L but for future applications we include it in our considerations. Otherwise using continuity of d it follows that there exists $\rho > 0$ with

$$\{y : \rho \leq |y - z| \leq (\tfrac{5}{4} + \tfrac{k+1/2}{4j})r\} \subset \{w : d(w) \geq \epsilon' r/100\}$$

and $d(x) = \epsilon' r/100$ for some $x \in \partial B(z, \rho)$. Applying the same analysis as previously we find first that if (4.27) is valid, then $\omega(E, \cdot) \geq c(\hat\epsilon)^{-1}$ on $\partial B(z, \rho)$ and thereupon from Harnack's inequality that (4.31) is still valid for suitably large $c(\hat\epsilon)$. Thus (4.31) is true in all cases.

From (4.31) and the maximum principle for weak solutions to L we find for $1 \leq k \leq j-1$ and $w \in D \setminus B(z, 4r)$ that

$$\omega(B(z,r) \cap \partial D, w) \leq c\omega(U_k \cap \partial D, w) + c(\hat\epsilon)\omega(E, w). \tag{4.32}$$

Summing (4.32) over $1 \leq k \leq j-1$ we get

$$(j-1)\omega(B(z,r) \cap \partial D, w) \leq \; c\omega(B(z,2r) \cap \partial D, w) + (j-1)c(\hat\epsilon)\omega(E, w). \tag{4.33}$$

Dividing (4.33) by $j-1$ and choosing c^* large enough we conclude that Lemma 4.25 is true. □

Next we state

Lemma 4.34. *Let D^* be an Ahlfors regular domain with constants M^*, r^* (i.e (3.2) holds with M replaced by M^* for all $z \in \partial D^*$ and $0 < r \leq r^*$). Let $z^* \in \partial D^*, 0 < \rho \leq r^*/4$, and suppose that ν is a positive Borel measure on ∂D^* with $\nu(\partial D^*) = 1$. Assume for $0 < \hat\epsilon \leq 1/2$, that there exists $\kappa = \kappa(\hat\epsilon), 0 < \kappa < 1$, $\bar c(\hat\epsilon) < \infty$, for which the following statement is true. If E is a Borel set, $\hat z \in \partial D^*, \hat r > 0$, and $E \subset B(\hat z, 2\hat r) \cap \partial D^* \subset B(z^*, 2\rho) \cap \partial D^*$ with*

$$\frac{H^{n-1}(E)}{H^{n-1}(B(\hat z, 2\hat r) \cap \partial D^*)} \geq 1 - \kappa(\hat\epsilon),$$

we have

$$\nu(B(\hat z, \hat r) \cap \partial D^*) \leq \; \hat\epsilon \nu(B(\hat z, 2\hat r) \cap \partial D^*) \; + \; \bar c(\hat\epsilon)\, \nu(E).$$

Then ν restricted to $\partial D^* \cap B(z^*, 2\rho)$ is absolutely continuous with respect to H^{n-1} measure on $\partial D^* \cap B(z^*, 2\rho)$. Moreover if $d\nu(\cdot)/dH^{n-1} = h$ on $\partial D^* \cap B(z^*, 2\rho)$ then for some $\lambda > 0$, depending on $M^*, \kappa, \bar c$ and n,

$$\int_{\partial D \cap B(z^*, \rho)} h^{1+\lambda} dH^{n-1} \; \leq c \left(\frac{\nu(B(z^*, 4\rho) \cap \partial D^*)}{\nu(B(z^*, \rho) \cap \partial D^*)} \int_{\partial D \cap B(z^*, 2\rho)} h \, dH^{n-1} \right)^{1+\lambda} \rho^{-n\lambda}.$$

Proof: Lemma 4.34 follows from arguments originally used in [G], although the modifications needed for an Ahlfors regular domain are somewhat tricky. A complete proof of Lemma 4.34 for harmonic measure can be deduced from [BL, Theorem 1]. However, our proof of Theorem 1.27 only uses absolute continuity of ν in Lemma 4.34 and this statement follows easily from the assumptions in Lemma 4.34. To outline the proof of absolute continuity, we first note for $K > 0$, sufficiently large, that for ν almost every $y \in \partial D^*$, we have $\nu[B(y, s)] > 0$ when $s > 0$ and

$$\liminf_{t \to 0} \frac{\nu[B(y, 2t) \cap \partial D^*]}{\nu[B(y, t) \cap \partial D^*]} \; < \; K, \tag{4.35}$$

since otherwise we could iterate this inequality to deduce that $t^{1-n}\, \nu[B(y, t) \cap \partial D^*] \to 0$ as $t \to 0$ for $y \in G$ Borel $\subset \partial D^*$ with $\nu(G) > 0$. Using a covering lemma it would then follow from Ahlfors regularity of ∂D^* that $\nu(G) = 0$, which is a contradiction. Fix K so that (4.35) is true. Next from a standard argument using the Besicovitch covering Lemma (see [Ma, Cor 2.14]) we deduce for ν almost every $y \in F$ Borel $\subset B(z^*, 2\rho) \cap \partial D^*$, that

$$\lim_{t \longrightarrow 0} \frac{\nu(B(y,t) \cap \partial D^* \setminus F)}{\nu(B(y,t) \cap \partial D^*)} = 0.$$

Now if ν were not absolutely continuous with respect to H^{n-1} measure on $B(z^*, 2\rho) \cap \partial D^*$, then for some F Borel $\subset B(z^*, 2\rho) \cap \partial D^*$ we would have $H^{n-1}(F) = 0$ and $\nu(F) > 0$. Choose $y \in F$ so that (4.35) and the above limit hold. To get a contradiction we use the middle display in Lemma 4.34 with $\hat{z}, \hat{r}$, replaced by y, t and $E = B(y, 2t) \cap \partial D^* \setminus F$. We obtain for some arbitrarily small $t > 0$ that

$$\nu[B(y,t) \cap \partial D^*] \leq \hat{\epsilon}\, \nu[B(y,2t) \cap \partial D^*] + \bar{c}(\hat{\epsilon})\, \nu[B(y,2t) \cap \partial D^* \setminus F]$$

$$\leq \hat{\epsilon} K \nu[B(y,t) \cap \partial D^*] \,+\, \bar{c}(\hat{\epsilon})\, \nu[B(y,2t) \cap \partial D^* \setminus F].$$

Dividing this inequality by $\nu[B(y,2t) \cap \partial D^*]$ we get a contradiction for some small $t > 0$ provided $\hat{\epsilon} K < 1/2$. Thus ν is absolutely continuous with respect to H^{n-1} on ∂D^*. $\Box$

To complete the proof of Theorem 1.27 under assumption (4.1) we modify an argument in [LV3, section 3]. Choose $r > 0$, $0 < r \leq (1/4) \min(\operatorname{diam} D, 1)$, $z \in \partial D$, $c = c(n) \geq 1$ (to be chosen later), $x \in B(z, r/100) \cap D$ with $B(z, cr) \subset N_1$ and

$$|\nabla u(x)| \geq b - \tau/2. \tag{4.36}$$

We can also assume $|\nabla u| \leq b + \tau$ on $B(z, cr) \cap D$ which for v (defined after (4.9)) implies

$$0 \,\leq v \,\leq\, (b+\tau)^2 - (b-\tau)^2 = 4b\tau \tag{4.37}$$

on $B(z, cr) \cap D$. Eventually we shall let $x \rightarrow \partial D$ keeping z, r fixed which is permissible as follows from the definition of b in (4.8). Let $\{Q_j\}, (\eta_m), \xi$ be defined as in section 2 following (2.14) with $\xi << d(x)/r$. Given $r' \in (r, 2r)$, let $\Lambda = \{j : Q_j \cap B(z, 2r') \neq \emptyset$ and $r_j \geq \xi r\}$. Define Λ_1 relative to Λ exactly as in section 2 following (2.17). Next let $\Lambda_{11} \subset \Lambda_1$ be as defined below (2.18) with r replaced by r' and set $\Lambda_{12} = \Lambda_1 \setminus \Lambda_{11}$. We also choose η_m so that $|\frac{\partial^2 \eta_m}{\partial y_i \partial y_j}| \leq c r_m^{-2}$ for $1 \leq i, j \leq n$. Finally we write g for $g(x, \cdot)$. We note that (4.17) remains valid with $\theta = v \sum_{m \in \Lambda} \eta_m$. Integrating

(4.17) by parts for this θ and using (4.15), (4.36), we find that

$$
\begin{aligned}
(\tau/4)(4b-3\tau) \,\le\, v(x) &= \sum_{m\in\Lambda}\int_D [\sum_{i,j=1}^n b_{ij}\,(\eta_m v)_{y_i}\, g_{y_j}\,]\,dy \\
&= -2\sum_{m\in\Lambda}\int_D [\sum_{i,j=1}^n b_{ij}\,(\eta_m)_{y_i}\, v_{y_j}\, g\,]\,dy \\
&- \sum_{m\in\Lambda}\int_D [\sum_{i,j=1}^n \{b_{ij}\,(\eta_m)_{y_i}\}_{y_j}\, v\, g\,]\,dy \\
&- \sum_{m\in\Lambda}\int_D f\,\eta_m\, g\,dy \\
&= P_1 + P_2 + P_3.
\end{aligned}
\tag{4.38}
$$

From (2.5), (2.6), (4.16) we get as in (2.18) that

$$
|P_1| + |P_2| \le c \sum_{m\in\Lambda_1}\int_{Q_m} [\, r_m^{-1}\,|\nabla v| + r_m^{-2}\, v\,]\, g\, dy. \tag{4.39}
$$

Recall that $\Lambda_1 = \Lambda_{11}\cup\Lambda_{12}$ where Λ_{11} consists of cubes which intersect $\partial B(z,2r')$ while the cubes in Λ_{12} have sidelength $\approx \xi r$. We claim for some $r'\in(r,2r)$ that

$$
\sum_{m\in\Lambda_{11}}\int_{Q_m} [\, r_m^{-1}\,|\nabla v| + r_m^{-2}\, v\,]\, g\, dy \,\le\, c\left(\frac{d(x)}{r}\right)^{\sigma}. \tag{4.40}
$$

Indeed, writing $\Lambda_{11} = \Lambda_{11}(r')$, integrating with respect to r' and interchanging the order of integration we get using (2.5), (2.6), (4.10), and (4.22) with $w = z, \rho = r/10$,

$$
\begin{aligned}
&\int_r^{2r}\left(\sum_{m\in\Lambda_{11}(r')}\int_{Q_m}[r_m^{-1}|\nabla v| + r_m^{-2}\, v]g\,dy\right)dr' \\
&\le c\int_{B(z,6r)\cap D\setminus B(z,r)} d(y)^{-1} g(x,y)\,dy \,\le\, cr\left(\frac{d(x)}{r}\right)^{\sigma}.
\end{aligned}
$$

This inequality and weak type estimates imply that (4.40) holds for some $r'\in(r,2r)$.

With r' now fixed let Λ' be the subfamily of cubes $Q_m = Q_m(y_m, r_m)\in\Lambda_{12}$ for which $v\not\equiv 0$ on Q_m and let $F = \bigcup_{m\,\in\,\Lambda'} B(y_m, cr_m)\cap\partial D$. Next using $\zeta = v\eta_m$ as a test function in (4.14), we find in view of (2.6), (2.11), (2.13), (4.10), (4.12), (4.15), (4.37) for $c = c(n)$ large enough and $\xi r \le \tau b$ that

$$
\int_{Q_m} |\nabla v|^2\, dy \,\le\, c\, r_m^{-2}\int_{B(y_m, cr_m)\cap D} (v + r_m)^2\, dy \,\le\, c\, r_m^{n-2}\,(\tau b)^2
$$

whenever $Q_m \in \Lambda'$. Here we have also used the fact that $|\Delta u(x)| \le c\,(|\nabla v(x)|+1)$ for $x \in \operatorname{supp} v$ to estimate the term in (4.14) involving $\hat{e}$. This fact follows for example from (2.29). Using the above inequality, Hölder's inequality, (4.20), (4.37), and arguing as in (2.21), we deduce for $c = c(n)$ large enough and $\xi r \le \tau b$ that

$$\sum_{m\in\Lambda_{12}} \int_{Q_m} [\, r_m^{-1}\,|\nabla v| \,+\, r_m^{-2}\,v\,]\,g\,dy$$

$$\le\, c\sum_{m\in\Lambda'} r_m^{-2}\,\left(\int_{B(y_m,cr_m)\cap D} (v+r_m)^2\,dy\right)^{1/2} \cdot \left(\int_{B(y_m,cr_m)\cap D} g^2\,dy\right)^{1/2} \tag{4.41}$$

$$\le c\tau b \sum_{m\in\Lambda'} \omega(B(y_m, cr_m)\cap \partial D, x) \;\le\; c\tau\, b\,\omega(F,x).$$

With $c = c(n)$ now fixed so that (4.41) holds, we can use (4.41), (4.42) in (4.39) to conclude

$$|P_1| + |P_2| \le\, c\left(\frac{d(x)}{r}\right)^{\sigma} + c\tau\, b\,\omega(F,x) \tag{4.42}$$

provided $\xi r \le \tau b$. To estimate P_3 we again use (2.5), (2.6), (4.10) and (4.12) to write

$$|P_3| = \left|\sum_{m\in\Lambda}\int_D f\,\eta_m\,g\,dy\right| \,\le\, c\int_{D\cap B(z,4r)} d(y)^{-1} g(x,y)dy$$

$$= \int_{[B(x,10d(x)\setminus B(x,d(x)/2)]\cap D} \ldots dy \;+\; \int_{B(x,d(x)/2)} \ldots dy \;+\; \int_{[B(z,4r)\cap D]\setminus B(x,10d(x))} \ldots dy \tag{4.43}$$

$$= J_1 + J_2 + J_3.$$

We shall show that

$$J_1 \le cd(x). \tag{4.44}$$

In fact if

$$I_k^* = \{y \in B(x, 10d(x)) \setminus B(x, d(x)/2) : 10^{1-k}d(x) < d(y) \le 10^{2-k}d(x)\} \text{ for } k = 1, 2, \ldots,$$

then as in (4.23) we deduce using (4.20),

$$\int_{I_k^*} d(y)^{-1}\, g(x,y)dy \;\le\; c\,\int_{I_k^*} d(y)^{1-n}\,\omega[B(y, 4d(y))\cap \partial D, x]\,dy \;\le\; c10^{-k}\,d(x).$$

Summing this inequality over $k = 1, 2, \ldots$, we obtain (4.44). Next if $100r > 10^q d(x) \ge 10r$, we can use (4.22) with $\rho = 10^k d(x)$ for $k = 0, 1, 2, \ldots, q$, to estimate J_3. That is,

$$J_3 \le c\sum_{\nu=1}^{q} \int_{\{10^\nu d(x)<|x-y|\le 10^{\nu+1}d(x)\}\cap D} d(y)^{-1}\, g(x,y)\,dy$$

$$\le\; c\,\sum_{\nu=1}^{q} 10^{\nu(1-\sigma)}\,d(x) \;\le\; c\,r^{1-\sigma}\,d(x)^{\sigma}. \tag{4.45}$$

Finally from (4.21) we deduce

$$J_2 \le c \log(\text{diam } D/d(x))\, d(x).$$

Combining this estimate with (4.44), (4.45) we find in view of (4.43) that

$$|P_3| \le cd(x) \log(\text{diam } D/d(x)) + cr^{1-\sigma}\, d(x)^\sigma. \tag{4.46}$$

Using (4.46), (4.42) in (4.38) we see for $\xi r \le \tau b$ that

$$(\tau/4)(4b - 3\tau) \le cd(x) \log(\text{diam } D/d(x)) + cr^{-\sigma}\, d(x)^\sigma + c\tau b\, \omega(F, x).$$

With r fixed we now suppose that $\xi r \le \tau b$ and $d(x)$ is so small that the first two terms on the righthand side of the above display are $\le \frac{1}{2} \cdot$ the lefthand side of this display. Then

$$1 \le c\, \omega(F, x). \tag{4.47}$$

To avoid confusion, we write $F = F(\xi r)$ to indicate the dependence of F on ξ and put $\xi = 2^{-k}$ for $k = 1, \dots$. Next we observe for any $w \in D$ that

$$\omega(\cdot, x) \le c(D, x, w)\omega(\cdot, w)$$

thanks to Harnack's inequality and connectivity of D. From this observation, Lemmas 4.25 and 4.34 with r replaced by $cr/4$, $\nu = \omega(\cdot, w)$, and w a point in $D \setminus B(z, cr)$, as well as (4.47), we see there exists $a > 0$ small and k_0 large so that

$$2a \le H^{n-1}[F(2^{-k} r)] \tag{4.48}$$

for $k \ge k_0$ where a is independent of k and k_0 depends on various quantities including $\tau, d(x), w, x, r, D$ and the data. Also (4.48) only requires absolute continuity of $\omega(\cdot, w)$ with respect to H^{n-1} measure on ∂D.

To continue our proof we need the fact that for H^{n-1} almost every $y \in \partial D \cap B(z, cr)$, we have

$$\limsup_{\rho \longrightarrow 0} \alpha(n-1)^{-1} \rho^{-(n-1)} H^{n-1}[B(y, \rho) \cap \partial D] \le 1$$

which follows from basic measure theory type arguments (see [Ma, Thm 6.6]). Using this fact, (4.48), and once again basic measure theory we see that if $k_1 \ge k_0$ is large enough and

$$G_k = F(2^{-k} r) \cap \{y \in \partial D \cap B(z, cr) : H^{n-1}(B(y, \rho) \cap \partial D) \le (\alpha(n-1) + \tau)\rho^{n-1} \text{ for } 0 < \rho \le 2^{-k_1} r\},$$

then for $k \ge k_1$ we have

$$H^{n-1}(G_k) \ge a. \tag{4.49}$$

Recall the definition of a good tangent ball in (3.6). Given $\delta, 0 < \delta \le \delta_1$, we replace δ_1 by δ in (3.6) (α). That is $B(x, d(x)) \subset D$ is a good (δ) tangent ball provided $|\nabla u(x)| \ge \delta$ and the chain condition (3.6) (β) holds for ϵ sufficiently small, say $0 < \epsilon \le \epsilon_0 = \epsilon_0(\delta)$ where $\epsilon_0 \le \delta^{100}$. With this change one easily checks that the argument after (3.7) can be repeated verbatim except that now constants can also depend on δ. We claim there exists $k_2 = k_2(k_1, \epsilon, \delta) \geq k_1$ such that if $k \geq k_2$, then for some $\hat{z} \in G_k$ it is not true that

$$\text{There is a bad tangent ball } B(y, d(y)) \subset B(\hat{z}, 2^{-k}cr) \cap D \text{ with } d(y) \geq 2^{-k}\epsilon^2\, r \text{ and } |\nabla u(y)| \geq \delta. \tag{4.50}$$

Indeed, let K be defined as in section 3 to be the set of all $(w, s) \in B(z, 2cr) \times (0, r)$ for which (4.50) holds with $\hat{z}, 2^{-k}r, c$ replaced by $w, s/10, 10c$. If (4.50) is true for all $\hat{z} \in G_k$, observe that

$$\int_{2^{-k}r}^{2^{1-k}r} H^{n-1}(K \cap [\mathbb{R}^n \times \{t\}])\, t^{-1}\, dt \ \geq\ a.$$

On the other hand, summing this inequality over all positive integers k we see as in (3.15) that the resulting sum is finite. Thus there are only a finite number of positive integers k for which (4.50) holds for all $\hat{z} \in G_k$.

Fix $k \geq k_2$ and let $\hat{z} \in G_k$ be the guaranteed point where (4.50) is false. If $r^* = 2^{-k}cr$, then from the definition of v, G_k, Λ', and a good (δ) tangent ball, we see as in section 3 that there exists $B(y, d(y)) \subset B(\hat{z}, r^*)$ with

$$|\nabla u|(y) \geq b - \tau \tag{4.51}$$

and $d(y) \geq r^*/c$. We assume as in section 3 that $0 \in \bar{B}(y, d(y)) \cap \partial D$ and $y = d(y)e_n$. Using the argument following (3.15), as well as, (4.51) and (4.37) we get that

$$|\nabla u(w) - b e_n| \ \leq\ |\nabla u(w) - |\nabla u(y)| e_n| + \tau \ \leq\ c(\delta)\, \epsilon^5 + \tau \tag{4.52}$$

whenever $w \in O = B(\hat{z}, 2r^*/\epsilon^2) \cap \{\hat{w} : \hat{w}_n \geq c(\delta)\epsilon^2 r^*\}$. More specifically, (4.52) is just (3.23) together with the claim after (3.25).

We now proceed as in [LV3] (see the argument after (3.34) of this paper). As in the discussion following (3.27) we can suppose ϵ is small enough, say $0 < \epsilon \leq \epsilon_0(\delta)$, so that one of the following two possibilities occurs. First suppose that

$$\partial D \cap B(\hat{z}, r^*) \subset S = \{w : |w_n| \leq \epsilon r^*/4\}. \tag{4.53}$$

Given $\eta, 0 < \eta \leq 1/2$, let $\psi \in C_0^\infty[(-1, 1)]$ be an even function with $\psi = 1$ on $(\eta - 1, 1 - \eta)$ and $|\psi'| \leq c/\eta$. Put

$$\phi(x', x_n) = \psi[2|x' - \hat{z}'|/r^*]\, \psi[x_n/(\epsilon\, r^*)] \text{ whenever } x' \in \mathbb{R}^{n-1} \text{ and } x_n \in \mathbb{R}$$

$$H = \{(x', x_n) : |x' - \hat{z}'| < r^*/2, |x_n| < \epsilon r^*\}$$

$$T_1 = \partial H \cap \{x : x_n = \epsilon r^*\}$$

$$T_2 = \partial H \cap \{x : x_n = -\epsilon r^*\}$$

$$T_3 = \partial H \cap \{x : |x' - \hat{z}'| = r^*/2\}.$$

Note that supp $\phi \subset H$ and $0 \leq \phi \leq 1$. Also $T_1 \subset O$ for $\epsilon_0 = \epsilon_0(\delta)$ sufficiently small. Using these notes, (1.6), $\hat{z} \in G_k$, and (1.28) we obtain

$$\begin{aligned} M_1 = |\int_H [A(u, |\nabla u|^2)\langle \nabla u\,,\, \nabla\phi\rangle - C(u, |\nabla u|^2)\phi]\ dy| &= \int \phi\, d\mu \\ &\leq \mu[B(\hat{z}, (1 + 4\epsilon)r^*/2)] \leq \beta_1\, (\alpha(n-1) + \tau)\, [(1/2 + 2\epsilon)r^*]^{n-1} \end{aligned} \tag{4.54}$$

From boundedness of C we deduce that

$$M_2 = |\int_H A(u, |\nabla u|^2) \langle \nabla u \, , \, \nabla \phi \rangle \, dy| \, \leq \, M_1 + c\epsilon \, (r^*)^n. \tag{4.55}$$

Moreover,

$$|\int_H A(u, |\nabla u|^2) u_{y_n} \phi_{y_n} \, dy| \, - \, |\int_H A(u, |\nabla u|^2) \langle \nabla' u, \nabla' \phi \rangle dy \, | \, = M_3 - M_4 \, \leq \, M_2 \tag{4.56}$$

where ∇' denotes the gradient in $x' = (x_1, \ldots, x_{n-1})$ only. From (4.53) we see that either (a) $\hat{O} = \{w : w_n < -\epsilon r^*/2\} \cap B(\hat{z}, r^*) \subset \mathbb{R}^n \setminus D$ or (b) $\hat{O} \subset D$. In either case letting $\eta \to 0$ we find

$$M_3 \to |\int_{T_1} A(u, |\nabla u|^2) \, u_{y_n} \, dH^{n-1}y \, - \, \int_{T_2} A(u, |\nabla u|^2) \, u_{y_n} \, dH^{n-1}y \, | \, = \, M_5 \tag{4.57}$$

where the integral involving T_2 is zero in case (a). In case (b) we see from our choice of $\hat{z}$ that either $(+)$ $|\nabla u(y)| \leq \delta$ for all $y \in \hat{O}$ with $d(y) \geq \epsilon^2 r^*$ or (-) $\hat{O}$ contains a good (δ) tangent ball of radius $\geq r^* \epsilon^2$. If $(+)$ occurs then from (1.1) (a) we deduce that

$$|\int_{T_2} A(u, |\nabla u|^2) u_{y_n} \, dH^{n-1}y| \leq c \, \delta^{p-1} \, (r^*)^{n-1}. \tag{4.58}$$

If (-) occurs, then from basic geometry and the argument after (3.19), it follows that $u_{y_n} < 0$ on T_2. Since $u_{y_n} \geq 0$ on T_1 we conclude that in either scenario,

$$M_6 = |\int_{T_1} A(u, |\nabla u|^2) u_{y_n} \, dH^{n-1}| \leq M_5 + c\delta^{p-1} \, (r^*)^{n-1}. \tag{4.59}$$

Using our smoothness assumptions on A and (4.52) we deduce from (4.59) that

$$b A(0, b^2) \alpha(n-1)(r^*/2)^{n-1} \leq M_6 + c(\tau + \epsilon + \delta^{p-1}) \, (r^*)^{n-1}. \tag{4.60}$$

Finally,

$$\limsup_{\eta \to 0} \, M_4 \leq \, c\epsilon \, (r^*)^{n-1} \tag{4.61}$$

as we see from boundedness of $|\nabla u|$ and simple estimates. Combining (4.54) - (4.61) we get after dividing by $\alpha(n-1) \, (r^*/2)^{n-1}$

$$b \, A(0, b^2) \leq \, \beta_1 \, + c \, (1 + \beta_1)(\tau + \delta^{p-1} + \epsilon) \, . \tag{4.62}$$

for given $\delta > 0, 0 < \epsilon \leq \epsilon_0(\delta)$, and c depending only on the data. Since ϵ, τ, δ can be arbitrarily small we conclude that (4.9) is true when (4.53) is valid. If (4.53) does not hold we can use the argument in the last paragraph of section 3 to get that if $r' = r/\epsilon^{1/2}$, then $\partial D \cap B(\hat{z}, r') \subset S' = \{w : |w_n| \leq \epsilon^{1/4} r'/4\}$ for $0 < \epsilon \leq \epsilon_0$. We can now repeat the argument after (4.53) with r^* replaced by r' and ϵ by $\epsilon^{1/4}$. In this case we do not need to introduce δ. Also case (a) and $(+)$ of case (b) can be omitted as they cannot occur. Once again we obtain (4.62). Thus Theorem 1.27 is true when (4.1) holds. □

5. Preliminary Reductions for Theorem 1.27. In this section we let c and ' depending on the data ' have the same meaning as in section 4. We also use the same notation as in section 4. Our goal is to show that Theorem 1.27 is valid without the Carleson measure chain assumption, (4.1). The strategy for obtaining this goal is contained in the following two propositions.

Proposition 5.1. *There exists $D_1 \subset D$ such that ∂D_1 is locally uniformly rectifiable and*

$$\begin{aligned} (\alpha) \quad & D \cap \partial D_1 = \cup\, \partial B(w_j, d(w_j)/\bar{c}) \text{ for some } \bar{c} \geq 10^5 \text{ depending only on the data} \\ & \text{and for } i \neq j,\ B(w_j, 10d(w_j)/\bar{c}) \cap B(w_i, 10d(w_i)/\bar{c}) = \emptyset, \\ (\beta) \quad & \text{If } \tau_0 > 0 \text{ is small enough (depending only on the data),} \\ & \text{then } v \equiv 0 \text{ on } D \cap \partial D_1 \text{ for } 0 < \tau \leq \tau_0. \end{aligned} \tag{5.2}$$

Proposition 5.3. *Let ω_1 be elliptic measure corresponding to L in (4.15) and D_1. Then Lemma 4.34 is true for $D^* = D_1, r^* = \tau_0^2$, and $\nu = \omega_1(\cdot, w)$ provided $w \in D_1 \setminus B(z^*, 4\rho)$.*

Remark: Armed with Propositions 5.1 and 5.3 we get Theorem 1.27 in the following way. First, D_1 has the same properties as D thanks to Proposition 5.1, so (4.17) - (4.23) are valid for ω_1 and g_1, the Green's function corresponding to L and D_1. Second we choose $cr \leq \tau_0^4$ and x so that (4.36), (4.37) hold. Third we replace g, ω by g_1, ω_1 in (4.17) and again use $\theta = v \sum_{m \in \Lambda} \eta_m$, as a test function. Integrating by parts as in (4.38) we get P_1, P_2, P_3, with g replaced by g_1. The additional term involving ω_1 arising in the integration by parts is equal to 0 (so can be omitted), since $v \equiv 0$ on $\partial D_1 \cap D$ for $0 < \tau \leq \tau_0$. We can now repeat verbatim the argument after (4.39) with g, ω replaced by g_1, ω_1. We get (4.49) (thanks to Proposition 5.3) and finally Theorem 1.27. Thus to complete the proof of Theorem 1.27 we need only prove Propositions 5.1 and 5.3.

Proof of Proposition 5.1. Let δ_1, v, b, τ, be as in (3.4), (4.8), and the display below (4.9). Also assume that (4.37) holds in $D \cap N_1$. We first allow $\epsilon, 0 < \epsilon \leq 10^{-10n}$, to vary, put $\hat{\tau}(\epsilon) = \exp(-1/\epsilon^2)$ and construct $D_1(\epsilon)$. Eventually we shall fix $\epsilon = \epsilon_1, \tau_0 = \hat{\tau}(\epsilon_1)^3$, satisfying several conditions, to get $D_1 = D_1(\epsilon_1)$ as in Propositions 5.1 and 5.3. We assume, as we may, that $\max\{\delta_1, \epsilon\} < b/100$ (otherwise redefine δ_1). To begin the proof observe from (4.37) that if $\eta = (\eta_1, \ldots, \eta_n)$ with $|\eta| = 1$ then $w = b + \tau - \langle \nabla u, \eta \rangle \geq 0$ in $D \cap N_1$. Moreover if $|\nabla u| \geq \delta_1$ on $\bar{B}(x, r) \subset D$ then from (4.11), (4.12), we see that

$$Lw + \langle \hat{d}, \nabla w \rangle + \hat{e}\, w = \hat{e}\,(b + \tau) \text{ on } \bar{B}(x, r).$$

Using this fact we deduce

$$\max_{B(x,r/2)} w \leq c \min_{B(x,r/2)} w + c\, br. \tag{5.4}$$

To sketch the proof of (5.4), write $w = \lambda + q$ where $L\lambda = 0$ in $B(x, 3r/4)$, $\lambda = w$ on $\partial B(x, 3r/4)$. We note that q can be written as a Green's potential in $B(x, 3r/4)$. Using Harnack's inequality for λ, our note, and making estimates on q using (4.19)-(4.21), as in the estimates for J_1, J_2 following (4.43), we get (5.4). We write D_1 for $D_1(\epsilon)$. To construct D_1 we examine again the argument leading to (3.7)-(3.12). Suppose $\hat{y} \in D$, $d(\hat{y}) \leq \hat{\tau}^{1/2}(\epsilon)$, and for k a fixed positive integer that

$$\max_{z_1, z_2 \in B(\hat{y}, (1-\epsilon^{100}) d(\hat{y}))} ||\nabla u|^k \nabla u(z_1) - |\nabla u|^k \nabla u(z_2)| \geq \epsilon^{100}. \tag{5.5}$$

In this case we claim for $\epsilon > 0$ small enough that there exists $c_* \geq 1$ (depending only on the data) and w such that

$$B(w, d(w)/c_*) \subset B(\hat{y}, (1-\epsilon^{100}/4)\, d(\hat{y})) \text{ and } v \equiv 0 \text{ on } B(w, d(w)/c_*). \tag{5.6}$$

To prove this claim we consider two cases. First if $|\nabla u(w)| \leq 4\delta_1$ (δ_1 as in (3.4)) at some point in $B(\hat{y}, (1-\epsilon^{100}/2)d(\hat{y}))$, then from (2.5), we see that (5.6) is valid. Otherwise, $|\nabla u| > 4\delta_1$ on this ball and we will show that $v \equiv 0$ on $B(\hat{y}, d(\hat{y})/4)$. Thus in this case (5.6) is true with $w = \hat{y}$. The proof is by contradiction. Assume $v(y) > 0$ or equivalently $|\nabla u(y)| > b - \tau$ for some $y \in B(\hat{y}, d(\hat{y})/4)$. Rotating coordinate systems if necessary we may also suppose that $u_{y_n} = |\nabla u(y)| > b - \tau$. Thus if $w = b + \tau - u_{y_n}$, then $0 \leq w(y) \leq 2\tau$ and we can apply the weak Harnack inequality in (5.4) to w. If $\tau \leq \hat{\tau}(\epsilon)$, $d(\hat{y}) \leq \hat{\tau}^{1/2}(\epsilon)$, we get after applying this inequality about $\log(1/\epsilon)$ times that

$$u_{y_n} \geq b - \exp(-1/\epsilon) \text{ on } B(\hat{y}, (1-\epsilon^{100})d(\hat{y})). \tag{5.7}$$

Since $|\nabla u| \leq b + \tau$ in this ball it follows that

$$|\nabla u|^2 - u_{y_n}^2 \; \leq \; 8b\, \exp(-1/\epsilon) \text{ in } B(\hat{y}, (1-\epsilon^{100})d(\hat{y})). \tag{5.8}$$

From (5.5)-(5.8) and a ball park estimate we see for some $z_1, z_2 \in B(\hat{y}, (1-\epsilon^{100})d(\hat{y}))$ that

$$\epsilon^{100} \; \leq \; ||\nabla u|^k \nabla u(z_1) - |\nabla u|^k \nabla u(z_2)| \; \leq \; c|u_{y_n}^{k+1}(z_1) \; - \; u_{y_n}^{k+1}(z_2)| \; + \; c\exp[-1/(2\epsilon)]$$

$$\leq c|u_{y_n}(z_2) - u_{y_n}(z_1)| + c\exp[-1/(2\epsilon)]$$

where c depends on the data (including k). We conclude from this inequality that

$$\min(u_{y_n}(z_1), u_{y_n}(z_2)) \leq b + \tau - \epsilon^{100}/c,$$

which contradicts (5.7) for ϵ sufficiently small. Thus in this case $v \equiv 0$ on $B(\hat{y}, d(\hat{y})/4)$. In either case claim (5.6) is true. Next let $\Theta_1 = \{\hat{y} \in D \cap N_1 : (5.5) \text{ holds and } d(\hat{y}) \leq \hat{\tau}^{1/2}(\epsilon)\}$. Set

$$\Theta = \{\hat{w} \in N_1 \cap D : v \equiv 0 \text{ on } B(\hat{w}, 1000d(\hat{w})/\tilde{c}) \text{ and for some } \hat{y} \in \Theta_1,\, |\hat{w} - \hat{y}| \, \leq \epsilon^{-100}\, d(\hat{y})$$

$$\text{while } \epsilon^{200} d(\hat{w}) \leq d(\hat{y}) \leq \epsilon^{-200} d(\hat{w})\}.$$

Here $\tilde{c} \geq 2000c_*$ (c_* as in (5.6)) is chosen so large that if $|\nabla u(w)| \leq 4\delta_1$ then $v \equiv 0$ on $B(w, 1000d(w)/\tilde{c})$. Thus $w \in \Theta$ when either (5.6) holds or $|\nabla u(w)| \leq 4\delta_1$ and there exists $\hat{y} \in \Theta_1$ with $|w - \hat{y}| \leq \epsilon^{-100}\, d(\hat{y})$ while $\epsilon^{200} d(\hat{w}) \leq d(\hat{y}) \leq \epsilon^{-200} d(\hat{w})$. From a well known covering lemma we see there exists, $\{B(y_i, d(y_i)/\tilde{c})\}$, with $y_i \in \Theta$, such that

$$\begin{aligned} &(a) \quad \Theta \subset \cup B(y_i, 1000d(y_i)/\tilde{c}) \\ &(b) \quad B(y_i, 10d(y_i)/\tilde{c}) \cap B(y_j, 10d(y_j)/\tilde{c}) = \emptyset \\ &(c) \quad v \equiv 0 \text{ on } \cup\, B(y_i, 1000d(y_i)/\tilde{c}). \end{aligned} \tag{5.9}$$

Put $D_1 = D \setminus [\cup \bar{B}(y_i, d(y_i)/\tilde{c})]$. Clearly D_1 has properties $(\alpha), (\beta)$ in Proposition 5.1. To prove Ahlfors regularity of ∂D_1 first suppose $z \in \partial D$. Then

$$H^{n-1}(D \cap \partial D_1 \cap B(z, r)) \leq c(\epsilon) r^{n-1}, \tag{5.10}$$

as we see from the argument after (3.7) leading to (3.13). Indeed, if $B(y_i, d(y_i)/\tilde{c}) \cap B(z,r) \neq \emptyset$, then there exists $\hat{y} \in \Theta_1$ corresponding to y_i as in the definition of Θ. From a Poincáre type argument (see (3.12)) and the fact that

$$\tfrac{1}{2}\epsilon^{100/(k+1)} \leq \max\{|\nabla u(z_1)|, |\nabla u(z_2)|\} \leq c \max\{d(z_1)^{-1}u(z_1), d(z_2)^{-1}u(z_2)\}$$

for $d(\hat{y}) \leq \hat{\tau}(\epsilon)^{1/2}$, we obtain

$$d(y_i)^{n-1} \leq c(\epsilon) \int_{B(\hat{y},(1-\epsilon^{100})d(\hat{y}))} u \sum_{i,j=1}^{n} (u_{w_i w_j})^2 \, dw.$$

Summing and using Theorem 1.10 we get (5.10). If $z \in D \cap \partial D_1$, then $z \in \partial B(y_i, d(y_i)/\tilde{c})$ for some i and for $0 \leq r \leq 4d(y_i)$, it follows from (5.9) as well as the above mentioned argument that (5.10) is true. If $r > 4d(y_i)$, then there exists $\hat{z} \in \partial D$ with $B(z,r) \subset B(\hat{z}, 2r)$, so we can use (5.10) again with z, r replaced by $\hat{z}, 2r$. We conclude in all cases from (5.10) and (3.2) that for some $M' = M'(\epsilon) < \infty$,

$$H^{n-1}(B(z,r) \cap \partial D_1) \leq M' r^{n-1} < \infty$$

whenever $z \in \partial D$ and $0 < r \leq$ diam D. The lower bound in the definition of Ahlfors regularity for $\partial D_1 \cap \partial B(z,r), z \in \partial D_1$, is essentially trivial as we see from dividing the proof into two cases and using the corresponding lower bound for $\partial D \cap B(z,r)$ when $z \in \partial D$. Given Ahlfors regularity, local uniform rectifiability of ∂D_1 follows from the so called big pieces functor in [DS1]. That is, given $0 < r \leq$ diam D, $z \in \partial D$, we show the existence of a uniformly rectifiable set U with ‘ bounded constants ’ and

$$c\, H^{n-1}[U \cap \partial D_1 \cap B(z,r)] \geq r^{n-1} \tag{5.11}$$

where c is independent of $r, z \in \partial D$. By ‘ bounded constants’ we mean the Ahlfors regularity constant (see (3.2)) and the norm of the Carleson measure associated with the exceptional set in one of the definitions of uniform rectifiability are bounded independently of r, z. (5.11) implies local uniform rectifiability of ∂D_1 (see [DS1, Part IV]). To prove (5.11) for $z \in \partial D$, we can simply take $U = D \cup P$ (P is the plane whose distance from $D \approx$ diam D). If $z \in \partial B(y_i, d(y_i)/\tilde{c})$ for some i and $0 \leq r \leq 4d(y_i)$, we take $U = \partial B(y_i, d(y_i)/\tilde{c}) \cup P_i$ where P_i is a plane whose distance from $B(y_i, d(y_i)/\tilde{c})$ is equal to 100 $d(y_i)$. If $r > 4d(y_i)$, we can again take $U = \partial D \cup P$. Since ϵ eventually will be fixed, the proof of Proposition 5.1 is now complete. □

Proof of Proposition 5.3. Again we allow ϵ to vary but at the end of the proof of this proposition will fix $\epsilon = \epsilon_1$. To prove Proposition 5.3 we claim it suffices to show for given $x \in D_1 = D_1(\epsilon)$ with $d_1(x) = d(x, \partial D_1) \leq \hat{\tau}(\epsilon)^{3/2}$, that there exists positive numbers $\xi_1 = \xi_1(\epsilon), \xi_2 = \xi_2(\epsilon)$, such that whenever $E \subset \partial D_1 \cap B(x, d_1(x)/\epsilon^{40})$ is Borel with

$$\frac{H^{n-1}(E)}{H^{n-1}[B(x, d_1(x)/\epsilon^{40}) \cap \partial D_1]} \geq 1 - \xi_1(\epsilon) \tag{5.12}$$

then

$$\omega_1(E, x) \geq \xi_2(\epsilon). \tag{5.13}$$

Indeed once (5.13) is proved we can repeat the argument in Lemma 4.25 for given $\hat{\epsilon} > 0$ and

$r \leq \hat{\tau}(\epsilon)^2$ with j replaced by the largest integer $\leq (\hat{\epsilon}\epsilon^{100})^{-1}, \epsilon' = \hat{\epsilon}\epsilon^{100}$ and U_k, S_k unchanged. We get Lemma 4.25 for ω_1 and $0 < \hat{\epsilon} \leq 1/2$, provided $c(\hat{\epsilon}, \epsilon)$ is sufficiently large. We then take $\epsilon = \epsilon_1$ and conclude Proposition 5.3. Thus we prove only (5.13). A key ingredient in the proof will be to develop an algorithm which produces a Lipschitz domain. We then start a process in which we either stop or apply the algorithm once again. In all cases we show, using Theorem 1.8, and a corona decomposition type argument as in [Se], [DS], [DS1], that after at most

$$N = c^{\#}\epsilon^{-3/4} \tag{5.14}$$

times (where $c^{\#}$ depends only on the data for D) our process must stop with an end result that produces at the N th step a family of Lipschitz domains, $\{\Omega_j\}$, for which the members of a subsequence, say $\{\Omega'_k\}$, contain big pieces of ∂D_1 in the sense of (4.4)-(4.6). Using the theorem of Kenig and Pipher mentioned earlier it will then follow that $\omega_1(E, w'_k) \geq \xi_3 > 0$ for some ξ_3 and $w'_k \in \Omega'_k$. Next we consider the Lipschitz domains produced in the $N-1$ st step and estimate in a quantitative way the number of domains whose boundaries contain big pieces on which $\omega_1(E, \cdot) \geq \xi_3/c$. For these domains we can apply the above mentioned theorem and get that $\omega_1(E, \cdot) \geq \xi_4 > 0$ at a certain distinguished point in each domain. Because the process is finite and we are essentially at liberty to choose ξ_1 our argument will ultimately arrive at the first step and x, yielding (5.13).

To begin the development of the algorithm recall the definition of a good ϵ^2 tangent ball in section 3 (i.e replace ϵ by ϵ^2 in (3.6)). As usual $d(x) = d(x, \partial D), d_1(x) = d(x, \partial D_1)$, and tangent balls will always mean with respect to D. We consider the following possibilities when $\hat{x} \in D_1$ and $0 < d_1(\hat{x}) \leq \hat{\tau}(\epsilon)^{3/2}$.

$$\begin{aligned}
&(a) \quad |\nabla u(\hat{x})| \geq \delta_1 \text{ and } B(\hat{x}, d(\hat{x})) \text{ is a good } \epsilon^2 \text{ tangent ball.}\\
&(b) \quad |\nabla u(\hat{x})| \geq \delta_1 \text{ and } B(\hat{x}, d(\hat{x})) \text{ is a bad } \epsilon^2 \text{ tangent ball.}\\
&(c) \quad |\nabla u(\hat{x})| < \delta_1 \text{ and every tangent ball } B(y, d(y)) \subset B(\hat{x}, 10d(\hat{x})) \cap D \text{ with}\\
&\qquad d(y) \geq \epsilon d(\hat{x}) \text{ and } |\nabla u(y)| \geq \delta_1 \text{ is a good } \epsilon^2 \text{ tangent ball.}\\
&(d) \quad |\nabla u(\hat{x})| < \delta_1 \text{ and some tangent ball } B(y, d(y)) \subset B(\hat{x}, 10d(\hat{x})) \cap D \text{ with}\\
&\qquad d(y) \geq \epsilon d(\hat{x}) \text{ and } |\nabla u(y)| \geq \delta_1 \text{ is a bad } \epsilon^2 \text{ tangent ball.}
\end{aligned} \tag{5.15}$$

We claim that (5.12) implies (5.13) whenever $(b), (d)$ are valid. To prove this claim first suppose (b) holds. If $d_1(\hat{x}) \leq d(\hat{x})/\tilde{c}$, where $\tilde{c}$ is as in (5.9), then for some $y \in D$ we have $\hat{x} \in B(y, 5d(y)/\tilde{c})$ and $\partial B(y, d(y)/\tilde{c}) \subset \partial D_1$. If $d_1(\hat{x}) \geq d(\hat{x})/\tilde{c}$, then from (3.6) (β), (5.6), and the definition of D_1 in (5.9) we see for ϵ small enough that there exists

$$B(y, d(y)/\tilde{c}) \subset B(\hat{x}, \epsilon^{-20}d_1(\hat{x})) \cap D \text{ with } d(y) \geq \epsilon^{200}d(\hat{x}) \text{ and } \partial B(y, d(y)/\tilde{c}) \subset \partial D_1. \tag{5.16}$$

If $\xi_1(\epsilon)$ is small enough then clearly we can guarantee that

$$H^{n-1}(E \cap A) \geq (1/2)\, H^{n-1}(A). \tag{5.17}$$

where $A = B(\hat{x}, 2d_1(\hat{x})) \cap \partial B(y, d(y)/\tilde{c})$ when $d_1(\hat{x}) \leq d(\hat{x})/\tilde{c}$ while $A = \partial B(y, d(y)/\tilde{c})$, otherwise. Using (5.17), (5.9) (b), Theorem 4.24, and Harnack's inequality we deduce that if $d_1(\hat{x}) \geq d(\hat{x})/\tilde{c}$, then

$$\omega_1(E \cap \partial B(y, d(y)/\tilde{c}), \cdot) \geq c(\epsilon)^{-1} \text{ on } B(y, 5d(y)/\tilde{c}). \tag{5.18}$$

Using the chain condition in (3.6) (β) and Harnack's inequality we now obtain (5.13) for ξ_2 small enough. A similar argument applies if $d_1(\hat{x}) \leq d(\hat{x})/\tilde{c}$. Thus (5.13) is valid when (b) of (5.15) occurs. If (d) of (5.15) holds, then from (3.6) (β), our choice of $\tilde{c}$, and (5.9) (a) we deduce that $\hat{x} \in B(y, 1000d(y)/\tilde{c}) \subset B(y, d(y)/2)$ for some $y \in D$ satisfying (5.16). We can then choose ξ_1 so that (5.17) holds and after that argue as below (5.17) to get (5.13) for $\xi_2 > 0$ suitably small.

We note as in the argument after (3.25) that if (a) or (c) in (5.15) are valid, then there exists $\sigma \in \mathbb{R}^n$ with $|\sigma| = 1$ such that $\hat{x} - d(\hat{x})\sigma \in \partial B(\hat{x}, d(\hat{x})) \cap \partial D$ and

$$O = O(\hat{x}, \epsilon/2, \sigma) = \{w : \langle w - \hat{x} + d(\hat{x})\sigma, \sigma\rangle \geq \epsilon^2\, d(\hat{x})/4\} \cap B(\hat{x}, 2d(\hat{x})/\epsilon)\} \subset D \tag{5.19}$$

From (5.19) we see that if

$$\partial D_1 \cap O(\hat{x}, \epsilon, \sigma) \neq \emptyset, \tag{5.20}$$

then either $d_1(\hat{x}) \leq d(\hat{x})/\tilde{c}$ or (5.16) holds for ϵ sufficiently small, so once again we can argue as below this inequality to get that (5.12) implies (5.13). In view of the above discussion we will say that $B(\hat{x}, d(\hat{x}))$ is a fine tangent ball if either (a) or (c) of (5.15) is valid and (5.20) is false. Otherwise $B(\hat{x}, d(\hat{x}))$ is a not so fine tangent ball. Observe from our discussion that (5.12) implies (5.13) for a not so fine tangent ball and $0 < \xi_1(\epsilon), \xi_2(\epsilon)$ suitably small. Also if $B(\hat{x}, d(\hat{x}))$ is a fine tangent ball then from (3.27) with ϵ replaced by ϵ^2 we deduce for some $\sigma \in \mathbb{R}^n$ (as in (5.19)) with $|\sigma| = 1, \hat{x} - d(\hat{x})\sigma \in \partial B(\hat{x}, d(\hat{x})) \cap \partial D$ that

(+) $O(\hat{x}, \epsilon, \sigma) \subset D_1$

(++) If $P(\hat{x}, \sigma) = \{w : \langle w - \hat{x} + d(\hat{x})\sigma, \sigma\rangle = 2\epsilon^2 d(\hat{x})\}$ and $w \in P(\hat{x}, \sigma) \cap B(\hat{x}, d(\hat{x})/\epsilon)$, then $B(w, 4\epsilon^2\, d(\hat{x})) \cap \partial D \neq \emptyset$.

(5.21)

We continue under the assumption that $B(\hat{x}, d(\hat{x}))$ is a fine tangent ball and assume, as we may, that $\sigma = e_n$ in (5.21) while $\hat{x} = d(\hat{x})e_n$ (so $0 \in \partial B(\hat{x}, d(\hat{x})) \cap \partial D$). Let Q be an $n-1$ dimensional Lipschitz domain with connected boundary and

$$\{w : w_n = 3d(\hat{x})/2\} \cap B(\hat{w}, d(\hat{x})/c_+) \subset Q \subset \{w : w_n = 3d(\hat{x})/2\} \cap B(\hat{w}, 10d(\hat{x}))$$

for some $\hat{w}, c_+ = c_+(n) \geq 1$, with $|\hat{w} - \hat{x}| \leq c_+\, d(\hat{x})$ and $\hat{w}_n = 3d(\hat{x})/2$. We also assume that $\bar{Q} \setminus Q$ can be covered by at most $l \leq c_+^2$ balls $\{B(z_i, s_i)\}_1^l$ with $z_i \in \bar{Q} \setminus Q$, $s_i \geq d(\hat{x})/c_+^2$, and the property that $B(z_i, 2s_i) \cap (\bar{Q} \setminus Q)$ coincides with the graph of a Lipschitz function ϕ_i from $\mathbb{R}^{n-2}$ into $\{w : w_n = \frac{3}{2}d(\hat{x})\}$. Moreover in the proper coordinate system $Q \cap B(z_i, 2s_i)$ lies above the graph of ϕ_i and $|\nabla\phi_i| \leq c_+^2$ for $1 \leq i \leq l$. In fact it will turn out in later iterations that $\bar{Q}$ (see (6.13)) is at worst essentially the affine (nearly conformal) image of a union of Whitney cubes and $\bar{Q} \setminus Q$ is contained in the image of cubes whose sidelengths are proportional (depending only on n). Let $C = \{x = (x', x_n) \in \mathbb{R}^n : (x', 3d(\hat{x})/2) \in Q\}$ be the infinite cylinder containing Q. Let $\{B(y_j, d(y_j))\}$ be a pairwise disjoint collection of tangent balls with $y_j \in P(\hat{x}, e_n) \cap \bar{C}$ and $P(\hat{x}, e_n) \cap \bar{C} \subset \cup B(y_j, 10d(y_j))$. Note from (5.21) that for each j,

$$\epsilon^2\, d(\hat{x}) < d_1(y_j) \leq d(y_j) \leq 8\epsilon^2 d(\hat{x}). \tag{5.22}$$

If $B(y_j, d(y_j))$ is a not so fine tangent ball we do nothing further to this ball. Otherwise this ball is a fine tangent ball so as above we deduce the existence of $\sigma_j \in \mathbb{R}^n$ with $|\sigma_j| = 1, y_j - d(y_j)\sigma_j \in$

$\partial B(y_j, d(y_j)) \cap \partial D$ and

$$\begin{aligned} &(-) \qquad O(y_j, \epsilon, \sigma_j) \subset D_1 \text{ while } O(y_j, \epsilon, \sigma_j) \cap O(\hat{x}, \epsilon, e_n) \neq \emptyset \\ &(--) \quad \text{ If } w \in P(y_j, \sigma_j) \cap \bar{C} \cap B(y_j, d(y_j)/\epsilon) \text{ then } B(w, 8\epsilon^2 d(y_j)) \cap \partial D \neq \emptyset. \end{aligned} \tag{5.23}$$

If

$$\arccos(\langle \sigma_j, e_n \rangle) \geq \epsilon^{1/10}, \tag{5.24}$$

where $0 \leq \arccos(\cdot) \leq \pi$, we do nothing further to $B(y_j, d(y_j))$. Otherwise we put $y_0 = \hat{x}$, write j_1 for j in the above definitions, and use (5.21)-(5.23) to continue by induction. Assume after $l \geq 1$ repetitions we have obtained $y_{j_1 \dots j_l}, \sigma_{j_1 \dots j_l}$ [with $y_{j_1 \dots j_l} - d(y_{j_1 \dots j_l})\sigma_{j_1 \dots j_l} \in \partial B(y_{j_1 \dots j_l}, d(y_{j_1 \dots j_l})) \cap \partial D$], $O(y_{j_1 \dots j_l}, \epsilon, \sigma_{j_1 \dots j_l})$, $P(y_{j_1 \dots j_l}, \sigma_{j_1 \dots j_l})$ satisfying (5.22), (5.23) and not (5.24) with j replaced by $j_1 \dots j_l$ while $\hat{x}$ in (5.22) is replaced by $y_{j_1 \dots j_{l-1}}$. Under this inductive assumption, we choose a disjoint collection $\{B(y_{j_1 \dots j_{l+1}}, d(y_{j_1 \dots j_{l+1}}))\}$ of tangent balls with centers in $P(y_{j_1 \dots j_l}, \sigma_{j_1 \dots j_l}) \cap \bar{C} \cap B(y_{j_1 \dots j_l}, 20 d(y_{j_1 \dots j_l}))$ and such that $\{B(y_{j_1 \dots j_{l+1}}, 10 d(y_{j_1 \dots j_{l+1}}))\}$ is a covering of $P(y_{j_1 \dots j_l}, \sigma_{j_1 \dots j_l}) \cap \bar{C} \cap B(y_{j_1 \dots j_l}, 20 d(y_{j_1 \dots j_l}))$. If $B(y_{j_1 \dots j_{l+1}}, d(y_{j_1 \dots j_{l+1}}))$ is a not so fine tangent ball, we quit. Otherwise we argue as below (5.21) to get (5.22), (5.23) with j replaced by $j_1 \dots j_{l+1}$. Thus by induction we obtain $\sigma_{j_1 \dots j_{l+1}}$ with $y_{j_1 \dots j_{l+1}} - d(y_{j_1 \dots j_{l+1}})\sigma_{j_1 \dots j_{l+1}} \in \partial B(y_{j_1 \dots j_{l+1}}, d(y_{j_1 \dots j_{l+1}})) \cap \partial D$. Also we get $O(y_{j_1 \dots j_{l+1}}, \epsilon, \sigma_{j_1 \dots j_{l+1}}), P(y_{j_1 \dots j_{l+1}}, \sigma_{j_1 \dots j_{l+1}})$, satisfying

$$\epsilon^2 \, d(y_{j_1 \dots j_l}) \; < \; d_1(y_{j_1 \dots j_{l+1}}) \leq \; d(y_{j_1 \dots j_{l+1}}) \leq 8\epsilon^2 \, d(y_{j_1 \dots j_l}). \tag{5.25}$$

and

$$\begin{aligned} &(*) \qquad O(y_{j_1 \dots j_{l+1}}, \epsilon, \sigma_{j_1 \dots j_{l+1}}) \subset D_1 \text{ while } O(y_{j_1 \dots j_{l+1}}, \epsilon, \sigma_{j_1 \dots j_{l+1}}) \cap O(y_{j_1 \dots j_l}, \epsilon, \sigma_{j_1 \dots j_l}) \neq \emptyset \\ &(**) \quad \text{ If } w \in P(y_{j_1 \dots j_{l+1}}, \sigma_{j_1 \dots \hat{j}_{l+1}}) \cap \bar{C} \cap B(y_{j_1 \dots j_{l+1}}, d(y_{j_1 \dots j_{l+1}})/\epsilon) \\ &\qquad\quad \text{then } B(w, 8\epsilon^2 d(y_{j_1 \dots j_{l+1}})) \cap \partial D \neq \emptyset. \end{aligned} \tag{5.26}$$

If

$$\arccos(\langle \sigma_{j_1 \dots j_{l+1}}, e_n \rangle) \geq \epsilon^{1/10} \tag{5.27}$$

we stop. Otherwise the inductive process continues.

To simply our notation, if $B(\hat{x}, d(\hat{x}))$ is a fine tangent ball, and as above, $y_0 = \hat{x}, \sigma_0 = \sigma$, we let $L = \{j_1 \dots j_l\} \cup \{0\}$ be the subscripts used in the above induction. Also if $\alpha = j_1 \dots j_l$ or 0 let $|\alpha| = l$ or 0 be the length of α. If $\alpha, \alpha' \in L \setminus \{0\}$, we write, $\alpha < \alpha'$ provided $l = |\alpha| < |\alpha'| = m$ and if $\alpha = j_1 \dots j_l$, $\alpha' = j'_1 \dots j'_m$, then $j_i = j'_i$ for $1 \leq i \leq l$. We say that α is an ancestor of α' or that α' is a descendant of α if $|\alpha| < |\alpha'|$. If $m = l+1$ we call α the father of α' or refer to α' as the child of α. Likewise by definition $0 < \alpha$ and α is the descendant of 0 or 0 is the ancestor of α whenever $\alpha \in L \setminus \{0\}$. Next suppose that $\alpha, \alpha^* \in L$ and $y = y_{\alpha^*} \in B(y_\alpha, d(y_\alpha)/\epsilon) \cap D$ with

$$\epsilon^{2k} d(y_\alpha) \leq d(y) < \epsilon^{2k-2} \, d(y_\alpha). \text{ for } k = 1, 2, \dots$$

We write $\sigma(y)$ for σ_{α^*} and suppose also that $B(y, d(y)), B(y_\alpha, d(y_\alpha))$, are fine tangent balls. Under these assumptions we claim there exists $c = c(n) \geq 1$ such that

$$|\sigma(y) - \sigma_\alpha| \leq ck \, \epsilon. \tag{5.28}$$

To prove this claim, we note that if $k = 1$ in the above display, then (5.28) follows easily from (5.25) - (5.26). In fact for $k = 1$ the worst case scenario occurs when $d(y) \approx \epsilon^2\, d(y_\alpha)$ and y lies within $\approx \epsilon^2 d(y_\alpha)$ of $P(y_\alpha, d(y_\alpha))$. In this case $P(y, \sigma(y)) \cap B(y, d(y)/\epsilon)$ is of $\approx$ diameter $\epsilon d(y_\alpha)$ and so each point of this set must stay within $\approx \epsilon^2\, d(y_\alpha)$ distance from $P(y_\alpha, \sigma_\alpha) \cap B(y_\alpha, 2d(y_\alpha)/\epsilon)(\subset D)$ in order to avoid ∂D, thanks to (5.26). Using this fact and some high school trigonometry, we get (5.28) for $k = 1$. (5.28) for $k = 2$ follows from applying the same argument as in the $k = 1$ case to y_α and the father of y, then after that to y and the father of y.

Continuing in this fashion we obtain claim (5.28).

Next if $x = (x', x_n)$, let $\pi(x) = x' = (x_1, \dots, x_{n-1})$ be the projection of $\mathbb{R}^n$ onto $\mathbb{R}^{n-1}$ and put $Q' \subset \mathbb{R}^{n-1} = \{x' : (x', 3d(\hat{x})/2) \in Q\}$. Set

$$\mathcal{B} = \{w : w \in B(y_\alpha, 20d(y_\alpha)) \text{ for some } \alpha \text{ and } B(y_\alpha, d(y_\alpha)) \text{ is a not so fine tangent ball }\}.$$

$$\mathcal{G} = \{w : w \in B(y_\alpha, 20d(y_\alpha)) \text{ for an infinite number of } \alpha \text{ with } B(y_\alpha, d(y_\alpha)) \text{ a fine tangent ball }\}.$$

$$\lambda(w) = \sup\{|\alpha| : w \in B(y_\alpha, 20d(y_\alpha)),\ \ B(y_\alpha, d(y_\alpha)) \text{ is a fine tangent ball and } \sigma_\alpha \text{ satisfies (5.27) }\}.$$

$$\mathcal{G}_l = \{w : \lambda(w) = l\} \text{ for } l = 0, 1, \dots$$

From our construction we note for $\epsilon > 0$ small enough that

$$\bar{Q}' \subset \pi[\mathcal{B} \cup \mathcal{G} \cup (\bigcup_{l=0}^{\infty} \mathcal{G}_l)]. \tag{5.29}$$

With this notation we prove

Lemma 5.30. *If*

$$H^{n-1}(Q' \cap [\pi(\mathcal{B} \cup \mathcal{G})])\ \geq\ 2\,\epsilon^n\, H^{n-1}(Q')$$

then there exists $\psi : \bar{Q}' \to \mathbb{R}$ *and* $c = c(n)$ *with* $\|\psi\|_\infty \leq c\epsilon^2 d(\hat{x})$ *while*

$$|\psi(x') - \psi(y')| \leq\ c|x' - y'| \text{ whenever } x', y' \in \bar{Q}'. \tag{5.31}$$

Moreover if $\Omega_1 = \{(x', t) : \psi(x') < t < 3d(\hat{x})/2 \text{ for } x' \in Q'\}$, *then* $\Omega_1 \subset D_1$ *and either (a) or (b) holds for* $\epsilon > 0$ *small enough:*

(a) $H^{n-1}(\partial\Omega_1 \cap C \cap \partial D) \geq \epsilon^n H^{n-1}(Q')$.

(b) *There exists* $L_1 \subset L$ *such that if* $\alpha \in L_1$, *then* $B(y_\alpha, d(y_\alpha))$ *is a not so fine tangent ball,* $d_1(y_\alpha) \geq d(y_\alpha)/8$ *and* $\{B(y_\alpha, 5d(y_\alpha))\}$ *are pairwise disjoint. Also if* $\mathcal{B}_1 = \bigcup_{\alpha \in L_1} P(y_{\tilde{\alpha}}, \sigma_{\tilde{\alpha}}) \cap B(y_\alpha, d(y_\alpha)/16)$ *where* $\tilde{\alpha}$ *is the father of* α, *then* $\{x \in \mathcal{B}_1 : \pi(x) \in \bar{Q}'\} \subset \{(x', \psi(x')) : x' \in \bar{Q}'\} \cap D_1$ *while for some* $c^* = c^*(n) \geq 1$,

$$H^{n-1}(\pi(\mathcal{B}_1) \cap \bar{Q}') \geq \epsilon^n H^{n-1}(Q')/c^*. \tag{5.32}$$

Proof: We consider two cases. First, suppose

$$H^{n-1}[\pi(\mathcal{B}) \cap Q'] \geq \epsilon^n\, H^{n-1}(Q'). \tag{5.33}$$

In this case we use a well known covering lemma to get $\tilde{L}_1 \subset L$ such that for each $\alpha \in \tilde{L}_1$, $B(y_\alpha, d(y_\alpha))$ is a not so fine tangent ball while $\pi(\mathcal{B}) \cap \bar{Q}' \subset \cup_{\alpha \in \tilde{L}_1} \pi[B(y_\alpha, 100d(y_\alpha))]$ and $\{\pi[B(y_\alpha, 5d(y_\alpha))] : \alpha \in \tilde{L}_1\}$ are pairwise disjoint. Using (5.33) we see there exists L_1 a finite subset of $\tilde{L}_1$ with

$$H^{n-1}\{\bar{Q}' \cap \pi[\mathcal{B} \cap \cup_{\alpha \in L_1} B(y_\alpha, 100d(y_\alpha))]\} \geq (1/2)\epsilon^n H^{n-1}(Q').$$

For $\alpha \in L$ with $B(y_\alpha, d(y_\alpha))$ a fine tangent ball we write P_α for $P(y_\alpha, \sigma_\alpha)$ and define $f_\alpha : \mathbb{R}^{n-1} \to P_\alpha$ by $f_\alpha(x') = (x', x_n) \in P_\alpha$. Let $\{Q'_i = Q(w'_i, r_i)\}$ be a Whitney cube decomposition of $\mathbb{R}^{n-1} \setminus \{\pi(y_\alpha) : \alpha \in L_1\}$ and $\{\eta'_j\}$ a partition of unity adapted to this decomposition defined as in (2.15) with $y_i, Q_i, \eta_i, \partial D, n$, replaced by $w'_i, Q'_i, \eta'_i, \{\pi(y_\alpha) : \alpha \in L_1\}, n-1$. To define ψ let W be the set of all i such that

$$\text{supp } \eta'_i \cap \left[\cup_{\alpha \in L_1} \pi(B(y_\alpha, d(y_\alpha))\right] = \emptyset$$

and supp $\eta'_i \cap \bar{Q}' \neq \emptyset$. If $i \in W$ choose $\pi(y_{\hat{\beta}}) \in \{\pi(y_\alpha) : \alpha \in L_1\}$ with

$$d(\pi(y_{\hat{\beta}}), \bar{Q}'_i) = d(\{\pi(y_\alpha) : \alpha \in L_1\}, \bar{Q}'_i).$$

We note that $d(\bar{Q}'_i, \pi(y_{\hat{\beta}})) > d(y_{\hat{\beta}})$ since $\bar{Q}'_i \subset$ supp η'_i. From this note, (5.25), (5.26) we see that if $d(\bar{Q}'_i, \pi(y_{\hat{\beta}})) < d(\hat{x})$, then there exists at least one ancestor β' of $\hat{\beta}$ (and at most two ancestors) with

$$\epsilon^2 \, d(y_{\beta'}) < d(\bar{Q}'_i, \pi(y_{\hat{\beta}})) \leq d(y_{\beta'}). \tag{5.34}$$

Then from our construction, $B(y_{\beta'}, d(y_{\beta'}))$ is a fine tangent ball and

$$\bar{Q}'_i \subset \pi[B(y_{\beta'}, 40d(y_{\beta'})]. \tag{5.35}$$

In this case we let $f'_i = f_{\beta'}$ while if $d(\bar{Q}_i, \pi(y_{\hat{\beta}})) > d(\hat{x})$, we put $f'_i = f_0$ ($f_0(x') = (x', 2\epsilon^2 d(\hat{x})$)). Finally if $\tilde{\alpha}$ is the father of $\alpha \in L_1$ we set

$$f'_i = f_{\tilde{\alpha}} \text{ whenever } \text{ supp } \eta'_i \cap \pi[B(y_\alpha, d(y_\alpha))] \neq \emptyset \text{ and } \alpha \in L_1.$$

To define ψ first let $\psi(\pi(y_\alpha)) = f_{\tilde{\alpha}}(\pi(y_\alpha))$ for $\alpha \in L_1$ and put

$$(x', \psi(x')) = \sum_i f'_i(x') \eta_i(x') \text{ otherwise in } \bar{Q}'. \tag{5.36}$$

It is easily checked that ψ is infinitely differentiable on $\bar{Q}'$. Next we note again from (5.25), (5.26) and (5.34), (5.35) that for some $c = c(n)$ and $x' \in$ supp η'_i, $i \in W$, that

$$c^{-1}\epsilon^2 d(\bar{Q}'_i, \pi(y_{\hat{\beta}})) \leq d(f'_i(x'), \partial D) \leq c d(\bar{Q}'_i, \pi(y_{\hat{\beta}})) \tag{5.37}$$

which implies in view of (5.28), (5.26) that if $i, j \in W$, and supp $\eta'_i \cap$ supp $\eta'_j \neq \emptyset$, then for $x' \in$ supp $\eta'_i \cup$ supp η'_j

$$|f'_i(x') - f'_j(x')| \leq c r_i. \tag{5.38}$$

This inequality also remains valid if either i, j is not in W as is readily checked using once again (5.25), (5.26) as well as disjointness of $\{\pi(B(y_\alpha, 5d(y_\alpha)) : \alpha \in L_1\}$. Using (5.38) it is easily shown that (5.31) of Lemma 5.30 is valid. In fact if $x' \in \bar{Q}' \cap$ supp η'_i, then

$$|\nabla\psi|(x') \leq \sum_{\{j : x' \in \text{ supp } \eta'_j\}} [|\nabla f_j|(x') + |f_i - f_j|(x') \, |\nabla \eta'_j|(x')] \leq c. \tag{5.39}$$

Hence (5.31) is true. Next we note that $\Omega_1 \subset D_1$. Indeed if $B(y_\alpha, d(y_\alpha)), \alpha \in L$, is a fine tangent ball, $x \in B(y_\alpha, 80d(y_\alpha))$, and $x' = \pi(x) \in \bar{Q}'$, then it follows from (5.26) $(*)$ that the closed vertical line segment joining $(x', \frac{3}{2}d(\hat{x}))$ to $f_\alpha(x')$ is contained in D_1. Since $(x', \psi(x'))$ is a convex sum of such segments, we conclude that the closed vertical line segment joining $(x', \frac{3}{2}d(\hat{x}))$ to $(x', \psi(x'))$ is contained in D_1. Thus $\Omega_1 \subset D_1$. Finally, from our choice of L_1, the fact that $y_\alpha \in \bar{C}$, and basic geometry we see that

$$H^{n-1}\{\pi[\bigcup_{\alpha \in L_1} B(y_\alpha, d(y_\alpha)/16)] \cap \bar{Q}'\} \geq \epsilon^n H^{n-1}(\bar{Q}')/c$$

for some $c = c(n)$. (b) follows easily from this inequality and the definition of $\psi, \mathcal{B}$. Thus Lemma 5.30 is valid when (5.33) holds.

Now suppose that (5.33) is false. In this case we note from (5.25), (5.26) and a continuity type argument that if $y \in \mathcal{G}$, then $y \in \partial D$ and the open line segment from $(\pi(y), 3d(\hat{x})/2)$ to y is contained in D_1. Clearly this statement remains valid when $y \in \bar{\mathcal{G}}$. Thus if $x' \in \pi[\bar{\mathcal{G}}] \cap \bar{Q}'$, then $(x', \psi(x')) \in \bar{\mathcal{G}}$ is well defined. We claim that (5.31) is valid for $x', y' \in \pi(\bar{\mathcal{G}}) \cap \bar{Q}'$. If $|x' - y'| \geq d(\hat{x})/100$, then (5.31) follows directly from (5.25) and the fact that diam $\bar{Q}' \leq 20\, d(\hat{x})$. Otherwise, let L_2 denote the set of all $\alpha \in L$ such that $B(y_\alpha, d(y_\alpha))$ is a fine tangent ball. Choose $\alpha, \beta \in L_2$ so that

$$d(y_\alpha) + d(y_\beta) + |x' - \pi(y_\alpha)| + |y' - \pi(y_\beta)| < |x' - y'|/2. \tag{5.40}$$

We can then choose ancestors α', β' of α, β, such that if $\tilde{y} \in \{y_{\alpha'}, y_{\beta'}\}$, then

$$\epsilon^2 d(\tilde{y}) < |x' - y'| \leq d(\tilde{y}). \tag{5.41}$$

Using (5.40), (5.41) we deduce first as in (5.37) that

$$|f_{\alpha'}(x') - (x', \psi(x'))| + |f_{\beta'}(y') - (y', \psi(y'))| \leq c|x' - y'| \tag{5.42}$$

and second as in (5.38) that

$$|f_{\alpha'}(x') - f_{\beta'}(x')| + |f_{\alpha'}(y') - f_{\beta'}(y')| \leq c|x' - y'|. \tag{5.43}$$

From (5.42), (5.43) and the triangle inequality we get (5.31) when $x', y' \in \pi(\bar{\mathcal{G}})$. Clearly (a) of Lemma 5.30 will be valid once we extend the definition of ψ to all of $\bar{Q}'$. To complete the proof of Lemma 5.30 (a) we argue as in the proof of Lemma 5.30 (b). That is, let $\{Q_i' = Q(w_i', r_i)\}$ be a Whitney cube decomposition of $\mathbb{R}^{n-1} \setminus \pi(\bar{\mathcal{G}})$ and $\{\eta_j'\}$ a partition of unity adapted to this decomposition. To define ψ on $\bar{Q}' \setminus \pi(\bar{\mathcal{G}})$ suppose supp $\eta_i' \cap \bar{Q}' \neq \emptyset$ and choose $\alpha \in L_2$ with

$$2d(y_\alpha) < d(\pi(y_\alpha), \bar{Q}_i') \leq 2d(\pi(\bar{\mathcal{G}}), \bar{Q}_i'). \tag{5.44}$$

If $d(\bar{Q}_i', \pi(\bar{\mathcal{G}})) < d(\hat{x})$, let $\hat{\alpha}$ be an ancestor of α with

$$\epsilon^2\, d(y_{\hat{\alpha}}) < d[\bar{Q}_i', \pi(\bar{\mathcal{G}})] \leq d(y_{\hat{\alpha}}) \tag{5.45}$$

and put $f_i' = f_{\hat{\alpha}}$. If $d(\bar{Q}_i', \pi(\bar{\mathcal{G}})) > d(\hat{x})$, set $f_i' = f_0$. Finally let

$$\psi = \sum_i f_i'\, \eta_i' \text{ on } \bar{Q}' \setminus \pi(\bar{\mathcal{G}}). \tag{5.46}$$

From (5.45), (5.46) we obtain as in (5.37)-(5.39) that

$$|\nabla\psi| \leq c \text{ on } \bar{Q}' \setminus \pi(\bar{\mathcal{G}}) \tag{5.47}$$

Using (5.47), and (5.31) for points in $\pi(\bar{\mathcal{G}})$, it is easily checked that (5.31) is valid. Also from our construction and (5.26) $(*)$ we conclude first that $\Omega_1 \subset D_1$ and second that Lemma 5.30 (a) is valid. The proof of Lemma 5.30 is now complete. □

To continue our proof of (5.13) we prove

Lemma 5.48. *Under the hypotheses of Lemma 5.30, (5.12) implies (5.13) for $\xi_1(\epsilon), \xi_2(\epsilon) > 0$ sufficiently small with x replaced by $\hat{x}$.*

Proof: If (a) of Lemma 5.30 is valid, then Lemma 5.48 can be deduced from a more general version of Theorem 4.24, Harnack's inequality and the fact that $B(\hat{x}, d(\hat{x}))$ is a fine tangent ball. Another way is to observe that there exists $z' \in Q'$ such that if $B = \{w' : \ |w' - z'| < \epsilon^{2n} d(\hat{x})\}$ then

$$B \subset \bar{Q}' \text{ and } H^{n-1}[B \cap \pi(\bar{\mathcal{G}})] \geq \epsilon^{5n^3} d(\hat{x})^{n-1}. \tag{5.49}$$

Indeed from our assumptions on $\bar{Q}'$ we deduce that

$$H^{n-1}[\{w' \in \bar{Q}' : d(w', \bar{Q}' \setminus Q') < 100\epsilon^{2n} d(\hat{x})\}] \leq c\epsilon^{2n} H^{n-1}(\bar{Q}').$$

Using this fact and covering $\bar{Q}'$ by a union of balls $\{B(z_i, \epsilon^{2n} d(\hat{x}))\}$ with $\{B(z_i, \epsilon^{2n} d(\hat{x})/10)\}$ pairwise disjoint we find from the usual volume argument that if (5.49) were false then for $\epsilon > 0$ small enough we would have a contradiction to

$$H^{n-1}[\pi(\bar{\mathcal{G}}) \cap \bar{Q}'] \geq \epsilon^n \, H^{n-1}[Q'].$$

Thus (5.49) is true. Let $\Gamma = \{(x', \psi(x')) : x' \in B\}$ and choose a so that $d(B \times \{a\}, \Gamma) = 2\epsilon^{2n} d(\hat{x})$ and $B \times \{a\} \subset \Omega_1$. Let Ω' be the domain obtained by drawing line segments parallel to the e_n axis from points in $B \times \{a\}$ to points in Γ. Extend ψ to be Lipschitz on $\mathbb{R}^{n-1}$ with Lipschitz constant as in (5.31). We can now apply Theorem 4.24 in Ω' and use the maximum principle to conclude first that

$$\omega_1[\Gamma \cap \partial D, (z', t)] \geq c(\epsilon)^{-1}$$

for some (z', t) in D_1 lying $\epsilon^{2n} d(\hat{x})$ from Γ. Using Harnack's inequality we then get Lemma 5.48 when (a) of Lemma 5.30 is valid.

If (b) of Lemma 5.30 holds let E be as in (5.12) and let L_3 be the set of all $\alpha \in L_1$ such that

$$H^{n-1}[B(y_\alpha, \epsilon^{-40} d_1(y_\alpha)) \cap E] \geq (1 - \hat{\xi}(\epsilon)) H^{n-1}[B(y_\alpha, \epsilon^{-40} d_1(y_\alpha)) \cap \partial D_1]. \tag{5.50}$$

As in (5.17) we deduce from (5.50) (using Lemma 5.30 (b)) that for some small $\tilde{\xi}(\epsilon) > 0$,

$$\omega_1[B(y_\alpha, \epsilon^{-20} d(y_\alpha)) \cap E, y_\alpha] \geq \tilde{\xi}(\epsilon) \tag{5.51}$$

whenever $\alpha \in L_3$. It follows from Harnack's inequality and our construction that if

$$\mathcal{B}_2 = \bigcup_{\alpha \in L_3} P(y_{\tilde{\alpha}}, \sigma_{\tilde{\alpha}}) \cap B(y_\alpha, d(y_\alpha)/16),$$

then

$$\omega_1(E,\cdot) \geq \hat{\xi}(\epsilon)/c \text{ on } \mathcal{B}_2. \tag{5.52}$$

We claim that if $\xi_1(\epsilon)$ is small enough then there exists $\xi'(\epsilon) > 0$ such that

$$H^{n-1}[\pi(\mathcal{B}_2) \cap \bar{Q}'] \geq \xi'(\epsilon)\, H^{n-1}(Q'). \tag{5.53}$$

Once (5.53) is proved we can use this inequality, (5.32) (b), (5.52), the boundary maximum principle for weak solutions to the pde in (4.15) and Theorem 4.24 as in case (a) to conclude that Lemma 5.48 is valid. Thus we only prove (5.53). To prove (5.53) first note from (5.22), (5.25) that $B(y_\alpha, \epsilon^{-40} d(y_\alpha)) \subset B(\hat{x}, \epsilon^{-40} d_1(\hat{x}))$ whenever $\alpha \in L_1$. Second observe from disjointness of $\{B(y_\alpha, 5d(y_\alpha)) : \alpha \in L_1\}$, Ahlfors regularity of ∂D_1, and a Vitali type covering argument that there exists $L_4 \subset L_1$ such that $\{B(y_\alpha, \epsilon^{-40} d(y_\alpha)) : \alpha \in L_4\}$ are disjoint and for some large $\hat{c}(\epsilon) \geq 1$,

$$\hat{c}(\epsilon) \sum_{\alpha \in L_4} d(y_\alpha)^{n-1} \geq \sum_{\alpha \in L_1} d(y_\alpha)^{n-1} \geq \hat{c}(\epsilon)^{-1} H^{n-1}(Q'), \tag{5.54}$$

where the last inequality follows from Lemma 5.30 (b). Let

$$F = [\cup_{\alpha\, \in\, L_4 \setminus L_3} \; B(y_\alpha, \epsilon^{-40} d_1(y_\alpha)) \cap \partial D_1] \setminus E.$$

Using the above note, the definition of L_3, L_4, (5.12) and Ahlfors regularity of ∂D we see for some $c_- \geq 1$, depending only on the data, that

$$\xi_1(\epsilon) H^{n-1}[B(\hat{x}, d_1(\hat{x})/\epsilon^{40}) \cap \partial D_1] \geq H^{n-1}(F) \geq \, c_-^{-1}\, \hat{\xi}(\epsilon) \sum_{\alpha \in L_4 \setminus L_3} d_1(y_\alpha)^{n-1}. \tag{5.55}$$

From (5.54), (5.55), the fact that

$$d(\hat{x})^{n-1} \,\leq\, c H^{n-1}(Q'),$$

and

$$d(y_\alpha) \,\leq\, 8 d_1(y_\alpha), \alpha \in L_1,$$

we conlude for $\xi_1(\epsilon) > 0$, small enough

$$\sum_{\alpha \in L_4 \setminus L_3} d(y_\alpha)^{n-1} \,\leq\, (1/2) \sum_{\alpha \in L_4} d(y_\alpha)^{n-1}.$$

which implies that

$$2 \sum_{\alpha \in L_3 \cap L_4} d(y_\alpha)^{n-1} \,\geq\, \sum_{\alpha \in L_4} d(y_\alpha)^{n-1}. \tag{5.56}$$

Finally observe that

$$H^{n-1}[\pi\{P(y_{\tilde{\alpha}}, \sigma_{\tilde{\alpha}}) \cap B(y_\alpha, d(y_\alpha)/16)\} \cap \bar{Q}'] \geq d(y_\alpha)^{n-1}/c \tag{5.57}$$

for some $c = c(n)$ as follows from Lipschitzness of Q', falseness of (5.27) for σ_α, and $\pi(y_\alpha) \in \bar{Q}'$. From (5.57), (5.56), (5.54), we conclude that (5.53) is true. The proof of Lemma 5.48 is now complete. □

6. Proof of Theorem 1.27. In this section we complete the description of our algorithm and use it to show that (5.12) implies (5.13). We then get Proposition 5.3. Using this proposition we obtain Theorem 1.27 as in the remark after Proposition 5.3. To complete the construction of our algorithm it remains to consider the situation when

$$A' = \pi\left(\bigcup_{l=0}^{\infty} \mathcal{G}_l\right) \setminus \pi(\mathcal{B}\cup\mathcal{G})$$

(see the display above (5.29)) has large measure. More specifically, if Lemma 5.30 is false, we can choose a finite subset $\hat{L}$ of L such that for each $\alpha \in \hat{L}$, $B(y_\alpha, d(y_\alpha))$ is a fine tangent ball and $\{\pi[B(y_\alpha, 5d(y_\alpha))] : \alpha \in \hat{L}\}$ are disjoint. Also, if $A = A' \cap \pi[\bigcup_{\alpha\in\hat{L}} B(y_\alpha, 100d(y_\alpha))]$, then

$$H^{n-1}(Q'\cap A) \geq H^{n-1}(Q'\cap A') - \epsilon^n\, H^{n-1}(Q') \geq (1-3\epsilon^n)H^{n-1}(Q'). \tag{6.1}$$

Moreover from (5.27), (5.28), we have

$$\epsilon^{1/10} \leq \arccos(\langle\sigma_\alpha, e_n\rangle) \leq \epsilon^{1/10} + c(n)\epsilon \tag{6.2}$$

whenever $\alpha \in \hat{L}$. Again we essentially repeat the argument after (5.33). To this end let $\{Q'_i = Q(w'_i, r_i)\}$ be a Whitney cube decomposition of $\mathbb{R}^{n-1}\setminus\{\pi(y_\alpha) : \alpha\in\hat{L}\}$ and $\{\eta'_j\}$ a partition of unity adapted to this decomposition. Let $\hat{W}$ be the set of all i such that

$$\text{supp } \eta'_i \cap \left[\bigcup_{\alpha\in\hat{L}} \pi(B(y_\alpha, d(y_\alpha))\right] = \emptyset$$

and supp $\eta'_i \cap \bar{Q}' \neq \emptyset$. If $i\in\hat{W}$ choose $\pi(y_{\hat{\beta}}) \in \{\pi(y_\alpha) : \alpha\in\hat{L}\}$ with

$$d(\pi(y_{\hat{\beta}}), \bar{Q}'_i) = d(\bar{Q}'_i, \{\pi(y_\alpha) : \alpha\in\hat{L}\}).$$

Then, $d(\pi(y_{\hat{\beta}}), \bar{Q}'_i) < d(\hat{x})$ for ϵ sufficiently small so there exists β' an ancestor of $\hat{\beta}$ with

$$\epsilon^2\, d(y_{\beta'}) < d(\bar{Q}'_i, \pi(y_{\hat{\beta}})) \leq d(y_{\beta'}). \tag{6.3}$$

Also, $B(y_{\beta'}, d(y_{\beta'}))$ is a fine tangent ball and

$$\bar{Q}'_i \subset \pi[B(y_{\beta'}, 40d(y_{\beta'})]. \tag{6.4}$$

Let $f'_i = f_{\beta'}$ in this case and set

$$f'_i = f_\alpha \text{ whenever } \ \text{supp } \eta'_i \cap \pi[B(y_\alpha, d(y_\alpha))] \neq \emptyset \text{ and } \alpha\in\hat{L}.$$

Next let $\psi(\pi(y_\alpha)) = f_\alpha(\pi(y_\alpha))$ for $\alpha\in\hat{L}$ and put

$$(x', \psi(x')) = \sum_i f'_i(x')\eta_i(x') \text{ otherwise in } \bar{Q}'. \tag{6.5}$$

As in (5.37), (5.38) we deduce for some $c = c(n)$ and $x' \in$ supp η'_i, $i\in\hat{W}$, that

$$c^{-1}\epsilon^2\, d(\bar{Q}'_i, \pi(y_{\hat{\beta}})) \,\leq\, d(f'_i(x'), \partial D_1) \,\leq\, d(f'_i(x'), \partial D) \,\leq\, c\, d(\bar{Q}'_i, \pi(y_{\hat{\beta}})). \tag{6.6}$$

Also if supp $\eta_i' \cap$ supp $\eta_j' \cap \bar{Q}' \neq \emptyset$, then for $x' \in$ supp $\eta_i' \cup$ supp η_j'

$$|f_i'(x') - f_j'(x')| \leq c r_i. \tag{6.7}$$

Using (6.3)- (6.7) it follows that $\|\psi\|_\infty \leq c\epsilon^2 d(\hat{x})$ and

$$|\psi(x') - \psi(y')| \leq c\,|x' - y'| \text{ when } x', y' \in \bar{Q}'. \tag{6.8}$$

Moreover, if $x' \in$ supp $\eta_i \cap \bar{Q}'$ and $i \in \hat{W}$ then for some $c = c(n)$,

$$c^{-1}\,\epsilon^2\, r_i \;\leq\; d(\{(x', \psi(x'))\}, \partial D_1) \leq\; d(\{(x', \psi(x'))\}, \partial D) \leq c r_i \tag{6.9}$$

while if $x' \in$ supp E_α for some $\alpha \in \hat{L}$, where

$$E_\alpha = \cup\, \{\bar{Q}_j' :\ \text{supp } \eta_j' \cap \pi(B(y_\alpha, d(y_\alpha))) \neq \emptyset\},$$

then

$$\tfrac{1}{2}\epsilon^2 \text{ diam } E_\alpha \;\leq\; d(\{(x', \psi(x'))\}, \partial D_1) \;\leq d(\{(x', \psi(x'))\}, \partial D) \;\leq\; 16\epsilon^2 \text{ diam } E_\alpha. \tag{6.10}$$

Thus if Ω_1 is defined as in Lemma 5.30, then $\Omega_1 \subset D_1$.

Let $Q'' = \{x' \in \bar{Q}' : d(x', \bar{Q}' \setminus Q') >\ 2\epsilon \text{ diam } Q'\}$. We note from (6.3), (6.2), and (5.28) that if either $M = \bar{Q}_i'$ with $\bar{Q}_i' \cap Q'' \cap A \neq \emptyset$ or $M = \pi[B(y_\alpha, 100d(y_\alpha))]$, $\alpha \in \hat{L}$, and $M \cap Q'' \neq \emptyset$, then for $\epsilon > 0$, small enough

$$\text{diam } M \leq \exp\left[-\frac{1}{\epsilon^{1/2}}\right] \text{ diam } Q', \qquad \text{so} \qquad M \subset \{x' \in Q' : d(x', \bar{Q}' \setminus Q') > \epsilon \text{ diam } Q'\}. \tag{6.11}$$

Set

$$W' = \{i : \bar{Q}_i' \cap A \cap Q'' \neq \emptyset \text{ and } \bar{Q}_i' \cap [\textstyle\bigcup_{\alpha \in \hat{L}}\ \pi(B(y_\alpha, d(y_\alpha)))] = \emptyset\}.$$

If $i \in W'$ choose w_i with $\pi(w_i) = w_i'$ and $d(w_i) = c\, d(w_i', \{\pi(y_\alpha) : \alpha \in \hat{L}\})$ where $c = c(n) \geq 1$ is chosen large enough so that $w_i \in \Omega_1$ and $d(w_i, \partial\Omega_1) \geq d(w_i)/c$. This choice is possible as we see from (6.9). Let W'' be the set of $i \in W'$ such that $B(w_i, d(w_i))$ is a not so fine tangent ball. We consider two cases. If

$$H^{n-1}[\textstyle\bigcup_{i \in W''}\ \pi\{B(w_i, d(w_i))\}\,] \geq \epsilon^n\, H^{n-1}(Q') \tag{6.12}$$

we can repeat the argument in case (b) of Lemma 5.48 with L_1 replaced by $W_1 \subset W''$ where $\{\pi[B(w_i, 5d(w_i))] : i \in W_1\}$ are pairwise disjoint and

$$\pi[\textstyle\bigcup_{i \in W''}\ B(w_i, 20d(w_i))] \subset \pi[\bigcup_{i \in W_1}\ B(w_i, 100d(w_i))].$$

Define W_3 relative to W_1 as in (5.50) with y_α replaced by w_i. Using (6.9) and arguing as in (5.50), (5.51), we get (5.52) with $\mathcal{B}_2$ replaced by

$$\{(x', \psi(x')) : x' \in \textstyle\bigcup_{i \in W_3}\ Q_i'\}.$$

The rest of the argument is unchanged. Thus (5.12) implies (5.13) when (6.12) holds.

To complete the description of our algorithm it remains to consider the case when (6.12) is false. Let $W^* = W' \setminus W''$ be the subset of W' with $B(w_i, d(w_i))$ a fine tangent ball when $i \in W^*$. If $i \in W^*$ let $\sigma_i, P(w_i, \sigma_i)$ be defined as in (5.23) with y_j replaced by w_i. We note from (6.2), (6.3) and the same argument as in (5.28) that $\arccos(\langle \sigma_i, e_n \rangle) \leq c\epsilon^{1/10}$ for some $c = c(n)$. Given $\alpha \in \hat{L}$, let

$$E'_\alpha = \{\pi(y_\alpha)\} \cup \{Q'_i : Q'_i \cap \pi(B(y_\alpha, d(y_\alpha))) \neq \emptyset\}$$

Next let $L^* = W^* \cup \{\alpha \in \hat{L} : E'_\alpha \cap \bar{Q}'' \neq \emptyset\}$ and $Q''_i = Q'_i(w'_i, (1-\epsilon)r_i)$. If $m \in W^*$ define $f_m : \mathbb{R}^{n-1} \to P(w_m, \sigma_m)$ by $f_m(w') = (w', w_n) \in P(w_m, \sigma_m)$ while if $m \in \hat{L}$ let f_m be as defined earlier. Put $T_m = f_m(Q''_m)$ for $m \in W^*$ and set $T_m = f_m(E'_m)$. when $m \in \hat{L}$. Then from our construction and the definition of Ω_1 we see for some $c = c(n) \geq 1$,

$$\begin{aligned}
&(a) \quad d(v_m)/c \leq d(v_m, \partial\Omega_1) \leq c d(v_m) \\
&(b) \quad \epsilon^2 d(v_m) \leq d(T_m, \partial D_1) \leq 8\epsilon^2 d(v_m) \\
&(c) \quad \epsilon^2\, d(v_m)/c \leq d[\pi^{-1}(\pi(T_m)) \cap \partial\Omega_1, \partial D_1] \leq c\, d(v_m)
\end{aligned} \tag{6.13}$$

where $v_m = w_m$ when $m \in W^*$ while $v_m = y_m$ when $m \in \hat{L}$. From (6.2) we see for ϵ small enough that

$$(1/2) H^{n-1}[T_m] \leq H^{n-1}[\pi(T_m)] \leq (1 + \epsilon^{1/4})\, H^{n-1}[T_m] \text{ whenever } m \in \hat{L} \cap L^*. \tag{6.14}$$

If $m \in L^*$ let C_m be the closed cylinder with height $16\epsilon^2 d(v_m)$, base $\bar{T}_m$, axis parallel to σ_m and $C_m \cap \partial D \neq \emptyset$. Let $T^*_m = \{y + (-\epsilon^2 + \frac{3}{2}) d(v_m)\sigma_m : y \in T_m\}$. We note that

$$C_m \cap C_l = \emptyset \text{ and } C_m \cup C_l \subset C \text{ whenever } m, l \in L^*. \tag{6.15}$$

Indeed from our construction we have $\pi(C_m)$ contained in the interior of $Q'_m(w'_m, (1 - \frac{1}{2}\epsilon) r_m)$ when $T_m = f_m(Q''_m)$. Using this fact and basic geometry one sees that $\pi(C_m) \cap \pi(C_l) = \emptyset$ whenever $m, l \in L^*$ and that all projections are contained in Q' provided $\epsilon > 0$ is small enough. Thus (6.15) is valid.

Next from the definition of $\hat{L}, Q'', E'_\alpha$, the fact that (6.12) is false, and (6.14) we deduce

$$2\, H^{n-1}(\bar{Q}') \geq H^{n-1}[\cup_{m \in L^*} T_m] \geq (1 + \epsilon^{1/3})\, H^{n-1}(\bar{Q}') \tag{6.16}$$

for ϵ sufficiently small. This completes the description of our algorithm.

Let $Q = T^*_0$ and $L'_0 = \{0\}, L^* = L^*_1$. For each $m_1 \in L^*_1$ we can repeat our algorithm with Q replaced by $T^*_{m_1}$ and $\hat{x}, e_n$ replaced by v_{m_1}, σ_{m_1} defined as in (6.13). Let E be as in (5.12). If $m_1 \in L^*_1$ and

$$H^{n-1}[B(v_{m_1}, \epsilon^{-40} d_1(v_{m_1})) \cap E] \geq (1 - \hat{\xi}(\epsilon)) H^{n-1}[B(v_{m_1}, \epsilon^{-40} d_1(v_{m_1})) \cap \partial D_1] \tag{6.17}$$

then for $\hat{\xi}(\epsilon) > 0$ and sufficiently small it follows from our algorithm (see Lemma 5.48 and the discussion following (5.15)) that either

$$\omega_1[B(v_{m_1}, \epsilon^{-20} d_1(v_{m_1})) \cap E, v_{m_1}] \geq \tilde{\xi}(\epsilon) \tag{6.18}$$

or there is a projection π_{m_1} onto a plane through the origin with normal, σ_{m_1} (which we now regard as $\mathbb{R}^{n-1}$), a Lipschitz domain $\Omega_1(m_1)$, and $(x', \psi_{m_1}(x')) : \pi_{m_1}(T^*_{m_1}) \to \partial\Omega_1(m_1)$ satisfying (6.8) with Q' replaced by $\pi_{m_1}(T^*_{m_1})$. We also get indices $\{m_1 m_2\}$, centers $\{v_{m_1 m_2}\}$, linear mappings $\{f_{m_1 m_2}\}$ of certain skeletal complexes in $\pi_{m_1}(T^*_{m_1})$ onto $\{T_{m_1 m_2}\}$, cylinders $\{C_{m_1 m_2}\}$ satisfying (6.13)-(6.15), with m replaced by $m_1 m_2$ and Ω_1 by $\Omega_1(m_1)$ (replace $C = C_0$ in (6.15) by C_{m_1}). Moreover we define $\{T^*_{m_1 m_2}\}$ and

$$(1/2)\, H^{n-1}[\cup_{m_1 \in L_1^*} T_{m_1 m_2}] \leq H^{n-1}[\pi_{m_1}(T_{m_1})] \leq (1+\epsilon^{1/3})\, H^{n-1}[\cup_{m_1 \in L_1^*} T_{m_1 m_2}]. \tag{6.19}$$

Let L_1^+ be the subset of L_1^* for which (6.17) is false. We note that if $\xi_1(\epsilon) > 0$ is small enough then we can repeat the argument following (5.52) to get (see (6.29) for a more general argument)

$$\sum_{m_1 \in L_1^+} H^{n-1}(T_{m_1}) \leq \epsilon^n \, H^{n-1}(Q'). \tag{6.20}$$

Let L_1^- be the subset of L_1^* for which (6.18) is true. If

$$\sum_{m_1 \in L_1^-} H^{n-1}(T_{m_1}) \geq 2\epsilon^n H^{n-1}(Q') \tag{6.21}$$

we stop and note from (6.20), (6.21) that

$$\sum_{m_1 \in L_1^- \setminus L_1^+} H^{n-1}(T_{m_1}) \geq \epsilon^n \, H^{n-1}(Q'). \tag{6.22}$$

Using Harnack's inequality, (6.18), and (6.13) we see first that

$$\omega_1(E, \cdot) \geq \tilde{\xi}(\epsilon) \text{ on } \cup_{m \in L_1^- \setminus L_1^+} \pi^{-1}(\pi(T^*_m)) \cap \partial\Omega_1 \tag{6.23}$$

for some small $\tilde{\xi}(\epsilon) > 0$. Second from (6.22) and a use of Theorem 4.24 as in the proof of Lemma 5.48 we conclude that (5.13) is valid. If (6.21) is false we let $L_1' = L_1^* \setminus (L_1^+ \cup L_1^-)$ and observe from (6.16), (6.20) that

$$\begin{aligned} 2\, H^{n-1}(\bar{Q}') &\geq H^{n-1}[\cup_{m \in L_1'} T_m] = \cup_{m \in L_1'} H^{n-1}(T_m) \\ &\geq (1+\epsilon^{1/3} - 3\epsilon^n)\, H^{n-1}(\bar{Q}') \geq (1+\epsilon^{1/2}) H^{n-1}(Q') \end{aligned} \tag{6.24}$$

for $\epsilon > 0$ sufficiently small. To continue we use an inductive argument and ancestor notation. Recall that $\alpha < \beta$ if α is a descendant of β. Suppose after $k \geq 1$ times that we have obtained indices $L_k' = \{m_1 m_2 \ldots m_k\}$, centers $\{v_\theta : \theta \in L_k'\}$, as well as $\{T_\theta,\, T^*_\theta : \theta \in L_k'\}$ and corresponding cylinders

$\{C_\theta : \theta \in L_k'\}$. Assume that

$$\begin{array}{ll} (i) & \{B(v_\theta, d(v_\theta)) : \theta \in L_k'\} \text{ consists of fine tangent balls satisfying (6.17)} \\ & \text{but not (6.18) with } m_1 \text{ replaced by } \theta. \\ & \\ (ii) & \text{(6.13) holds with } m \text{ replaced by } \theta \in L_k' \text{ and } \Omega_1 \text{ by } \Omega_1(\gamma) \\ & \text{where } \gamma \text{ is the father of } \theta. \\ & \\ (iii) & \text{(6.15) is valid with } m, l \text{ replaced by } \theta, \phi \in L_k' \text{ and } C \text{ by } C_\gamma \\ & \text{where } \gamma \in L_{k-1}' \text{ is the father of } \theta, \phi. \\ & \\ (iv) & \bigcup_{\{\theta \in L_k' \,:\, \theta < \gamma\}} B(v_\theta, \epsilon^{-40}\, d(v_\theta)) \subset B(v_\gamma, \epsilon^{-40}\, d_1(v_\gamma)) \quad \text{for each } \gamma \in L_{k-1}'. \end{array} \tag{6.25}$$

We also assume that

$$c\,\epsilon^{2-2n}\, d(\hat{x})^{n-1} \geq H^{n-1}[\bigcup_{\theta \in L_k'} T_\theta] \geq (1+\epsilon^{1/2})^k\, H^{n-1}(\bar{Q}') \tag{6.26}$$

where c depends only on the data for D so is independent of ϵ and k. Under these assumptions we apply our algorithm to each T_θ^* with $\theta \in L_k'$. Let σ_θ be the normal to the plane containing T_θ and let π_θ be the projection of T_θ^* onto a plane through the origin with normal σ_θ (which we now regard as $\mathbb{R}^{n-1}$). As in the discussion following (6.18) there exists a Lipschitz domain $\Omega_1(\theta), (x', \psi_\theta(x')) : \pi(T_\theta^*) \to \partial\Omega_1(\theta)$, where ψ_θ satisfies (6.8) on $\pi_\theta(T_\theta^*)$ with ψ replaced by ψ_θ. We also get indices $L_{k+1}^* = \{m_1 \ldots m_k m_{k+1}\}$, centers $\{v_\delta : \delta \in L_{k+1}^*\}$, linear mappings $\{f_\delta : \delta \in L_{k+1}^*\}$ of certain skeletal complexes in $\pi_\theta(T_\theta^*)$ onto $\{T_\delta : \delta < \theta,\, \delta \in L_{k+1}^*\}$, as well as cylinders $\{C_\delta\}$ and $\{T_\delta^* : \delta < \theta,\, \delta \in L_{k+1}^*\}$. Moreover (6.25) $(iii), (iv)$ hold with L_k' replaced by L_{k+1}^* and L_{k-1}' replaced by L_k'. Given $\theta \in L_k'$ let $Z(\theta) = \{\delta \in L_{k+1}^* : \delta < \theta\}$. Then as in (6.16) we see for $\theta \in L_k'$ that

$$2H^{n-1}(T_\theta) \geq H^{n-1}(\bigcup_{\delta \in Z(\theta)} T_\delta) \geq (1+\epsilon^{1/3})\, H^{n-1}(T_\theta). \tag{6.27}$$

Given $\theta \in L_k'$, let $Z_1(\theta)$ be the subset of $Z(\theta)$ for which (6.17) is false with m_1 replaced by δ. Let $\tilde{L}_k$ be the subset of L_k' with the property that for $\theta \in \tilde{L}_k$

$$\sum_{\delta \in Z_1(\theta)} H^{n-1}(T_\delta) \leq \epsilon^n H^{n-1}(T_\theta). \tag{6.28}$$

We claim as in (6.20) that we can choose $\xi_1(\epsilon)$ so small that

$$\sum_{\theta \in \tilde{L}_k} H^{n-1}(T_\theta) \geq (1-\epsilon^n) \sum_{\theta \in L_k'} H^{n-1}(T_\theta). \tag{6.29}$$

To verify this claim we essentially repeat the argument following (5.53). In fact suppose $\theta \in L_k' \setminus \tilde{L}_k$. We note from the definition of $\{C_\delta : \delta \in L_{k+1}^*\}$ and disjointness of these sets as well as fineness of $\{B(v_\delta, d(v_\delta)) : \delta \in L_{k+1}^*\}$ that for some $q_\delta \in C_\delta \cap \partial D$ and c depending only on the data for D, we have

$$B(q_\delta, \epsilon^2\, d(v_\delta)/c) \subset C_\delta. \tag{6.30}$$

Using (6.30), Ahlfors regularity of ∂D, and (6.25)(iii) we find $Z_2(\theta) \subset Z_1(\theta)$ such that

$$\sum_{\delta \in Z_1(\theta)} d(v_\delta)^{n-1} \, \leq \, c(\epsilon) \sum_{\delta \in Z_2(\theta)} d(v_\delta)^{n-1} \tag{6.31}$$

and so that $\{B(v_\delta, \epsilon^{-40} d(v_\delta)) : \delta \in Z_2(\theta)\}$ are disjoint. Then

$$H^{n-1}[B(v_\theta, \epsilon^{-40} d_1(v_\theta)) \cap \partial D_1 \setminus E] \geq \alpha(n-1)\, \hat{\xi}(\epsilon) \sum_{\delta \in Z_2(\theta)} d_1(v_\delta)^{n-1} \geq \hat{\xi}(\epsilon) \tilde{c}(\epsilon)^{-1} H^{n-1}(T_\theta) \tag{6.32}$$

as we see from (6.31) and the fact that $\theta \in L'_k \setminus \tilde{L}_k$. Next we use the same reasoning to choose $\hat{L}_k \subset L'_k \setminus \tilde{L}_k$ with

$$\sum_{\theta \in L'_k \setminus \tilde{L}_k} H^{n-1}(T_\theta) \, \leq \, c(\epsilon) \sum_{\theta \in \hat{L}_k} H^{n-1}(T_\theta) \tag{6.33}$$

and so that $\{B(v_\theta, \epsilon^{-40} d(v_\theta)) : \theta \in \hat{L}_k\}$ are disjoint. Summing (6.32) over $\theta \in \hat{L}_k$ and using (5.12), (6.25) (iv), (6.33) we obtain

$$\begin{aligned} \xi_1(\epsilon) H^{n-1}[B(\hat{x}, d_1(\hat{x})/\epsilon^{40}) \cap \partial D_1] &\geq H^{n-1}[B(\hat{x}, \epsilon^{-40} d_1(\hat{x})) \cap (\partial D \setminus E)] \\ &\geq \sum_{\theta \in \hat{L}_k} H^{n-1}[B(v_\theta, \epsilon^{-40} d_1(v_\theta)) \cap \partial D_1 \setminus E] \geq \hat{\xi}(\epsilon)\, \tilde{c}(\epsilon)^{-1} \sum_{\theta \in \hat{L}_k} H^{n-1}(T_\theta) \\ &\geq \bar{c}(\epsilon)^{-1} \sum_{\theta \in L'_k \setminus \tilde{L}_k} H^{n-1}(T_\theta). \end{aligned} \tag{6.34}$$

(6.34), Ahlfors regularity of ∂D_1 and (6.26) imply claim (6.29) for $\xi_1(\epsilon) > 0$, sufficiently small.

Next given $\theta \in \tilde{L}_k$, let $Z_3(\theta)$ be the set of all $\delta \in Z(\theta) \setminus Z_1(\theta)$ such that (6.18) is valid with v_m replaced by v_δ. Let $L_k^{\#}$ be the subset of $\theta \in \tilde{L}_k$ for which

$$\sum_{\delta \in Z_3(\theta)} H^{n-1}(T_\delta) \geq \epsilon^n \, H^{n-1}(T_\theta). \tag{6.35}$$

If

$$\sum_{\theta \in L_k^{\#}} H^{n-1}(T_\theta) \geq \epsilon^n \sum_{\theta \in \tilde{L}_k} H^{n-1}(T_\theta) \tag{6.36}$$

we stop. Otherwise, set

$$L'_{k+1} = \cup_{\theta \, \in \, \tilde{L}_k \, \setminus \, L_k^{\#}} \; Z(\theta) \setminus [Z_1(\theta) \cup Z_3(\theta)].$$

Then from (6.27) - (6.29) as well as falsity of (6.35), (6.36), we see that

$$\begin{aligned} \sum_{\delta \in L'_{k+1}} H^{n-1}(T_\delta) &= \sum_{\theta \in \tilde{L}_k \setminus L_k^{\#}} \left(\sum_{\delta \in Z(\theta) \setminus [Z_1(\theta) \cup Z_3(\theta)]} H^{n-1}(T_\delta) \right) \\ &\geq (1 + \epsilon^{1/3} - 2\epsilon^n) \sum_{\theta \in \tilde{L}_k \setminus L_k^{\#}} H^{n-1}(T_\theta) \geq (1 - \epsilon^n)^2 (1 + \epsilon^{1/3} - 2\epsilon^n) \sum_{\theta \in L'_k} H^{n-1}(T_\theta). \end{aligned} \tag{6.37}$$

Using (6.26), (6.37) it follows that

$$H^{n-1}\left(\bigcup_{\delta\,\in\, L'_{k+1}} T_\delta\right) = \sum_{\delta\in L'_{k+1}} H^{n-1}(T_\delta) \geq (1+\epsilon^{1/2})^{k+1}\, H^{n-1}(\bar{Q}'). \tag{6.38}$$

Also from disjointness of $\{B(q_\delta, \epsilon^2 d(v_\delta)/c) : \delta \in L'_{k+1}\}$ and Ahlfors regularity of ∂D we see that

$$\sum_{\delta\in L'_{k+1}} H^{n-1}(T_\delta) \leq c\epsilon^{2-2n} \sum_{\delta\in L'_{k+1}} d(v_\delta)^{n-1} \leq c\,\epsilon^{2-2n}\, d(\hat{x})^{n-1}. \tag{6.39}$$

Combining (6.38), (6.39) we get (6.26) with k replaced by $k+1$. From (6.26) and induction we see for $c^{\#}$ large enough that our process must stop after at most $N \leq c^{\#}\,\epsilon^{-3/4}$ times. Thus (5.14) is true. Finally we show by a backward iteration process that (5.12) implies (5.13) with x replaced by $\hat{x}$. Indeed, from our induction assumption and the fact that the algorithm is now stopped we deduce the validity of (6.35), (6.36) with $k = N$. Using (6.35) and (6.25) (ii) we deduce for some $\tilde{c}$, depending only on the data, that

$$\omega_1(E,\cdot) \geq \tilde{\xi}(\epsilon)/\tilde{c} \text{ on } X(\theta) = \partial\Omega_1(\theta) \cap \left(\bigcup_{\delta\,\in\, Z_3(\theta)} \pi_\theta^{-1}[\pi_\theta(T^*_\delta)]\right) \tag{6.40}$$

and

$$H^{n-1}(X(\theta)) \,\geq\, \frac{\epsilon^n}{\tilde{c}}\, H^{n-1}(T_\theta). \tag{6.41}$$

Using (6.40), (6.41), Theorem 4.24 as in the proof of Lemma 5.40, and fineness of $B(v_\theta, d(v_\theta))$ we conclude that

$$\omega_1(E,\cdot) \geq \xi_+(\epsilon) \text{ on } T^*_\theta \tag{6.42}$$

provided $\xi_+(\epsilon) > 0$ is small enough. If $N = 2$, we can use (6.42), (6.36), (6.26) and once again Theorem 4.24 to conclude that (5.12) implies (5.13) for $\hat{x}$. Otherwise we proceed by induction. Suppose for some $k \geq 1$ that we have obtained (6.42) with ξ_+ replaced by ξ_- for some subset Y_k with $\theta \in Y_k \subset L'_{N-k}$ and $N-k > 2$. Moreover, assume for some large c_* depending only on the data that

$$\sum_{\theta\in Y_k} H^{n-1}(T_\theta) \geq \frac{\epsilon^n}{c_*^k} \sum_{\theta\in L'_{N-k}} H^{n-1}(T_\theta). \tag{6.43}$$

Let Y_{k+1} be the set of all $\gamma \in L'_{N-k-1}$ for which

$$\sum_{\theta\in Y_k,\,\theta<\gamma} H^{n-1}(T_\theta) \geq \frac{\epsilon^n}{c_*^{k+1}} H^{n-1}(T_\gamma). \tag{6.44}$$

We claim for c_* large enough that

$$\text{(6.43) is valid with } k \text{ replaced by } k+1. \tag{6.45}$$

Indeed, otherwise from (6.27) (see also (6.37)), we have

$$\begin{aligned}\sum_{\gamma\in Y_{k+1}} \left(\sum_{\theta\in Y_k,\theta<\gamma} H^{n-1}(T_\theta)\right) &\leq 2 \sum_{\gamma\in Y_{k+1}} H^{n-1}(T_\gamma) \\ &\leq 2\frac{\epsilon^n}{c_*^{k+1}} \sum_{\gamma\in L'_{N-k-1}} H^{n-1}(T_\gamma) \leq 2\frac{\epsilon^n}{c_*^{k+1}} \sum_{\theta\in L'_{N-k}} H^{n-1}(T_\theta)\end{aligned} \tag{6.46}$$

where c depends only on the data. Moreover from the definition of Y_{k+1}, we see that

$$\sum_{\gamma\in L'_{N-k-1}\setminus Y_{k+1}}\left(\sum_{\theta\in Y_k,\theta<\gamma} H^{n-1}(T_\theta)\right) \le \frac{\epsilon^n}{c_*^{k+1}}\sum_{\gamma\in L'_{N-k-1}} H^{n-1}(T_\gamma) \le \frac{\epsilon^n}{c_*^{k+1}}\sum_{\theta\in L'_{N-k}} H^{n-1}(T_\theta). \tag{6.47}$$

Combining (6.46), (6.47) we get

$$\sum_{\theta\in Y_k} H^{n-1}(T_\theta) \le \frac{3\,\epsilon^n}{c_*^{k+1}}\sum_{\theta\in L'_{N-k}} H^{n-1}(T_\theta). \tag{6.48}$$

which is a contradiction to (6.43) for c_* large enough. Thus claim (6.45) is true. Using (6.42), (6.45), we see first from (6.25) (ii) that

$$\omega_1(E,\cdot) \ge \xi_-(\epsilon)/c \text{ on } \partial\Omega_1(\gamma)\cap\left(\bigcup_{\theta\,\in\,Y_k,\,\theta\,<\,\gamma}\ \pi_\theta^{-1}[\pi_\theta(T_\theta^*)]\right). \tag{6.49}$$

Second from (6.49), (6.43), and Theorem 4.24 we conclude for $\gamma\in Y_{k+1}$ that (6.42) holds for $k+1,\theta$ replaced by γ and suitably small $\xi_+(\epsilon)$. Continuing this argument $k = N-2$ times we obtain that (5.12) implies (5.13). We now fix $\epsilon = \epsilon_1 > 0$ small enough so that (5.12) implies (5.13) and then conclude from the remark after (5.13) that Proposition 5.3 is valid. Finally from the remark after Proposition 5.3 we get Theorem 1.27. The proof of Theorem 1.27 is now complete. □

7. Proof of Theorems 1.30 and concluding remarks. Let u, A, D, be as in Theorem 1.30. From (1.1) (b) we see there exists exactly one positive number a with $a\,A(0,a^2) = \beta_1$ where β_1 is as in (1.28). Then from Theorem 1.27 and (1.1) (b), we deduce that

$$\limsup_{x\longrightarrow\partial D}|\nabla u|(x)\ \le\ a. \tag{7.1}$$

Next we note that if D is δ Reifenberg flat (δ small) then it follows as in (3.3), (3.11) that near ∂D,

$$c^{-1}\,d(x)\ \le\ u(x)\ \le\ c\,d(x). \tag{7.2}$$

Moreover by assumption we have

$$\mu(B(z,r)\cap\partial D) = \beta_1 H^{n-1}(B(z,r)\cap\partial D) \text{ for } 0<r\le r_0. \tag{7.3}$$

Using (7.1) - (7.3) it can be shown that the machine developed in [ACF, sec 5] can be applied to the situation when D is $\delta > 0$ Reifenberg flat and δ is small enough. In fact these authors consider slightly less general A and assume that u is a local minimizer for a certain variational problem. However a careful reading of [ACF] shows that these assumptions are essentially to guarantee that (7.1) - (7.3) and slightly weaker assumptions on D are all valid. Also one has to be careful when A has degenerate ellipticity (see for example [DP] for the p Laplacian) but this obstacle can be overcome by considering a related partial differential equation as in (4.13) and using estimates for subsolutions. We omit the details. Applying this machine we deduce first as in [V, sec 6] that $\partial\Omega$ is $C^{2,\alpha}$ and thereupon from the moving plane argument (see [V, sec 7]) that D is a ball. □

Remark. Here we make some remarks concerning possible generalizations of Theorems 1.8, 1.10, 1.27, and 1.30.

1) As regards Theorem 1.10, we would like to know if $u = 0$ (i.e (1.5)) can be replaced by a weaker condition. For example, assume that $C \equiv 0$ and $A(s,t) = t^{p/2-1}, t \in (0,\infty)$. Suppose D is a bounded domain and that u is a weak solution to the p Laplacian partial differential equation in $\mathbb{R}^n \setminus \partial D$ while u is a bounded Lipschitz supersolution to this equation in $\mathbb{R}^n$. Then there exists a positive positive Borel measure μ corresponding to u as in (1.6) with supp $\mu \subset \partial D$. Under these assumptions, (1.7), (1.11), we would like to know if ∂D is still locally uniformly rectifiable. As evidence for this query we note that if $p = 2$, then one can use the Riesz represesentation formula for superharmonic functions to get that certain truncated Riesz transforms of μ are bounded on the space of square integrable functions defined on ∂D and taken with respect to μ ($L^2_\mu(\partial D)$). In case $n = 2$ it is shown in [MMV] that boundedness of the Riesz transforms on $L^2_\mu(\partial D)$ for an Ahlfors regular domain D implies uniform rectifiability. Thus our query is true when $n = 2, p = 2$. The difficulty one encounters in trying to prove this query in general, is in finding a meaningful square function estimate similar to (1.9).

2) One could also attempt to prove generalizations of Theorem 1.10 for higher order partial differential equations and lower dimensional Ahlfors regular sets. For example suppose that

$$P(x) = \int_E K(x-y)\, d\mu(y), x \in \mathbb{R}^n \tag{7.4}$$

is the capacitary potential for a compact set $E \subset \mathbb{R}^n$, $n \geq 4$, corresponding to the kernel

$$K(x) = |x|^{4-n} \text{ when } n > 4 \text{ and } K(x) = \log \frac{1}{|x|} \text{ for } n = 4$$

(see for example [L] for definitions). Assume that

$$c^{-1}\, r^{n-3} \leq \mu(B(x,r)) \leq c\, r^{n-3} \tag{7.5}$$

whenever $x \in E$ and $0 < r \leq r_0$. It is easily seen from (7.4) that E is $n-3$ Ahlfors regular and a solution to the biharmonic equation in $\mathbb{R}^n \setminus E$. We would like to know if E is locally $n-3$ dimensional uniformly rectifiable (in the language of [DS]). This query is true when $n = 4$ as follows once again from [MMV]. Somewhat similar problems occur in [DS1, III, 3.8].

3) Another question is whether Theorem 1.27 remains valid when hypothesis (1.11) is removed. In order to do away with this assumption it appears that one should somehow make estimates more in terms of μ and probably also generalize Theorem 4.24. We note that somewhat similar questions for nondoubling measures have recently been studied in [NTV], [T], [T1], [MTV].

4) Does Theorem 1.30 remain valid without the Reifenberg flatness assumption? It appears likely from [F], [W] (or perhaps is even implied in [ACF]) that this theorem remains valid for Lipschitz domains. We did not pursue the proof of Theorem 1.31 for Lipschitz domains as we feel strongly that at least in two dimensions the above question is not beyond our reach.

5) A theorem in [BJ] states that if a set, say F, is added to a locally uniformly rectifiable set E in such a way that $E \cup F = \partial D \subset \mathbb{R}^2$, where D is a simply connected domain, then $\omega|_E$ is absolutely continuous with respect to $H^1|_E$ where ω denotes harmonic measure in D taken with respect to a fixed point. What is the analogue of this result in $\mathbb{R}^n, n \geq 3$? That is, what is the most general class of domains in $\mathbb{R}^n$ (eg NTA) for which this conclusion is valid. On a related note we believe that the technique in sections 5 and 6 can be used to show that if F is taken to be a certain union of balls then the above conclusion is valid. Moreover, ∂D is locally uniformly rectifiable.

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