Caloric Measure in Parabolic Flat Domains

Steve Hofmann[1]
Department of Mathematics
University of Missouri
Columbia MO 65211-0001
email: hofmann@math.missouri.edu

John L. Lewis[1,2]
Department of Mathematics
University of Kentucky
Lexington, KY 40506-0027
john@ms.uky.edu

Kaj Nyström[2]
Department of Mathematics
Umeå University
90187 Umeå, Sweden
email: Kaj.nystrom@math.umu.se

Current Address:
Föreningssparbanken
Group Financial Risk Control
S-10534 Stockholm, Sweden
kaj.nystrom@foreningssparbanken.se

ABSTRACT: In this paper we define parabolic chord arc domains and show that in a parabolic chord arc domain with vanishing constant, the logarithm of the density of caloric measure with respect to a certain projective Lebesgue measure is of vanishing mean oscillation. We also obtain a partial converse to this result. Our work generalizes to the heat equation, recent work of Kenig and Toro for the Laplacian.

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1. Background and Notation. In this paper we obtain parabolic analogues of results by Kenig-Toro in [KT], [KT1]. To give some history, a longstanding problem concerning solvability of the Dirichlet problem for Laplace's equation in a Lipschitz domain, $\tilde{\Omega} \subset \mathbb{R}^n$, was resolved by B. Dahlberg [D], who showed that in $\tilde{\Omega}$ harmonic measure with respect to a fixed point , $d\tilde{\omega}$, and surface measure, $d\tilde{\sigma}$, are mutually absolutely continuous. In fact if $\tilde{k} = d\tilde{\omega}/d\tilde{\sigma}$, then Dahlberg showed that $\tilde{k}$ is in a certain L^2 reverse Hölder class from which it follows that $\log \tilde{k} \in BMO(\partial\tilde{\Omega})$, the functions of bounded mean oscillation with respect to surface area on $\partial\tilde{\Omega}$. Jerison and Kenig [JK] showed for a C^1 domain that $\log \tilde{k} \in VMO(\partial\tilde{\Omega})$, the functions in $BMO(\partial\tilde{\Omega})$ of vanishing mean oscillation. In [KT] this result is generalized to ' chord arc domains with vanishing constant' while in [KT1] a partial converse to the work in [KT] is obtained. These authors show that if (i) harmonic measure is asymptotically optimally doubling, (ii) $\log \tilde{k} \in VMO(\partial\tilde{\Omega})$, and (iii) $\tilde{\Omega}$ is a chord arc domain with sufficiently small constant, then actually $\tilde{\Omega}$ is a chord arc domain with vanishing constant. Recently in [KT2] these authors have shown that (i) is not necessary to obtain the above conclusion.

R. Hunt proposed the problem of finding an analogue of Dahlberg's result for the heat equation in domains whose boundaries are given locally as graphs of functions $A(x,t)$ which are uniformly Lipschitz in the space variable. It was conjectured at one time that A should be $\mathrm{Lip}_{\frac{1}{2}}$ in the time variable, but subsequent counterexamples of Kaufmann and Wu [KW] showed that this condition does not suffice. Lewis and Murray [LM], made significant progress toward a solution of Hunt's question, by establishing mutual absolute continuity of caloric measure and a certain parabolic analogue of surface measure in the case that A has $\frac{1}{2}$ of a time derivative in $BMO(\mathbb{R}^n)$ (defined with respect to parabolic rectangles), a condition only slightly stronger than $\mathrm{Lip}_{\frac{1}{2}}$. Furthermore they showed that caloric measure is in a certain L^p reverse Hölder class for some $p > 1$. This result was later shown to be sharp in [HL] where examples are given of graph domains, as in [LM], with p arbitrarily near 1.

To state parabolic analogues of the results in [KT], [KT1], and relate our results to work in [LM], [HL], we shall need some definitions and notation. Let $(X,t), X = (x_0, \dots, x_{n-1}), t \in \mathbb{R}$ denote a point in $\mathbb{R}^{n+1}$ and for given $r > 0$ set $C_r(X,t) = \{(Y,s) : |Y - X| < r, |t-s| < r^2\}$. Given a Borel set $F \subset \mathbb{R}^{n+1}$ let $\bar{F}, \partial F$ denote the closure, boundary of F, respectively, and put $\sigma(F) = \int_F d\sigma_t \, dt$ where $d\sigma_t$ is $n-1$ dimensional Hausdorff measure on the time slice $F \cap (\mathbb{R}^n \times \{t\})$. Let Ω be a connected open set in $\mathbb{R}^{n+1}$. We say that $\partial\Omega$ separates $\mathbb{R}^{n+1}$ and is δ_0 Reifenberg flat, $0 < \delta_0 \leq 1/10$, if given any $(X,t) \in \partial\Omega, R > 0$, there exists a n dimensional plane $\hat{P} = \hat{P}(X,t,R)$ containing (X,t) and a line parallel to the t axis with unit normal $\hat{n} = \hat{n}(X,t,R)$ such that

$$\begin{gathered} \{(Y,s) + r\hat{n} \in C_R(X,t) : (Y,s) \in \hat{P}, r > \delta_0 R\} \subset \Omega \\ \{(Y,s) - r\hat{n} \in C_R(X,t) : (Y,s) \in \hat{P}, r > \delta_0 R\} \subset \mathbb{R}^{n+1} \setminus \Omega. \end{gathered} \tag{1.1}$$

For short we shall just say $\partial\Omega$ separates $\mathbb{R}^{n+1}$ when (1.1) holds for some δ_0. Note that if $\partial\Omega$ separates $\mathbb{R}^{n+1}$, then a line segment drawn parallel to $\hat{n}$ and with endpoints in each of the sets in (1.1), also intersects $\partial\Omega$.

We say that $\partial\Omega$ satisfies a (M,R) Ahlfors condition, $M \geq 4$, if for all $(X,t) \in \partial\Omega$ and $0 < r \leq R$,

$$\sigma(\partial\Omega \cap C_r(X,t)) \leq M r^{n+1}. \tag{1.2}$$

Using the above note, (1.1), (1.2), the fact that Hausdorff measure does not increase under a projection we deduce for $0 < r \leq R$, $(X,t) \in \partial\Omega$,

$$(r/2)^{n+1} \leq \sigma(\partial\Omega \cap C_r(X,t)) \leq M r^{n+1}, \tag{1.3}$$

whenever $\partial\Omega$ separates $\mathbb{R}^{n+1}$ and satisfies a (M, R) Ahlfors condition.

Next let

$$d(F_1, F_2) = \inf\{|X - Y| + |s - t|^{1/2} : (X, t) \in F_1,\ (Y, s) \in F_2\}, \tag{1.4}$$

denote the parabolic distance between the sets F_1, F_2 and for Ω as in (1.3) set

$$\gamma(Z, \tau, r) = \inf_P \left[r^{-n-3} \int_{\partial\Omega \cap C_r(Z,\tau)} d(\{(Y, s)\}, P)^2 d\sigma(Y, s) \right] \tag{1.5}$$

where the infimum is taken over all n dimensional planes P containing a line parallel to the t axis. Let

$$d\nu(Z, \tau, r) \; = \; \gamma(Z, \tau, r)\, d\sigma(Z, \tau)\, r^{-1} dr. \tag{1.6}$$

We say that ν is a Carleson measure on $[\partial\Omega \cap C_R(Y, s)] \times (0, R)$ if there exists $M_1 < \infty$ such that whenever $(X, t) \in \partial\Omega$ and $C_\rho(X, t) \subset C_R(Y, s)$, we have

$$\nu([C_\rho(X, t) \cap \partial\Omega] \times (0, \rho)) \; \leq \; M_1\, \rho^{n+1}. \tag{1.7}$$

The least such M_1 in (1.7) is called the Carleson norm of $[\,\partial\Omega \cap C_R(Y, s)] \times (0, R)$. We write $\|\nu\|_+$ for the Carleson norm of ν when (1.7) holds for all $\rho > 0$. As in [DS], [DS1] we say that $\partial\Omega$ is uniformly rectifiable (in the parabolic sense) if $\|\nu\|_+ < \infty$ and (1.3) holds for all $R > 0$. We remark that the work in [DS], [DS1] was in turn motivated by earlier work of [Jo], [Jo1] where L^∞ versions of (1.5) were introduced. If $\partial\Omega$ separates $\mathbb{R}^{n+1}$ and is uniformly rectifiable, we shall call Ω a parabolic regular domain. Finally in analogy with [KT] we say that Ω is a chord arc domain with vanishing constant if Ω is a parabolic regular domain and

$$\sup_{\substack{(X,t)\in\partial\Omega \\ 0<\rho\leq r}} \left[\rho^{-(n+1)}\, \nu([C_\rho(X, t) \cap \partial\Omega] \times (0, \rho)) \right] \to 0 \text{ as } r \to 0. \tag{1.8}$$

Next we note that if $\partial\Omega$ separates $\mathbb{R}^{n+1}$, then the bounded continuous Dirichlet problem for the heat equation in Ω always has a unique solution as follows from barrier type estimates, using solutions to the heat equation in rectangles, and a Perron-Wiener-Brelot type argument. For fixed $(\hat{X}, \hat{t}) \in \Omega$, let $\omega(\hat{X}, \hat{t}, \cdot)$ be the corresponding parabolic measure for the heat equation obtained by way of the Riesz representation theorem. Let $\Delta(X, t, r) \; = \; C_r(X, t) \cap \partial\Omega$ whenever $(X, t) \in \partial\Omega$ and $r > 0$. Given $(\hat{X}, \hat{t}) \in \Omega$, $(X, t) \in \partial\Omega$ suppose $|X - \hat{X}|^2 \; \leq \; A(\hat{t} - t)$ for some $A \geq 2$. We shall say that $\omega(\hat{X}, \hat{t}, \cdot)$ is a doubling measure provided there exists $a_1 = a_1(n, A)$ such that if $\hat{t} - t \; \geq \; 8r^2$, then

$$\omega\left(\hat{X}, \hat{t}, \Delta(X, t, 2r)\right) \; \leq \; a_1\, \omega\left(\hat{X}, \hat{t}, \Delta(X, t, r)\right). \tag{1.9}$$

To continue our definitions, suppose $\omega(\hat{X}, \hat{t}, \cdot)$ is a doubling measure which is absolutely continuous with respect to σ on $\partial\Omega$. If $k(\hat{X}, \hat{t}, \cdot) = d\omega(\hat{X}, \hat{t}, \cdot)/d\sigma$ and there exists $p > 1, a_2 = a_2(n, A, p) < \infty$, such that:

$$\begin{aligned} &\int_{\Delta(X,t,2r)} k(\hat{X}, \hat{t}, Y, s)^p\, d\sigma(Y, s) \\ &\quad \leq \; a_2\, \sigma[\Delta(X, t, 2r)]^{(1-p)} \left[\int_{\Delta(X,t,2r)} k(\hat{X}, \hat{t}, Y, s)\, d\sigma(Y, s) \right]^p \end{aligned} \tag{1.10}$$

then we say that $\omega(\hat{X}, \hat{t}, \cdot)$ is an A^∞ weight (with respect to σ). Here $(X,t), r, A$ are as in (1.9). (1.10) is referred to as an L^p reverse Hölder inequality. We will first prove the following theorem.

Theorem 1. *If Ω is a parabolic regular domain with Reifenberg constant $\delta_0 = \delta_0(M, \|\nu\|_+)$, sufficiently small, then ω is an A^∞ weight.*

Next let $a = a(\Delta(X,t,\rho), f)$, denote the average of $f = \log k(\hat{X}, \hat{t}, \cdot)$ on $\Delta(X,t,\rho)$ with respect to σ. Then we say that $f \in VMO(\partial\Omega)$ provided for each compact $K \subset \partial\Omega \cap \{(Y,s) : s < \hat{t}\}$,

$$\lim_{r \longrightarrow 0} \sup_{\substack{(X,t)\in K \\ 0<\rho\le r}} \sigma\left(\Delta(X,t,\rho)\right)^{-1} \int_{\Delta(X,t,\rho)} |\, f(Y,s) - \, a \,|\, d\sigma \, = \, 0. \tag{1.11}$$

In section 4 we obtain the following parabolic analogue of [KT].

Theorem 2. *If Ω is a chord arc domain with vanishing constant and $k(\hat{X}, \hat{t}, \cdot) = d\omega(\hat{X}, \hat{t}, \cdot)/d\sigma$, then* $\log k(\hat{X}, \hat{t}, \cdot) \in VMO(\partial\Omega)$.

We note that our definition of a chord arc domain with vanishing constant appears to bear no relationship to the one in [KT], [KT1]. However our definition is actually equivalent to their definition for the Laplacian while their definition is weaker than ours for the heat equation. In fact Theorem 2 is false under parabolic versions of their assumptions (see the remark at the end of section 3 in [HLN]). Given $(X,t) \in \partial\Omega$, and $r, \rho > 0$ put $\Delta(X,t,r,\rho) = \{(Y,s) \in \partial\Omega : |Y - X| < r, \; |s-t| < \rho^2\}$. So with this notation $\Delta(X,t,r) = \Delta(X,t,r,r)$. We say that $\omega(\hat{X}, \hat{t}, \cdot)$ is asymptotically optimal doubling if whenever $K \subset \partial\Omega \cap \{(Y,s) : s < \hat{t}\}$ is compact and $0 < \tau_1, \tau_2 < 1$, we have

$$\lim_{r \longrightarrow 0} \sup_{(X,t)\in K} \frac{\omega(\Delta(X,t,\tau_1 r, \tau_2 r))}{\omega(\Delta(X,t,r))} = \lim_{r \longrightarrow 0} \inf_{(X,t)\in K} \frac{\omega(\Delta(X,t,\tau_1 r, \tau_2 r))}{\omega(\Delta(X,t,r))} = \tau_1^{n-1}\tau_2^2. \tag{1.12}$$

In sections 5-7 we obtain as in [KT1]

Theorem 3. *Let Ω be as in Theorem 1 and put $k(\hat{X}, \hat{t}, \cdot) = d\omega(\hat{X}, \hat{t}, \cdot)/d\sigma$. If*

(a) $\quad \omega(\hat{X}, \hat{t}, \cdot)$ *is asymptotically optimal doubling,*

(b) $\quad \log k(\hat{X}, \hat{t}, \cdot) \in VMO(\partial\Omega)$,

(c) $\quad \|\nu\|_+$ *is small enough,*

and F is a compact subset of $\partial\Omega \cap \{(Y,s) : s < \hat{t}\}$, then (1.8) holds with $\partial\Omega$ replaced by F.

We have been unable to show as in [KT2] that the conclusion of Theorem 3 is valid without assumption (a). In fact the proof in [KT2] uses an important result of [AC] for elliptic pde, whose generalization to the heat equation is apparently an interesting open problem (see the remark at the end of section 5, for a discussion of this problem and other 'free boundary problems'). As for our proofs, if $X=(x_0,x)$, we note for $n>1$ that [LM], [HL] considered graph domains of the form $\tilde\Omega=\{(X,t)\in\mathbb{R}^{n+1}: x_0>\psi(x,t)\}$ where $\psi=\psi(x,t):\mathbb{R}^n\to\mathbb{R}$ had compact support and satisfied

$$|\psi(x,t)-\psi(y,s)|\leq b_1(|x-y|+|s-t|^{1/2})<\infty \text{ for all } (x,t),(y,s)\in\mathbb{R}^n, \tag{1.13}$$

$$\|D^t_{1/2}\psi\|_*\ \leq\ b_2<\infty. \tag{1.14}$$

Here $D^t_{1/2}\psi(x,t)$ denotes the 1/2 derivative in t of $\psi(x,\cdot), x$ fixed. This half derivative in time can be defined by way of the Fourier transform or by

$$D^t_{1/2}\psi(x,t)\equiv\hat c\int_{\mathbb{R}}\frac{\psi(x,s)-\psi(x,t)}{|s-t|^{3/2}}\,ds$$

for properly chosen $\hat c$. Also $\|\cdot\|_*$ denotes the bounded mean oscillation norm on $\mathbb{R}^n$ taken with respect to rectangles of side length r in the space direction and r^2 in the time direction (see the display following (2.9) for a definition). If $n=1$ the above authors considered graph domains Ω for which (1.13), (1.14) held with $\psi=\psi(t)$. In the proofs of Theorem 1-3 we make frequent use of the results of [HLN] on the existence of big and very big pieces of time-varying graphs on every scale. In particular concerning big pieces (Proposition 2.1 below), in [HLN] it is proved if Ω is parabolic regular and if δ_0 in (1.1) is sufficiently small, then $\partial\Omega$ contains big pieces of time varying graphs in the following sense. Given $(X,t)\in\partial\Omega$ and $R>0$, there exists after a possible rotation in the space variable, a function ψ as in (1.13), (1.14) and $\tilde\Omega$ as above such that for some $M_2=M_2(M,\|\nu\|_+,\delta_0)$

$$M_2\,\sigma(\partial\tilde\Omega\cap\Delta(X,t,R))\geq R^{n+1}. \tag{1.15}$$

In the proof of Theorem 2 and 3 we also make use of a refinement of this result (Proposition 2.2 below), which is a parabolic analogue of Semmes decomposition, [S], concerning very big pieces of time varying graph domains when $\|\nu\|_+$ is sufficiently small. Section 2 is devoted to the statement of these results but we also show (in case of a Lip(1,1/2) graph) how one can make use of the Carleson condition (1.7) to control the bound b_2 in (1.14). In section 3 we list some basic estimates for solutions to the heat - adjoint heat equation in certain parabolic NTA domains. From these estimates and Proposition 2.1, we then obtain Theorem 1. In sections 4 - 7 we follow the general outline of [JK], [J], [KT], [KT1] to get first Theorem 2 in section 4 and then Theorem 3 in sections 5-7. Proposition 2.2 will play a key role in our proofs. In the proof of Theorem 2 we shall need certain reverse Hölder type inequalities for the density of caloric measure in a time varying domain $\tilde\Omega$ (as above) when b_1,b_2 in (1.13), (1.14) are sufficiently small. In [JK], [KT], these inequalities are obtained from integration by parts. Although such an argument could be used in a time varying domain (see [HL]), it seems much easier for us to use estimates for solutions to the Neumann problem in $\tilde\Omega$ when the norms in (1.13), (1.14) are small. In section 5 we first define what is meant by Reifenberg flatness with vanishing constant for $\partial\Omega\cap C_r(X,t)$ when $(X,t)\in\partial\Omega$ and (1.1), (1.3) are valid. We then show that this property for domains satisfying (1.1), (1.3) is equivalent to (a) of Theorem 3. In Proposition 5.13 we show for parabolic regular domains $\subset\mathbb{R}^2$ that a weak form of asymptotic optimal doubling for $\omega(\hat X,\hat t,\cdot)$ ((1.12) with $\tau_1=\tau_2$) implies Reifenberg flatness with vanishing constant. We then discuss some open problems regarding these two properties at the end of section 5. Using Propositions 5.1, 2.2, singular integral estimates, and an iteration type argument we then obtain Theorem 3 in sections 6 and 7.

To avoid redundance we shall sometimes refer to arguments in [H], [KT], [KT1], [LM], [HLN] so the reader is advised to have these papers on hand. Finally we note that we could have proved local versions of Theorems 1-3. Also our theorems have exact analogues for the adjoint Green's function and adjoint caloric measure. The validity of both statements should be clear from our proofs.

2. Propositions on Big Pieces. In this section we state the results of [HLN] on the existence of big and very big pieces of time-varying graphs.

Given Ω as in (1.1) let $(X,t) \in \partial\Omega$ and let $\|\nu\|$ be the Carleson norm of ν on $\Delta(X,t,800R) \times (0,800R)$ (see (1.7)). The first proposition concerns the existence of big pieces.

Proposition 2.1. *If Ω satisfies (1.1), (1.3), and $\|\nu\| < \infty$, then there exists $c(M)$ such that if $0 < \delta_0 \leq c(M)^{-1}$ in (1.1), then $\partial\Omega$ contains big pieces of time varying graphs in the sense of (1.15) with $b_1 + b_2 \leq c(M)(1+\|\nu\|)^{1/2}$ in (1.13), (1.14). Moreover for $(X,t), R$ as in (1.15), we can make the construction so that either $\tilde{\Omega} \cap C_{2R}(X,t) \subset \Omega$ or $\Omega \cap C_{2R}(X,t) \subset \tilde{\Omega}$.*

The second proposition is a parabolic version of a theorem of Semmes for chord arc domains with small constants (see [S],[S1], [KT]).

Proposition 2.2. *Let $\Omega, \delta_0, (X,t)$, be as in Proposition 2.1 and suppose for some $\hat{\delta}, 0 < \hat{\delta} \leq \delta_0$, that $\|\nu\| = \delta^\kappa$ with $\delta \leq \hat{\delta}$ and $\kappa = 24(n+3)\max\{n-1,1\}$. If $\hat{\delta} = \hat{\delta}(M)$ is small enough, then $\Delta(X,t,R)$ contains very big pieces of time varying graphs with small constants in the following sense. There exists after a possible rotation in the space variable, $\psi, \tilde{\Omega}, c_2 = c_2(M)$ satisfying (1.13), (1.14) and*

(*a*) $\quad b_1 + b_2 \leq c_2\delta.$

(*b*) $\quad \sigma\left(\Delta(X,t,R) \setminus \partial\tilde{\Omega}\right) + \sigma\left(\partial\tilde{\Omega} \cap C_R(X,t) \setminus \partial\Omega\right) \leq e^{-1/(c_2\delta)} R^{n+1}.$

(*c*) $\quad \Delta(X,t,2R) = G \cup B$ *where* $G \subset \partial\tilde{\Omega}$ *and* $\sigma(B) \leq e^{-1/(c_2\delta)} R^{n+1}.$

(*d*) $\quad$ *If p denotes the orthogonal projection of $\mathbb{R}^{n+1}$ on $\{0\} \times \mathbb{R}^n, d(\cdot,\cdot)$ is as in (1.4), and $(Y,s) \in B$, then $|y_0 - \psi(y,s)| \leq c_2\,\delta\, d(\{p(Y,s)\}, p(G))$,*

(*e*) $\quad$ *We can make the construction so that either $\tilde{\Omega} \cap C_R(X,t) \subset \Omega$ or $\Omega \cap C_R(X,t) \subset \tilde{\Omega}$.*

The proofs of Proposition 2.1 and 2.2 can be found in [HLN]. These propositions are not a straight forward generalization of work in [KT], [S], since one must also estimate the parabolic BMO norm in time of an approximating graph function (i.e b_2 above). For the readers convenience we show how (in the case of a Lip(1,1/2) graph) one can make use of the Carleson measure condition to control the parabolic BMO-norm of the 1/2 time-derivative. I.e., let $\Gamma = \{(\psi(y,s),y,s) : (y,s) \in \mathbb{R}^n\}$ and let γ, ν be the measures in (1.5)-(1.7) corresponding to Γ. Assume furthermore that

$$|\psi(y,s) - \psi(z,\tau)| \leq cM(|y-z| + |s-\tau|^{1/2}) \tag{2.3}$$

and that $\|\nu\|_+ < \infty$. Note that (2.3) implies that

$$d\sigma(Y,s) = \sqrt{1 + |\nabla\psi(y,s)|^2}\, dyds \leq 2M dH^n(y,s) \text{ for } (Y,s) \in \Gamma. \tag{2.4}$$

Let for $r > 0, (\hat{x},\hat{t}) \in \mathbb{R}^n$

$$\kappa(\hat{x},\hat{t},r) \;=\; r^{-(n+3)} \inf_L \int_{C_r(\hat{x},\hat{t})} |\psi(y,s) - L(y)|^2 \, dyds$$

where the infimum is over all linear functions of y (only) and note that

$$(c(M))^{-1}\, \kappa(\hat{x},\hat{t},r) \;\le\; \gamma(\hat{X},\hat{t},r) \;\le\; c(M)\kappa(\hat{x},\hat{t},r) \tag{2.5}$$

as we see from (2.4), (2.3), and an elementary geometric argument.

Before proceeding we need to introduce some more notation. Let ϕ be an infinitely differentiable real valued function on $\mathbb{R}^n$ with compact support in $C_1(0,0)$ (i.e, $\phi \in C_0^\infty(C_1(0,0))$) and set

$$\phi_\lambda(z,\tau) \;\equiv\; \lambda^{-(n+1)}\phi(z/\lambda, \tau/\lambda^2).$$

$$\|\phi\|_\infty \;=\; \max\{|\phi(z,\tau)| : (z,\tau) \in \mathbb{R}^n\}.$$

Next for $\theta : \mathbb{R}^n \to \mathbb{R}$, define the convolution of ϕ with θ by

$$\phi * \theta(z,\tau) = \int_{\mathbb{R}^n} \phi(z-y, \tau-s)\theta(y,s)\, dyds$$

whenever this convolution makes sense. For short we write $\phi_\lambda \theta$ for $\phi_\lambda * \theta$. Suppose now that ϕ as above annihilates constants and the coordinate functions $y_i, 0 \le i \le n-1$. That is,

$$\int_{\mathbb{R}^n} \phi(y,s)dyds = 0 \;\text{ and }\; \int_{\mathbb{R}^n} y_i\phi(y,s)dyds = 0.$$

Observe for each linear function L of y only that

$$(\phi_\lambda * \psi)^2(\hat{z},\hat{\tau}) = [\phi_\lambda * (\psi - L)]^2(\hat{z},\hat{\tau}) \;\le\; c\,\|\phi\|_\infty\, \lambda^{-(n+1)} \int_{C_\lambda(\hat{z},\hat{\tau})} |\psi - L|^2 dyds.$$

From this observation, the definition of κ, (2.5), and our assumption on the Carleson measure we see that

$$\begin{aligned} &\int_0^\rho \int_{C_\rho(z,\tau)} \lambda^{-3}\, (\phi_\lambda * \psi)^2\,(y,s)dydsd\lambda \\ &\quad\le c\|\phi\|_\infty \int_0^\rho \int_{C_\rho(z,\tau)} \lambda^{-1}\kappa(y,s,\lambda)dydsd\lambda \;\le\; c(M)\|\phi\|_\infty \delta^2 \rho^{n+1}. \end{aligned} \tag{2.6}$$

Let $P \in C_0^\infty(C_1(0,0))$ be an even non negative function on $\mathbb{R}^n$ with $\int_{\mathbb{R}^n} P(w)\, dw \;\equiv\; 1$. We note that if $\phi = \phi(z,\tau)$ is any of the functions,

$$\lambda\frac{\partial P_\lambda}{\partial \lambda},\, \lambda^2\frac{\partial P_\lambda}{\partial \tau},\, \lambda^2\frac{\partial^2 P_\lambda}{\partial z_i^2}, 1 \le i \le n-1, \tag{2.7}$$

then ϕ annihilates linear functions of y and $\|\phi\|_\infty \le c$ as is easily seen. From (2.6), (2.7), we conclude

$$\begin{aligned} &\int_0^\rho \int_{C_\rho(z,\tau)} \left(\lambda^{-1}\,[\tfrac{\partial(P_\lambda\psi)}{\partial\lambda}]^2 \,+\, \lambda\,[\tfrac{\partial(P_\lambda\psi)}{\partial\tau}]^2 + \lambda[\tfrac{\partial^2(P_\lambda\psi)}{\partial z_i^2}]^2\right) dydsd\lambda \\ &\quad\le c\delta^2\, \rho^{n+1} \text{ for } 1 \le i \le n-1. \end{aligned} \tag{2.8}$$

We will now use (2.8) to show that

$$\|D^t_{1/2}\psi\|_* \le c(M)(1+\delta^2)^{1/2}. \tag{2.9}$$

Here $\|\cdot\|_*$ denotes the norm in the space BMO ($\mathbb{R}^n$) defined in the following way. Given $g:\mathbb{R}^n\to\mathbb{R}$, locally integrable with respect to Lebesgue n measure, let $C=C_r(z,\tau)$ and let $|C|$ be the Lebesgue n measure of C. Set

$$g_C = |C|^{-1}\int_C g\,dyds.$$

Then $g\in$BMO($\mathbb{R}^n$) with norm $\|g\|_*$ if and only if

$$\|g\|_* = \sup_C\,\{|C|^{-1}\int_C |g-g_C|\,dyds\,\} < \infty.$$

To prove (2.9), let $\beta\in C_0^\infty(C_{2\rho}(z,\tau)), 0\le\beta\le 1$, with $\beta\equiv 1$ on $C_{3\rho/2}(z,\tau)$ and $\rho^{-l}|\frac{\partial^l}{\partial y^l}\beta| + \rho^{-2l}\,|\frac{\partial^l}{\partial s^l}\beta| \le c(l,n)$ at each point of $\mathbb{R}^n$. If $C=C_\rho(z,\tau)$, we put

$$\psi = [(\psi-\psi_C)\beta] + [\beta\,\psi_C + (1-\beta)\psi] = \psi_1+\psi_2.$$

We first consider ψ_1. Given $\epsilon, 0<\epsilon<\rho/100$, we note that

$$\begin{aligned}
&\int_{\mathbb{R}^n}(D^t_{1/2}P_\epsilon\psi_1)^2dyds\\
&\quad= \int_{\mathbb{R}^n}(D^t_{1/2}P_\rho\psi_1)^2dyds - \int_\epsilon^\rho\int_{\mathbb{R}^n}2D^t_{1/2}(\tfrac{\partial}{\partial\lambda}P_\lambda\psi_1)D^t_{1/2}(P_\lambda\psi_1)dydsd\lambda\\
&\quad= \int_{\mathbb{R}^n}(D^t_{1/2}P_\rho\psi_1)^2dyds + a\int_\epsilon^\rho\int_{\mathbb{R}^n}\mathcal{H}[\tfrac{\partial}{\partial\lambda}(P_\lambda\psi_1)]\;\tfrac{\partial}{\partial t}(P_\lambda\psi_1)dydsd\lambda\\
&\qquad\qquad = V_1+V_2.
\end{aligned}\tag{2.10}$$

In (2.10), a is a constant and $\mathcal{H}$ is the Hilbert transform on $\mathbb{R}$ defined by

$$\mathcal{H}k(s) = \text{ PV }\int_{\mathbb{R}} k(\hat\tau)(s-\hat\tau)^{-1}\,ds$$

for k a real valued function. In the above display we have $k=\frac{\partial}{\partial\lambda}(P_\lambda\psi_1)(y,\cdot)$. From (2.3) we get $|\partial(P_\rho\psi_1)/\partial s|\le c(M)\rho^{-1}$ at each point in $C_{8\rho}(z,\tau)$ and this function vanishes elsewhere in $\mathbb{R}^n$. From this inequality we find for $|y-z|\le 4\rho$ that

$$|D^t_{1/2}P_\rho\psi_1|(y,s) = |c\int_{\mathbb{R}}\frac{P_\rho\psi_1(y,s)-P_\rho\psi_1(y,\hat t)}{|s-\hat t|^{3/2}}d\hat t| \le c(M)\,\min\{1,\frac{\rho^3}{|s-\tau|^{3/2}}\}.$$

If $|y-z|>4\rho$, then $D^t_{1/2}P_\rho\psi_1(y,s)\equiv 0$. Using these inequalities and integrating over $\mathbb{R}^n$, we conclude that

$$|V_1|\le c(M)\rho^{n+1}. \tag{2.11}$$

To handle V_2 we note that $\mathcal{H}$ is a bounded operator with norm $\leq c$ from $L^2(I\!\!R)$ to $L^2(I\!\!R)$ - the usual space of Lebesgue square integrable functions on $I\!\!R$. Using this note and Hölder's inequality we get from (2.9),

$$\begin{aligned} |V_2|^2 \;\leq\; c\left(\int_0^\rho \int_{I\!\!R^n} \lambda^{-1}[\tfrac{\partial}{\partial\lambda}(P_\lambda\psi_1)]^2\, dyds\right)\cdot\left(\int_0^\rho \int_{I\!\!R^n} \lambda\,[\tfrac{\partial}{\partial t}(P_\lambda\psi_1))]^2\, dyds\right) \\ = V_{21}\cdot V_{22}. \end{aligned} \tag{2.12}$$

To estimate V_{22} we observe for $(y,s)\in C_{4\rho}(z,\tau), 0<\lambda\leq\rho$, and

$$L(\hat{x}) = \beta(y,s) + \sum_{i=1}^{n-1}\frac{\partial}{\partial y_i}\beta(y,s)(y_i-\hat{x}_i)$$

that

$$\begin{aligned} |\frac{\partial}{\partial s}(P_\lambda\psi_1)|(y,s) \;\leq\; &|\int_{I\!\!R^n}\tfrac{\partial}{\partial t}(P_\lambda)(y-\hat{x},s-\hat{t})\,(\psi(\hat{x},\hat{t})-\psi_C)(\beta(\hat{x},\hat{t})-L(\hat{x}))\,d\hat{x}d\hat{t}\,| \\ &+\sum_{i=1}^{n-1}|\tfrac{\partial}{\partial y_i}\beta(y,s)\int_{I\!\!R}(y_i-\hat{x}_i)\tfrac{\partial}{\partial t}(P_\lambda)(y-\hat{x},s-\hat{t})\,(\psi(\hat{x},\hat{t})-\psi(y,s))d\hat{x}d\hat{t}| \\ &+|\beta(y,s)\,\tfrac{\partial}{\partial s}P_\lambda\psi(y,s)| \;\leq\; c(M)/\rho + c\,|\tfrac{\partial}{\partial s}P_\lambda\psi|(y,s) \end{aligned}$$

where we have used (2.3), (2.7), and Taylor's theorem to estimate the integrals. If $(y,s)\in I\!\!R^n\setminus C_{4\rho}(z,\tau)$, then $\frac{\partial}{\partial s}(P_\lambda\psi_1)(y,s)=0$. Similarly, for $(y,s)\in C_{4\rho}(z,\tau), 0<\lambda\leq\rho$,

$$|\tfrac{\partial}{\partial\lambda}(P_\lambda\psi_1)|(y,s)\;\leq\; c(M)\lambda/\rho + c\,|\tfrac{\partial}{\partial\lambda}P_\lambda\psi|(y,s)$$

and this function vanishes elsewhere in $I\!\!R^n$. Putting these inequalities into (2.12) and using (2.8) we obtain first that

$$|V_2|^2\;\leq\; c(M)(1+\delta^2)^2\rho^{2n+2}$$

and second in view of (2.10), (2.11) that

$$\int_{I\!\!R^n}(D^t_{1/2}P_\epsilon\psi_1)^2dyds\;\leq\; c(M)(1+\delta^2)\rho^{n+1}.$$

Using this inequality, the inequalities directly above (2.11), and letting $\epsilon\to 0$, we deduce that $D^t_{1/2}P_\epsilon\psi_1$ converges weakly in $L^2(I\!\!R^n)$ to a function ζ satisfying the same inequalities as $D^t_{1/2}P_\epsilon\psi_1$. Using weak convergence, we see that $\psi_1(y,\cdot)=cI_{1/2}*\zeta(y,\cdot)$ where $I_{1/2}(s)=|s|^{-1/2}$. This equality implies that $\zeta=D^t_{1/2}\psi_1$ exists and satisfies

$$\int_{I\!\!R^n}(D^t_{1/2}\psi_1)^2dyds\;\leq\; c(M)(1+\delta^2)\,\rho^{n+1}. \tag{2.13}$$

We now consider integrals involving ψ_2. Observe that ψ_2 is constant on $C_{3\rho/2}(z,\tau)$. Thus for $(y,s)\in C$,

$$D^t_{1/2}\psi_2(y,s)\;=\;c\int_{\{\hat{t}:|\hat{t}-\tau|\geq 9\rho^2/4\}}\frac{\psi_2(y,\hat{t})-\psi_2(y,s)}{|s-\hat{t}|^{3/2}}d\hat{t}.$$

Using this equality, $\psi_2 = (\psi_C - \psi)\beta + \psi$, and (2.3) we deduce from estimates similar to the above that if

$$\alpha = c \int_{\{\hat t : |\hat t - \tau| \geq 9\rho^2/4\}} \frac{\psi(z,\hat t) - \psi(z,\tau)}{|\tau - \hat t|^{3/2}} d\hat t,$$

then $|D^t_{1/2}\psi_2 - \alpha| \leq c(M)$ on C. Thus,

$$\int_C |D^t_{1/2}\psi_2 - \alpha| dyds \;\leq\; c(M)\rho^{n+1}. \tag{2.14}$$

From (2.14) and (2.13) we see that $D^t_{1/2}\psi$ exists and is integrable on C. Moreover, since

$$|\alpha - (D^t_{1/2}\psi)_C| \leq \; |C|^{-1} \int |\alpha - D^t_{1/2}\psi(y,s)| dyds$$

it follows from (2.13), (2.14) and Hölders inequality that

$$\begin{aligned} \int_C |D^t_{1/2}\psi - (D^t_{1/2}\psi)_C| dyds \;&\leq\; 2\int_C |D^t_{1/2}\psi_2 - \alpha| dyds + 2\int_C |D^t_{1/2}\psi_1| dyds \\ &\leq\; c(M)\rho^{n+1} \;+\; c\rho^{(n+1)/2} \left(\int_{\mathbb{R}^n} |D^t_{1/2}\psi_1|^2 dyds \right)^{1/2} \\ &\leq\; c(M)(1+\delta^2)^{1/2}\,\rho^{n+1}. \end{aligned} \tag{2.15}$$

Hence $D^t_{1/2}\psi \in \mathrm{BMO}(\mathbb{R}^n)$ with $\|D^t_{1/2}\psi\|_* \;\leq\; c(M)(1+\delta^2)^{1/2}$. This completes the argument.

3. Basic Estimates and Proof of Theorem 1. In this section we state some basic estimates for certain solutions to the heat and adjoint heat equation in parabolic NTA domains. Lemmas 3.3, 3.6, 3.10 are well known for Lip (1, 1/2) domains and generalize easily to our situation. An outline of the proofs of these lemmas valid in the current situation can be found in [LM, ch 3, section 6]. To begin, recall from [LM, ch 3, section 6] that $D \subset \mathbb{R}^{n+1}$ is an unbounded parabolic nontangentially accessible domain if the conditions in (3.1), (3.2) are satisfied for some $\lambda, \gamma \geq 100$. More specifically, given $(X^*, t^*) \in \partial D$ and $r > 0$, suppose there exists $(U_i^*, t_i^*) \in D \cap C_r(X^*, t^*)$, $(N_i^*, \tau_i^*) \in (\mathbb{R}^{n+1} \setminus \bar D) \cap C_r(X^*, t^*)$, $i = 1, 2$, for which

$$\begin{aligned} &(i) && \lambda^{-1} r^2 \leq \min\,(\, t_2^* - t^*,\; t^* - t_1^*\,) \leq\; \lambda r^2, \\ &(ii) && \lambda^{-1} r^2 \leq \min\,(\, \tau_2^* - t^*,\; t^* - \tau_1^*\,) \;\leq\; \lambda r^2, \\ &(iii) && r/\lambda \leq \min\,[\, d(\{(N_i^*, \tau_i^*)\},\, \partial D), d(\{(U_i^*, t_i^*)\},\, \partial D)\,]\;. \end{aligned} \tag{3.1}$$

Here $U_i^*, N_i^*, t_i^*, \tau_i^*$, depend on r, X^*, t^*, when $i = 1, 2$ and λ is independent of these quantities. We write $U_i^* = U_i^*(X^*, t^*, r)$ when we wish to indicate the dependence of U_i^* on these quantities. A similar statement holds for N_i^*. As in [JK1] we refer to (3.1) as the <u>corkscrew condition.</u> Next suppose $(U_i, s_i) \in D$, $i = 1, 2$, with $(s_2 - s_1)^{\frac{1}{2}} \;>\; \gamma^{-1}\, d(\{(U_1, s_1)\},\, \{(U_2, s_2)\})$. We say as in [JK1] that $\{C_{r_i}(X_i,t_i)\}_1^l$ is a Harnack chain from (U_1,s_1) to (U_2,s_2) with constant γ provided there exists $c(\gamma)\geq 1$ such that

$$
\begin{aligned}
&(A)\ \ (U_1,s_1)\in C_{r_1}(X_1,t_1),\ \ (U_2,s_2)\in C_{r_l}(X_l,t_l),\ \ \text{and}\\
&C_{r_{i+1}}(X_{i+1},t_{i+1})\cap C_{r_i}(X_i,t_i)\neq\emptyset \text{ for } i=1,2,\dots,l-1,\\
&(B)\ \ c(\gamma)^{-1}\, d(\{(X_i,t_i)\},\,\partial D)\ \leq\ r_i\ \leq\ c(\gamma)\, d(\{(X_i,t_i)\},\,\partial D),\\
&\quad \text{when}\ \ i=1,2,\dots,l,\\
&(C)\ \ t_{i+1}-t_i\ \geq\ c(\gamma)^{-1}\, r_i^2,\ \ \text{for } i=1,2,\dots,l.\\
&(D)\ \ l\leq c(\gamma)\log\left(2+\tfrac{d(\{(U_1,s_1)\},\{(U_2,s_2)\})}{\min[d(\{(U_1,s_1)\},\partial D),d(\{(U_2,s_2)\},\partial D)]}\right).
\end{aligned}
\tag{3.2}
$$

l is referred to as the length of the above Harnack chain. We now state our first lemma.

Lemma 3.3. *Let Ω be as in (1.1). If $\delta_0>0$ is sufficiently small, then Ω is a parabolic NTA domain for $\lambda=100$ and any $\gamma\geq 100$.*

Proof: Clearly (1.1) implies (3.1) for $\delta_0>0$ sufficiently small with $\lambda=100$, and

$$
\begin{aligned}
&(U_1^*,t_1^*)=(X^*+r\hat{n},t^*-r^2),\quad (U_2^*,t_2^*)=(X^*+r\hat{n},t^*+r^2),\\
&(N_1^*,\tau_1^*)=(X^*-r\hat{n},t^*-r^2),\quad (N_2^*,\tau_2^*)=(X^*-r\hat{n},t^*+r^2),
\end{aligned}
\tag{3.4}
$$

where $\hat{n}=\hat{n}(X^*,t^*,r)$. To prove (3.2) for any $\gamma\geq 100$, we follow [KT1] and for given $(U_1,s_1),(U_2,s_2)\in\Omega$, as in (3.2), choose points $(P_1^*,t_1^*),(P_2^*,t_2^*)\in\partial\Omega$ with

$$\rho_i=d(\{(P_i^*,t_i^*)\},\{(U_i,t_i)\})\ =d(\{(U_i,t_i)\},\partial\Omega),\ \ i=1,2.$$

If $\rho=d(\{(U_1,t_1)\},\{(U_2,t_2)\})$, then using (1.1) for $\delta_0>0$ small, we can choose a Harnack chain of length

$$l\ \leq\ c(\gamma)\ \log\left(2\ +\ \frac{\rho}{\min\{\rho_1,\rho_2\}}\right) \tag{3.5}$$

joining (U_1,s_1) to (U_2,s_2). If for example $\rho>1000\max(\rho_1,\rho_2)$ and l_0 is the least positive integer greater than $\log(\rho/\rho_1)$, then from (1.1) we can choose $\{(X_i,t_i)\}$ as in (3.2) with $(X_i,t_i)\in C_{e^i\rho_1}(P_1^*,t_1^*)$ for $2\leq i\leq l_0$ and then $(X_i,t_i)\in C_{e^{l_0-i}\rho}(P_2^*,t_2^*)$ for $l_0+1\leq i\leq l_0+l_1$, where $l_1\leq\log(2\rho/\rho_2)$.□

Next we state

Lemma 3.6. *Let Ω,δ_0 be as in (1.1) with $(X^*,t^*)\in\partial\Omega$ and suppose that u is a solution to either the heat or adjoint heat equation in $\Omega\cap C_{2r}(X^*,t^*)$ which vanishes continuously on $\partial\Omega\cap C_{2r}(X^*,t^*)$. If $\delta_0>0$ is small enough, there exists $\alpha=\alpha(\delta_0), 0<\alpha<\frac{1}{2}$, and $c=c(\delta_0)\ \geq\ 1$ such that*

$$u(Y,s)\ \leq\ c\ \left[\tfrac{d(\{(Y,s)\},\{(X^*,t^*)\})}{r}\right]^{\alpha}\ \sup_{(Z,\tau)\in\Omega\cap C_r(X^*,t^*)}\ \max\{u,0\}(Z,\tau),$$

whenever $(Y,s) \in \Omega \cap C_r(X^*,t^*)$.

We shall also need

Lemma 3.7. *Let $u \geq 0, \Omega, (X^*,t^*), \delta_0$ be as in Lemma 3.6. If $(Y,s) \in \Omega \cap C_{r/2}(X^*,t^*)$, then*

$$u(Y,s) \leq c\, u(\hat{U},\hat{t})$$

where $\hat{U} = U_2^*(X^*,t^*,r)$, $\hat{t} = t_2^*(X^*,t^*,r)$ when u is a solution to the heat equation while $\hat{U} = U_1^*(X^*,t^*,r)$, $\hat{t} = t_1^*(X^*,t^*,r)$ when u is a solution to the adjoint heat equation in $C_{2r}(X^*,t^*) \cap \Omega$.

Given $(Y,s) \in \Omega$, let $G(\cdot,Y,s)$ denote Green's function for the heat equation in Ω with pole at (Y,s). That is

$$\tfrac{\partial}{\partial t} G(X,t,Y,s) - \Delta\, G(X,t,Y,s) = \delta((X,t)-(Y,s)) \text{ in } \Omega \text{ and } G \equiv 0 \text{ on } \partial\Omega.$$

Here δ denotes the Dirac delta function and Δ is the Laplacian in X. Existence of G follows from Lemma 3.6 and a Perron - Wiener - Brelot type argument. We note that $G(Y,s,\cdot)$ is the Green's function for the adjoint heat equation with pole at $(Y,s) \in \Omega$ (i.e $-\frac{\partial}{\partial t}G(Y,s,\cdot) - \Delta G(Y,s,\cdot) = \delta(\cdot - (Y,s))$. Let $\omega, \hat{\omega}$ be the corresponding caloric and adjoint caloric measures for the heat - adjoint heat equation in Ω. We note that $\omega(Y,s,\cdot), \hat{\omega}(Y,s,\cdot)$ are the Riesz measures associated with $G(\cdot,Y,s), G(Y,s,\cdot)$ by way of the Riesz representation theorem for sub caloric - adjoint caloric functions in $\mathbb{R}^{n+1} \setminus \{(Y,s)\}$ (see [Do]). From this theorem one gets for example

$$\int_{\partial\Omega} \phi \, d\omega(Y,s,\cdot) = \int_{\Omega} G(Y,s,\cdot)(\Delta\phi - \tfrac{\partial\phi}{\partial s})\, dZ\, d\tau \tag{3.8}$$

when $\phi \in C_0^\infty(\mathbb{R}^{n+1} \setminus \{(Y,s)\})$. A similar formula holds for $\hat{\omega}$. We use Lemma 3.6 with r replaced by $\frac{r}{A}$, $u = 1-\omega$ or $1-\hat{\omega}$, as well as Harnack's inequality for the heat-adjoint heat equation, and the Harnack chain condition (3.5), to find for some $c(A) \geq 1$ that

$$\omega^*(Y,s,\Delta(X^*,t^*,r)) \geq c(A)^{-1} \tag{3.9}$$

where $\omega^* = \omega$ when either $(Y,s) \in C_{\frac{r}{2A}}(X^*,t^*) \cap \Omega$ or $(Y,s) \in \Omega$ with $s - t^* \geq -\frac{r^2}{4A^2}$ and

$$A^{-1} r \leq d(\{(Y,s)\}, \partial\Omega) \leq d(\{(Y,s)\},\{(X^*,t^*)\}) \leq A\, r.$$

Also $\omega^* = \hat{\omega}$ when either $(Y,s) \in C_{\frac{r}{2A}}(X^*,t^*) \cap \Omega$ or $(Y,s) \in \Omega$ with $t^* - s \geq -\frac{r^2}{4A^2}$ and

$$A^{-1} r \leq d(\{(Y,s)\}, \partial\Omega) \leq d(\{(Y,s)\},\{(X^*,t^*)\}) \leq A\, r.$$

Estimates for caloric - adjoint caloric measure in terms of the Green's function and vice versa are given by the following lemma.

Lemma 3.10. *Let $\Omega, \delta_0, (X^*,t^*)$ be as in Lemma 3.6. Suppose for some $A \geq 100$ that $(Y,s) \in \Omega$ with $|Y - X^*|^2 \leq A|s - t^*|$ and $|s - t^*| \geq 4r^2$. There exists $c = c(A) \geq 1$ such that if $s > t^*$, then*

$$c^{-1}\, r^n\, G(Y,s,U_2^*,t_2^*) \leq \omega(Y,s,\, \Delta(X^*,t^*,r/2)) \leq c\, r^n\, G(Y,s,U_1^*,t_1^*)$$

while if $s < t^*$,

$$c^{-1}\, r^n\, G(U_1^*, t_1^*, Y, s) \;\leq\; \hat{\omega}(Y, s,\, \Delta(X^*, t^*, r/2)) \,\leq c\, r^n\, G(U_2^*, t_2^*, Y, s).$$

Here $(U_i^*, t_i^*), i = 1, 2,$ *are as in (3.4).*

We note that the left hand inequality in Lemma 3.10 follows from (3.9) while the right hand inequality for ω is proved using (3.8). Next we have the following backward Harnack inequality.

Lemma 3.11. *Let* $\Omega, \delta_0, (X^*, t^*)$ *be as in Lemma 3.6. Suppose for some* $A \geq 100$ *that* $(Y, s) \in \Omega$ *with* $|Y - X^*|^2 \;\leq\; A|s - t^*|$ *and* $|s - t^*| \geq 5r^2$. *There exists* $c = c(A) \geq 1$ *such that*

$$r^n\, G(Y, s, U_1^*, t_1^*) \leq c r^n\, G(Y, s, U_2^*, t_2^*)$$

when $s > t^*$ *while if* $s < t^*$, *then*

$$r^n\, G(U_1^*, t_1^*, Y, s) \leq c r^n\, G(U_2^*, t_2^*, Y, s)$$

Here $(U_i^*, t_i^*), i = 1, 2,$ *are as in (3.4).*

Proof: A proof of Lemma 3.11 when Ω is a cylinder domain and for more general parabolic partial differential equations can be given as in [FS, FSY] (see also [HL, ch 1, Lemma 3.14] and [N]). For the reader's convenience and because Lemma 3.11 is not so well known we briefly indicate how the proof generalizes to our situation when $s > t^*$. Put $v(Z, \tau) = G(Y, s, Z, \tau)$, for $(Z, \tau) \in \Omega \setminus \{(Y, s)\}$ and $v \equiv 0$ on $\mathbb{R}^{n+1} \setminus \bar{\Omega}$. Note from Lemma 3.3 that $v \geq 0$ is a continuous subsolution to the adjoint heat equation on $\mathbb{R}^{n+1} \setminus \{(Y, s)\}$. Given $r_1, r_2 > 0$ put

$$C_{r_1, r_2}(X^*, t^*) \;=\; \{((Z, \tau) \in \Omega : |Z - X^*| < r_1,\; |\tau - t^*| < r_2\}.$$

Let $100 \;\leq\; k \;\leq\; (100\, \delta_0)^{-1}$, $(Z, \tau) \;\in\; C_{k\rho, 4\rho^2}(X^*, t^*)$, and $(Y, s) \;\notin\; C_{k\rho, 5\rho^2}(X^*, t^*)$. From Harnack's inequality for the adjoint heat equation (see [E, ch 7]), (3.5) and Lemma 3.7 we see for some $\beta,\, c \geq 1$, depending only on n that

$$e^{-ck^2}\; [(d(\{(Z, \tau)\}, \partial\Omega)/\, \rho]^\beta\; v(\tilde{U}_2', \tilde{t}_2') \leq v(Z, \tau) \;\leq\; c\, e^{ck^2}\; v(\bar{U}_1, \bar{t}_1) \tag{3.12}$$

where (U_2', t_2') is as in (3.4) with $r, *,$ replaced by $2\rho,\, '$, and

$$(\bar{U}_1, \bar{t}_1) = (X^* + k\rho\hat{n}(X^*, t^*, k\rho), t^* - 5\rho^2).$$

Next let

$$S = \{(Z, \tau) \in \partial C_{k\rho, 4\rho^2}(X^*, t^*) : |Z - X^*| \,=\, k\rho\}$$

Set

$$M_1 \;=\; \max\{v(Z, \tau) : (Z, \tau) \in \bar{C}_\rho(X^*, t^*)\,\}$$

$$M_2 \;=\; \max\{\, v(Z, \tau) : (Z, \tau) \in S\,\}.$$

We first show that if

$$M_2 \;\leq\; k^\beta\, M_1, \tag{3.13}$$

then for $k \geq 100$ large enough (i.e $\delta_0 > 0$ small enough) we have

$$M_3 \;=\; \max\{\, v(Z, t^* + 4\rho^2) : (Z, t^* + 4\rho^2) \in \partial C_{k\rho, 4\rho^2}(X^*, t^*)\} \;\geq\; \tfrac{1}{2}\, M_1. \tag{3.14}$$

To prove (3.13) one uses estimates for the least caloric majorant of v in a certain rectangle to show as in [FS] that

$$v(Z, \tau) \;\leq\; M_3 \;+\; e^{-k/c}\, M_2 \tag{3.15}$$

when $(Z, \tau) \in C_{k\rho/2, 4\rho^2}(X^*, t^*)$. Clearly (3.15) and (3.13) imply (3.14) for k large enough.

Fix k subject to the above requirement and put

$$\tilde{\rho} = \max[2r, A^{-2}\, d(\{(Y, s)\}, \{(X^*, t^*)\})]$$

$$M(\rho) = \max_{\bar{C}_\rho(X^*, t^*)} v$$

$$\sigma \;=\; \max_{\{2r \leq \rho^* \leq \tilde{\rho}\}} (\rho^*)^{-\beta}\, M(\rho^*).$$

Choose ρ, $2r \leq \rho \leq \tilde{\rho}$, such that $M(\rho) \;=\; \rho^\beta\, \sigma$. We consider two cases. First if $\rho \geq \tilde{\rho}/2k$ we note that for some $c = c(A) \geq 1$,

$$c^{-1} \tilde{\rho}^\beta\, \sigma \;\leq v(U_i^+, t_i^+) \;\leq\; c v(U_j^+, t_j^+)$$

whenever $i, j = 1, 2$ where the above points are as in (3.4) with $r, *$ replaced by $2\tilde{\rho}, +$. This inequality follows from Lemma 3.7 and the maximum principle for the adjoint heat equation. Using the above inequality and (3.12) we find for some $K = K(A, k) \geq 2$ that

$$\begin{aligned} K^{-1} \hat{\rho}^{-\beta}\, v(U_2^+, t_2^+) \;&\leq\; r^{-\beta}\, v(U_2^*, t_2^*) \;\leq\; c\, r^{-\beta} v(U_1^*, t_1^*) \\ &\leq\; c\,\sigma \;\leq K\, \hat{\rho}^{-\beta} v(U_2^+, t_2^+). \end{aligned} \tag{3.16}$$

Otherwise $\rho \leq \tilde{\rho}/(2k)$ and we have

$$M(k\rho) \;\leq\; k^\beta\, M(\rho),$$

so (3.13) is satisfied and (3.14) can be used to conclude for some $(Z, t^* + 4\rho^2) \in \partial C_{k\rho, 4\rho^2}(X^*, t^*)$ that

$$M(\rho) \;\leq\; 2\, v(Z, t^* + 4\rho^2).$$

From this inequality, (3.12), and Lemma 3.7 it follows that for some $K = K(A, k) \geq 2$

$$\begin{aligned} K^{-1} M(\rho) &\leq 2K^{-1} v(Z, t^* + 4\rho^2) \\ &\leq\; (\rho/r)^\beta\, v(U_2^*, t_2^*) \;\leq\; c(\rho/r)^\beta\, v(U_1^*, t_1^*) \\ &\leq\; K\, M(\rho). \end{aligned}$$

Thus in either case

$$v(U_1^*, t_1^*) \;\leq\; c(A)\, v(U_2^*, t_2^*)$$

which is Lemma 3.11. □

Using Lemmas 3.11, 3.10, and Harnack's inequality for the heat - adjoint heat equation, we get the following doubling property of caloric - adjoint caloric measure.

Lemma 3.17. *Let $\Omega, \delta_0, (X^*, t^*), (Y, s), A$ be as in Lemma 3.10. Then*

$$\omega^*(Y, s, \Delta(X^*, t^*, r)) \, \leq \, c(A)\omega^*(Y, s, \Delta(X^*, t^*, r/2))$$

where $\omega^* = \omega$ when $s > t^*$ while $\omega^* = \hat{\omega}$ when $s < t^*$.

Let $\Omega, \delta_0, (X^*, t^*)$ be as in Lemma 3.6 and $\rho > 0$. We shall say that $w > 0$ satisfies a strong Harnack inequality in $C_\rho(X^*, t^*) \cap \Omega$ provided that w is a solution to either the heat or adjoint heat equation in $C_\rho(X^*, t^*) \cap \Omega$ and

$$w(X, t) \leq \lambda w(\tilde{X}, \tilde{t}) \text{ whenever } (X, t), (\tilde{X}, \tilde{t}) \in C_r(Z, \tau) \text{ and } C_{2r}(Z, \tau) \subset C_\rho(X^*, t^*) \cap \Omega.$$

Here $\lambda, 1 \leq \lambda < \infty$, is independent of $C_{2r}(Z, \tau) \subset C_\rho(X^*, t^*) \cap \Omega$. From Lemmas 3.11, 3.10, and 3.7 we see that if $|s - t^*| \geq 5\rho^2$ and $|Y - X^*|^2 \leq A|s - t^*|$, then $G(Y, s, \cdot)$ satisfies a strong Harnack inequality in $C_\rho(X^*, t^*) \cap \Omega$ provided $s > t^*$ while if $s < t^*$, then $G(\cdot, Y, s)$ satisfies a strong Harnack inequality in $C_\rho(X^*, t^*) \cap \Omega$. Moreover, λ depends only on A once δ_0 in (1.1) is chosen sufficiently small. We shall use this note and the following lemmas throughout the proof of Theorems 2 and 3.

Lemma 3.18. *Let $u, v > 0$ be continuous in $\bar{C}_{2r}(X^*, t^*) \cap \bar{\Omega}$ with $u = v = 0$ on $\Delta(X^*, t^*, 2r)$ and solutions to either the heat or adjoint heat equation in $C_{2r}(X^*, t^*) \cap \Omega$ where $\Omega, \delta_0, (X^*, t^*)$ are as in Lemma 3.6. If u, v satisfy a strong Harnack inequality in $C_{2r}(X^*, t^*) \cap \Omega$ for some $\lambda \geq 1$, then*

$$\frac{u(Y, s)}{v(Y, s)} \, \leq \, c(\lambda) \, \frac{u(\hat{U}, \hat{t})}{v(\hat{U}, \hat{t})} \text{ in } C_{r/2}(X^*, t^*) \cap \bar{\Omega}$$

where $(\hat{U}, \hat{t}) = (U_2^*, t_2^*)$ (as defined in (3.4)) when u, v are solutions to the heat equation while $(\hat{U}, \hat{t}) = (U_1^*, t_1^*)$ when u, v are solutions to the adjoint heat equation in $\Omega \cap C_{2r}(X^*, t^*)$.

Lemma 3.18 follows from Lemmas 3.17, 3.11, 3.10 and the same argument as in [LM, ch 3, Lemma 6.5]. We omit the details. From Lemma 3.18 and an argument as in [JK1] (see also [FS, Theorem 2.2]), we get

Lemma 3.19. *Under the same hypotheses as in Lemma 3.18, there exists $\gamma = \gamma(\lambda)$, $0 < \gamma \leq 1/2$, and $c = c(\lambda) \geq 1$, such that*

$$\left| \frac{u(Z, \tau) \; v(Y, s)}{v(Z, \tau) \; u(Y, s)} \, - \, 1 \right| \, \leq \, c \, (\rho/r)^\gamma \text{ for } (Z, \tau), (Y, s) \in C_\rho(X^*, t^*) \cap \Omega$$

whenever $0 < \rho \leq r/2$.

Proof of Theorem 1: We now prove Theorem 1. Let Ω be as in Theorem 1 and fix $(\hat{X}, \hat{t}) \in \Omega$. Let $(X, t) \in \partial\Omega$ with $|X - \hat{X}|^2 \leq A(\hat{t} - t)$ and $\hat{t} - t \geq 8R^2$. To prove Theorem 1, we assume as we may, that $\hat{n}(X, t, R) = e_0$ and observe from Proposition 2.1 that there exists $\psi : \mathbb{R}^n \to \mathbb{R}$ satisfying (1.13), (1.14) with $b_1 + b_2 \leq c(M)(1 + \|\nu\|)^{1/2}$. Also if $\tilde{\Omega} = \{(Z,\tau) : z_0 > \psi(z,\tau)\}$, then $\tilde{\Omega} \cap C_{2R}(X,t) \subset \Omega$ and

$$\sigma(\partial\tilde{\Omega} \cap \Delta(X,t,R)) \;\geq\; \eta\sigma(\Delta(X,t,R)) \tag{3.20}$$

where $\eta = \eta(M, \|\nu\|), 0 < \eta < 1$. Let $\tilde{\omega}$ denote caloric measure for $\tilde{\Omega}$. To prove Theorem 1 we shall show the existence of $\theta = \theta(A, M, \|\nu\|, \delta_0), 0 < \theta \leq 1/2$, with the following property.

$$\begin{gathered}\text{If } E \subset \Delta(X,t,R) \text{ is a closed set with } \sigma(E) \geq (1-\eta/2)\,\sigma[\Delta(X,t,R)],\\ \text{then } \omega(\hat{X},\hat{t},E) \geq \theta\omega(\hat{X},\hat{t},\Delta(X,t,R)).\end{gathered} \tag{3.21}$$

Indeed, let E be as in (3.21) and put $F = E \cap \Delta(X,t,R) \cap \partial\tilde{\Omega}$. Then from (3.20) we see that

$$\sigma(F) \;\geq\; (1/2)\,\eta\,\sigma[\Delta(X,t,R)]. \tag{3.22}$$

We claim that there exists c_- (depending only on n, M) such that for each $r, 0 < r \leq R/2$, we have

$$\sigma(F \cap \Delta(X^*,t^*,r)) \;\geq\; (\eta/c_-)\,\sigma(\Delta(X^*,t^*,r)) \tag{3.23}$$

for some $(X^*,t^*) \in F$. Indeed otherwise we could use a covering argument to contradict the size of F in (3.22). From (3.23) and [LM, Theorem 2, ch 3] we see that if $w \geq 0$ is continuous on $\partial\tilde{\Omega}$ with $w \geq 1$ on $F \cap \Delta(X^*,t^*,r)$, and also a bounded solution to the heat equation in $\tilde{\Omega}$, then

$$w[U_2^*, t_2^*, F \cap \Delta(X^*,t^*,r)] \geq \theta_1, \tag{3.24}$$

where θ_1 depends only on $M, \|\nu\|$ and $U_2^* = U_2(X^*,t^*,r), , t_2^* = t_2^*(X^*,t^*,r)$ are as in (3.4). Next let $c' = c'(M, \|\nu\|)$ be so large that $\psi(y,s) < c'R + \psi(x,t)$ for all $(y,s) \in C_{3R/2}(x,t)$ and put

$$\Omega^+(X,t,R) \;=\; \{(Y,s) \in \tilde{\Omega} : \psi(y,s) < y_0 < 4c'R + \psi(x,t)\} \subset \Omega, \tag{3.25}$$

as follows from our assumption that $\hat{n}(X,t,R) = e_0$, (1.1), and (3.20) provided $\delta_0 > 0$ is small enough. We also observe that the continuous Dirichlet problem for $\Omega^+(X,t,R)$ and the heat equation always has a solution for continuous functions on $\partial\Omega^+(X,t,R) \setminus \{(Y,s) : s = t + 9R^2/4\}$ as we see from barrier type estimates and the method of Perron-Wiener-Brelot. Let $v \geq 0$ be a continuous function on $\mathbb{R}^{n+1}$ with $v \geq 1$ on F (as in (3.22)) and $v \equiv 0$ on $\partial\Omega^+(X,t,R) \setminus \tilde{\Omega}$. Let $\tilde{h}, h^+$ be the bounded solutions to the heat equation in $\tilde{\Omega}, \Omega^+(X,t,R)$ with continuous boundary values, v on the caloric boundary of each domain. Set $u = \tilde{h} - h^+ \geq 0$ in $\Omega^+(X,t,R)$ and observe from Lemma 3.6 that there exists $c_+ = c_+(M, \|\nu\|, \delta_0) \geq 1$, such that if $r = c_+R$, then

$$u(U_2^*, t_2^*) \leq (1/2)\theta_1, \tag{3.26}$$

where $X^*, t^*, U_2^*, t_2^*, \theta_1$ are as in (3.24). From (3.26) and (3.24) with $w = \tilde{h}$ we deduce

$$h^+(U_2^*, t_2^*) \geq (1/2)\tilde{h}(U_2^*, t_2^*) \;\geq\; (1/2)\theta_1$$

which in view of Harnack's inequality for the heat equation and (3.5) implies for some $\tilde{c} = \tilde{c}(M, \|\nu\|, \delta_0) \geq 1$ that

$$h^+(\bar{U}, \bar{t}) \geq \tilde{c}^{-1} \tag{3.27}$$

where $(\bar{U},\bar{t}) = (\psi(x,t)+2c'R, x, t+R^2)$. Let ω^+ be caloric measure for $\Omega^+(X,t,R)$. Taking the infimum over all such v's we conclude from (3.25), the maximum principle for the heat equation, and (3.27) that for some $\tilde{c} = \tilde{c}(M, \|\nu\|, \delta_0) \geq 1$,

$$\omega(\bar{U},\bar{t},F) \geq \omega^+(\bar{U},\bar{t},F) \geq \tilde{c}^{-1}. \tag{3.28}$$

Finally from (3.28), Harnack's inequality for the heat equation, and Lemma 3.10 we see that

$$\omega(\cdot, F) \geq c^{-1} R^n G(\cdot, U', t')$$

on $\partial C_{R/8}(U',t')$ where $(U',t') = (U_2^*(X,t,2R), t_2^*(X,t,2R))$ is as in (3.4) and c has the same dependence as in (3.28). Using this inequality, the maximum principle for the heat equation, and Lemmas 3.10, 3.11, we conclude that

$$\omega(\hat{X},\hat{t},F) \geq \theta\, \omega(\hat{X},\hat{t},\Delta(X,t,R))$$

which is (3.21). Theorem 1 follows from (3.21) and a slight elaboration of the argument in [CF] (see also [JK1], [HL, ch2]). We omit the details. □

Let

$$⨍_D \ldots d\lambda = \lambda(D)^{-1} \int_D \ldots d\lambda$$

denote the average over the Borel set $D \subset \partial\Omega$ when λ is a positive Borel measure with $\lambda(D) > 0$. Let $k(\hat{X},\hat{t},\cdot)$ denote the Radon - Nikodym derivative of $\omega(\hat{X},\hat{t},\cdot)$ with respect to σ. We note from Theorem 1 for use in sections 4, 5 that for $\delta_0 > 0$ small enough and fixed, there exists $\tilde{p} = \tilde{p}(A, M, \|\nu\|)$, $c = c(A, M, \|\nu\|) > 1$ such that

$$\begin{aligned} c\left(\frac{\omega[\hat{X},\hat{t},\Delta(X,t,R)]}{\sigma[\hat{X},\hat{t},\Delta(X,t,R)]}\right)^{\tilde{p}} &= c\left(⨍_{\Delta(X,t,R)} k(\hat{X},\hat{t},Y,s)\,d\sigma(Y,s)\right)^{\tilde{p}} \\ &\geq ⨍_{\Delta(X,t,R)} k(\hat{X},\hat{t},Y,s)^{\tilde{p}}\,d\sigma(Y,s) \end{aligned} \tag{3.29}$$

when $|X-\hat{X}|^2 \leq A(\hat{t}-t)$ and $\hat{t}-t \geq 8R^2$. From (3.29) and Hölder's inequality we deduce that if $D \subset \Delta(X,t,R)$, D Borel, then

$$\frac{\omega(\hat{X},\hat{t},D)}{\omega[\hat{X},\hat{t},\Delta(X,t,R)]} \leq c\left\{\frac{\sigma(\hat{X},\hat{t},D)}{\sigma[\hat{X},\hat{t},\Delta(X,t,R)]}\right\}^{(1-1/\tilde{p})}. \tag{3.30}$$

Let $(\hat{X},\hat{t}), (X,t)$ be as above and suppose also that $(X_1,t_1) \in \Omega$ with $|X_1 - X|^2 \leq A(t_1 - t)$, $t_1 - t \geq 8R^2$. If $0 < \rho \leq R$ and $(Y,s), (Z,\tau) \in C_\rho(X,t) \cap \Omega$, then from Lemma 3.19 with $u = G(\hat{X},\hat{t},\cdot)$, $v = G(X_1,t_1,\cdot)$ we see for some $c = c(A)$ and δ_0 small enough that

$$\left|\frac{G(\hat{X},\hat{t},Y,s)G(X_1,t_1,Z,\tau)}{G(X_1,t_1,Y,s)G(\hat{X},\hat{t},Z,\tau)} - 1\right| \leq c(\rho/R)^\gamma. \tag{3.31}$$

Using (3.31) we shall show that if $1 \le q \le \tilde{p}$, then

$$
\begin{aligned}
&[1-\bar{c}(\rho/R)^{\gamma}]^{q}\,\frac{\displaystyle\fint_{\Delta(X,t,\rho)} k^{q}(\hat{X},\hat{t},\cdot)d\sigma}{\left(\displaystyle\fint_{\Delta(X,t,\rho)} k(\hat{X},\hat{t},\cdot)d\sigma\right)^{q}} \le \frac{\displaystyle\fint_{\Delta(X,t,\rho)} k^{q}(X_1,t_1,\cdot)d\sigma}{\left(\displaystyle\fint_{\Delta(X,t,\rho)} k(X_1,t_1,\cdot)d\sigma\right)^{q}} \\
&\qquad \le [1+\bar{c}(\rho/R)^{\gamma}]^{q}\,\frac{\displaystyle\fint_{\Delta(X,t,\rho)} k^{q}(\hat{X},\hat{t},\cdot)d\sigma}{\left(\displaystyle\fint_{\Delta(X,t,\rho)} k(\hat{X},\hat{t},\cdot)d\sigma\right)^{q}}
\end{aligned}
\tag{3.32}
$$

where $\bar{c} = \bar{c}(A, M, \|\nu\|)$. To this end we note that (3.32) is implied by (3.31) and our claim that for σ almost every $(Z,\tau) \in \Delta(X,t,\rho)$, we have

$$
l(Z,\tau) = \lim_{(Y,s)\in\Omega \longrightarrow (Z,\tau)} \frac{G(X_1,t_1,Y,s)}{G(\hat{X},\hat{t},Y,s)} = \frac{k(X_1,t_1,Z,\tau)}{k(\hat{X},\hat{t},Z,\tau)}. \tag{3.33}
$$

For example assuming (3.33), we get from (3.31),

$$
\begin{aligned}
&\frac{\displaystyle\fint_{\Delta(X,t,\rho)} k^{q}(X_1,t_1,\cdot)d\sigma}{\left(\displaystyle\fint_{\Delta(X,t,\rho)} k(X_1,t_1,\cdot)d\sigma\right)^{q}} \le \frac{\displaystyle\max_{(Z,\tau)\in\Delta(X,t,\rho)} l^{q}}{\displaystyle\min_{(Z,\tau)\in\Delta(X,t,\rho)} l^{q}}\,\frac{\displaystyle\fint_{\Delta(X,t,\rho)} k^{q}(\hat{X},\hat{t},\cdot)d\sigma}{\left(\displaystyle\fint_{\Delta(X,t,\rho)} k(\hat{X},\hat{t},\cdot)d\sigma\right)^{q}} \\
&\qquad \le [1+c(\rho/R)^{\gamma}]^{q}\,\frac{\displaystyle\fint_{\Delta(X,t,\rho)} k^{q}(\hat{X},\hat{t},\cdot)d\sigma}{\left(\displaystyle\fint_{\Delta(X,t,\rho)} k(\hat{X},\hat{t},\cdot)d\sigma\right)^{q}}.
\end{aligned}
$$

Thus to get (3.32) we need only prove claim (3.33). To do this we first note from (3.31) that l exists and may be regarded as the Hölder γ extension of $G(X_1,t_1,\cdot)/G(\hat{X},\hat{t},\cdot)$ to $C_\rho(X,t) \cap \bar{\Omega}$. Next let $\phi \ge 0 \in C_0^\infty[C_2(0,0)]$ with $\phi \equiv 1$ on $C_1(0,0)$. Define ϕ_λ relative to ϕ as above display (2.6). Given $(Z,\tau) \in \partial\Omega$, let $\xi(Y,s) = \phi_r(Y-Z, s-\tau)$ and replace ϕ by ξ in (3.8). Following [KT, Lemma 4.3] we note from Lebesgue differentiation theory and (3.8) that for a.e $(Z,\tau) \in \partial\Omega$ with respect to σ we have

$$
\frac{k(X_1,t_1,Z,\tau)}{k(\hat{X},\hat{t},Z,\tau)} = \lim_{r\longrightarrow 0} \frac{\displaystyle\int_{\partial\Omega} \xi d\omega(X_1,t_1,\cdot)}{\displaystyle\int_{\partial\Omega} \xi d\omega(\hat{X},\hat{t},\cdot)} = \lim_{r\longrightarrow 0} \frac{\displaystyle\int_{\Omega} G(X_1,t_1,\cdot)(\Delta\xi - \tfrac{\partial}{\partial s}\xi)dY ds}{\displaystyle\int_{\Omega} G(\hat{X},\hat{t},\cdot)(\Delta\xi - \tfrac{\partial}{\partial s}\xi)\, dY ds} \tag{3.34}
$$

Now for fixed r,

$$
\int_{\Omega} G(X_1,t_1,\cdot)(\Delta\xi - \tfrac{\partial}{\partial s}\xi)dY ds = l(Z,\tau)\int_{\Omega} G(\hat{X},\hat{t},\cdot)(\Delta\xi - \tfrac{\partial}{\partial s}\xi)dY ds + e_r \tag{3.35}
$$

where

$$|e_r| = \left| \int_\Omega [l(Z,\tau) - \tfrac{G(X_1,t_1,\cdot)}{G(\hat{X},\hat{t},\cdot)}] G(\hat{X},\hat{t},\cdot)\,(\Delta\xi - \tfrac{\partial}{\partial s}\xi) dY ds \right|$$

$$\leq c(r/R)^\gamma\, r^n\, G(\hat{X},\hat{t}, U_1^*(X,t,4r), t_1^*(X,t,4r))$$

$$\leq\; c\,(r/R)^\gamma \int_{\partial\Omega} \xi\, d\omega(\hat{X},\hat{t},\cdot) \;=\; c\,(r/R)^\gamma \int_\Omega G(\hat{X},\hat{t},\cdot)(\Delta\xi - \tfrac{\partial}{\partial s}\xi) dY ds$$

and we have used Lemmas 3.7, 3.10, 3.11 and 3.17 to make our estimates. Here c may depend on $(\hat{X},\hat{t})$, (X_1,t_1), R but is independent of r. Using this inequality and (3.35) it follows easily that

$$l(Z,\tau) = \lim_{r\to 0} \frac{\int_\Omega G(X_1,t_1,\cdot)(\Delta\xi - \frac{\partial}{\partial s}\xi) dY ds}{\int_\Omega G(\hat{X},\hat{t},\cdot)(\Delta\xi - \frac{\partial}{\partial s}\xi)\, dY ds}$$

and thereupon from (3.34) that (3.33) is valid.

4. Proof of Theorem 2. If $n > 1$, let $\Omega_+ = \{(X,t) : x_0 > \psi_+(x,t)\}$ where ψ_+ satisfies (1.13), (1.14) and b_1, b_2 are small. If $n = 1$, replace $\psi_+(x,t)$ by $\psi_+(t)$ in the definiton of Ω_+. In proving Theorems 2 and 3, we shall need some results on singular integrals and the Neumann problem for the adjoint heat equation from [H], [HL], [LM]. Although theorems in these papers were stated only for the heat equation, they imply corresponding theorems for the adjoint heat equation, as follows from the observation that if u is a solution to the heat equation in Ω_+, as above, then $v(X,t) = u(X,-t)$ is a solution to the adjoint heat equation in $\{(Y,s) : y_0 > \psi_+(y,-s)\}$ and this graph function also satisfies (1.13), (1.14). For ease of notation we shall limit our discussion to the case $n > 1$. At the end of this discussion, we indicate the slight modifications that are necessary when $n = 1$. To begin, let $(X,t) \in \mathbb{R}^{n+1}$ and let

$$W(X,t) \;=\; \begin{cases} (4\pi t)^{-n/2}\ \exp\left[\frac{-|X|^2}{4t}\right] & \text{when } t > 0 \\[2ex] 0 \text{ when } t \leq 0, \end{cases}$$

denote the usual Gaussian in $\mathbb{R}^{n+1}$. Next for given $\epsilon > 0$, let $W_\epsilon(X,t) = W(X,t)$ when both $|X| > \epsilon$, $t > \epsilon^2$. Otherwise, $W_\epsilon \equiv 0$. Let

$$d\sigma_+(Y,s) = \sqrt{1 + |\nabla\psi_+|^2(y,s)}\, dy ds$$

$$n(Y,s) = \frac{(-1, \nabla\psi_+(y,s), 0)}{\sqrt{1+|\nabla\psi_+|^2(y,s)}}$$

$$\nabla_Y = (\tfrac{\partial}{\partial y_0}, \ldots, \tfrac{\partial}{\partial y_{n-1}}).$$

We note that as usual $\nabla\psi_+$ is only in the y variable so that $n(y,s)$ is an outer normal to the time slice $\Omega \cap (\mathbb{R}^n \times \{s\})$ considered as a subset of n dimensional Euclidean space. Let $L^p(\partial\Omega_+), 1 < p < \infty$, be the space of Lebesgue functions, f, defined on $\partial\Omega_+$ which are p th power integrable with respect to σ_+ and with norm $\|f\|_p$. Define the single layer adjoint heat potential of $f \in L^p(\partial\Omega_+)$ by

$$\mathcal{S}f(X,t) \equiv \int_{\partial\Omega_+} W(X-Y, s-t)\, f(Y,s)\; d\sigma_+(Y,s),$$

whenever this integral exists and note that $\mathcal{S}f$ is a solution to the adjoint heat equation in $\mathbb{R}^{n+1} \setminus \partial\Omega_+$. For given $\epsilon > 0$, define K_ϵ and associated maximal operator $\bar{K}$ on $\partial\Omega_+$ by

$$\begin{aligned} K_\epsilon f(X,t) &\equiv \int_{\partial\Omega_+} \langle\, \nabla_X\, W_\epsilon(X-Y, s-t)\,,\; n(X,t)\,\rangle\; f(Y,s)\, d\sigma_+(Y,s) \\ &\equiv \int_{\partial\Omega_+} \tfrac{\partial}{\partial n_t}\, W_\epsilon(X-Y, s-t)\, f(Y,s)\, d\sigma_+(Y,s) \\ \bar{K}f(X,t) &\equiv \sup_{\epsilon>0} |K_\epsilon f|(X,t) \end{aligned}$$

whenever $(X,t) \in \partial\Omega_+$.

We note from Sobolev type estimates that $Sf(X,t)$ exists for σ_+ a.e $(X,t) \in \partial\Omega_+$. In [H, Theorem 2] and [LM, chapters 1, 2] it is shown that

$$\|\bar{K}f\|_p \;\leq\; \alpha(p,b)\, \|f\|_p \tag{4.1}$$

whenever $f \in L^p(\partial\Omega_+)$, where $b = b_1 + b_2$ as in (1.13), (1.14) and $\alpha = \alpha(p,b) {\rightarrow} 0$ for fixed $p > 1$ as $b {\rightarrow} 0$. Moreover

$$Kf(X,t) \;=\; \lim_{\epsilon \longrightarrow 0} K_\epsilon f(X,t) \tag{4.2}$$

exists for almost every $(X,t) \in \partial\Omega_+$ with respect to σ_+. Next for given $a > 0$ let $(X,t) = (\psi_+(x,t), x, t) \in \partial\Omega_+$ and let $\Gamma^{\pm}(X,t) = \Gamma^{\pm}_a(X,t)$ be the parabolic cones :

$$\Gamma^+(X,t) \;\equiv\; \{\, (y_0, y, s) \in \Omega_+ : |x-y| + |t-s|^{1/2} \;<\; a(y_0 - \psi_+(x,t))\,\}$$

$$\Gamma^-(X,t) \;\equiv\; \{\, (y_0, y, s) \in \mathbb{R}^{n+1} \setminus \bar{\Omega}_+ : |x-y| + |t-s|^{1/2} \;<\; a(\psi_+(x,t) \;-\; y_0)\,\}.$$

If h is a real valued function on Ω_+ or $\mathbb{R}^{n+1} \setminus \bar{\Omega}_+$ define the nontangential maximal functions $N^+h, N^-h : \partial\Omega_+ {\rightarrow} \mathbb{R}$ by

$$N^+\, h(X,t) \;\equiv\; \sup_{(Y,s) \in \Gamma^+(X,t)} |h|(Y,s)$$

$$N^-\, h(X,t) \;\equiv\; \sup_{(Y,s) \in \Gamma^-(X,t)} |h|(Y,s).$$

If $(X,t) \in \partial\Omega_+$, then by $\overset{\pm}{\lim\limits_{(Y,s) \longrightarrow (X,t)}} h(Y,s)$ we mean the limit as $(Y,s) {\rightarrow} (X,t)$ in $\Gamma^{\pm}(X,t)$.

In [LM, ch 3, Theorem 6] we stated the solution to the Neumann problem for $L^p(\partial\Omega)$ when $p > 1$ is fixed and $b = b(p) > 0$ is small enough. To get our solution we used singular integral estimates and expansion in terms of a Neumann series to solve a certain integral equation. More specifically, if $b = b(p) > 0$ is sufficiently small, then given $g \in L^p(\partial\Omega_+), 1 < p < \infty$, there exists $f^+, f^- \in L^p(\partial\Omega_+)$ such that

$$\|f^{\pm}\|_p \le 4\,\|g\|_p \tag{4.3}$$

and

$$(\tfrac{1}{2} + K)f^+ \equiv g \equiv (-\tfrac{1}{2} + K)f^-. \tag{4.4}$$

Moreover,

$$\overset{\pm}{\lim_{(Y,s)\to(Z,\tau)}} \langle\, \nabla_Y \mathcal{S} f^{\pm}(Y,s), n(Z,\tau)\,\rangle \;=\; (\pm\tfrac{1}{2} \;+\; K)f^{\pm}(Z,\tau) = g(Z,\tau) \tag{4.5}$$

for σ_+ almost every $(Z,\tau) \in \partial\Omega_+$. $\mathcal{S}f^+, \mathcal{S}f^-$ are up to constants the unique solutions to the adjoint heat equation in Ω_+, $\mathbb{R}^{n+1} \setminus \bar{\Omega}_+$ with normal derivative g on $\partial\Omega_+$ (as in (4.5)) and with

$$\sum_{i=0}^{n-1} \|N^{\pm}\left(\tfrac{\partial \mathcal{S} f^{\pm}}{\partial x_i}\right)\|_p \;\le\; c(p,b)\|g\|_p. \tag{4.6}$$

For further results on this problem under a smallness assumption only on $D^t_{1/2}\psi$ see [HL], [HL2].

Let $G_+(\cdot,\cdot)$ be Green's function for Ω_+ and put $G_+(\cdot,\cdot) \equiv 0$ on $\mathbb{R}^{n+1} \setminus \Omega_+$. Next we note for $(\hat{X},\hat{t}), (\hat{Y},\hat{s}) \in \Omega_+$ and $(\hat{X},\hat{t}) \ne (\hat{Y},\hat{s})$, that

$$G_+(\hat{X},\hat{t},\hat{Y},\hat{s}) \equiv W(\hat{X}-\hat{Y},\hat{t}-\hat{s}) \;+\; h(\hat{X},\hat{t},\hat{Y},\hat{s}) \tag{4.7}$$

where $h(\cdot,\hat{Y},\hat{s}), h(\hat{X},\hat{t},\cdot)$ are solutions to the heat - adjoint heat equations in $\mathbb{R}^{n+1} \setminus \partial\Omega_+$. Also since $G_+ \equiv 0$ in $\mathbb{R}^{n+1} \setminus \Omega_+$ one sees from (4.7) that $h(\hat{X},\hat{t},\cdot) \equiv -W(\hat{X}-\cdot,\hat{t}-\cdot)$ on $\mathbb{R}^{n+1} \setminus \Omega_+$. Thus if $l^-(\hat{X},\hat{t},\cdot)$ is chosen so that (4.4), (4.5) hold with f^- replaced by $l^-(\hat{X},\hat{t},\cdot)$ and g by

$$-\langle \nabla_Z W(\hat{X}-Z,\hat{t}-\tau), n(Z,\tau)\rangle \text{ whenever } (Z,\tau) \in \partial\Omega_+,$$

then

$$h(\hat{X},\hat{t},\cdot) \equiv \mathcal{S}l^-(\hat{X},\hat{t},\cdot). \tag{4.8}$$

From (4.8) and (4.5) we now can get an integral equation for

$$\overset{+}{\lim_{(Y,s)\to(Z,\tau)}} \langle \nabla_Y G_+(\hat{X},\hat{t},Y,s)\,,\, n(Z,\tau)\rangle \;=\; \tfrac{\partial}{\partial n_\tau} G_+(\hat{X},\hat{t},Z,\tau)$$

on $\partial\Omega_+$. To sum up, we have for $(Z,\tau) \in \partial\Omega_+$,

$$\begin{aligned} (-\tfrac{1}{2} + K)l^-(\hat{X},\hat{t},Z,\tau) \;&=\; -\tfrac{\partial}{\partial n_\tau} W(\hat{X}-Z,\hat{t}-\tau) \\ (\tfrac{1}{2} + K)l^-(\hat{X},\hat{t},Z,\tau) \;&=\; \tfrac{\partial G_+}{\partial n_\tau}(\hat{X},\hat{t},Z,\tau) - \tfrac{\partial W}{\partial n_\tau}(\hat{X}-Z,\hat{t}-\tau) \end{aligned} \tag{4.9}$$

σ_+ a.e on $\partial\Omega_+$. Adding and subtracting these equations we find σ_+ a.e on $\partial\Omega_+$ that

$$\begin{aligned} &(a) \qquad l^-(\hat{X},\hat{t},\cdot) \equiv \tfrac{\partial}{\partial n_\tau} G_+(\hat{X},\hat{t},\cdot) \equiv -|\nabla_Z G_+(\hat{X},\hat{t},\cdot)| \\ &(b) \qquad K l^-(\hat{X},\hat{t},\cdot) \equiv \tfrac{1}{2}\tfrac{\partial G_+}{\partial n_\tau}(\hat{X},\hat{t},\cdot) - \tfrac{\partial W}{\partial n_\tau}(\hat{X}-\cdot,\hat{t}-\cdot). \end{aligned} \tag{4.10}$$

Now
$$G_+(\hat X,\hat t,\cdot) = W(X-\cdot,\hat t-\cdot) - \mathcal{S}(d\omega_+(\hat X,\hat t,\ ))(\cdot)$$
and $\omega_+(\hat X,\hat t,\cdot)$ is the unique measure with this property, as we see from the Riesz representation formula for sub and super caloric functions (see [Do]). Using this fact, (4.7), (4.8) and (4.10) we conclude that
$$d\omega_+/d\sigma_+(\hat X,\hat t,Z,\tau) = k_+(\hat X,\hat t,Z,\tau) = |\nabla_Z G_+(\hat X,\hat t,Z,\tau)|\,. \tag{4.11}$$
If $n=1, d\sigma_+(Y,s)=ds$ and $n(Y,s)=(-1,0)=-e_0$. Parabolic cones are defined as below (4.2) except that all x,y terms are removed. With these modifications, (4.1)-(4.11) are valid.

We now begin the proof of Theorem 2. Let Ω, M be as in Theorem 2, $(\hat X,\hat t)\in\Omega$, and δ_0 so small that Theorem 1 and the results in section 3 are valid. Fix δ_0, M, subject to these requirements and observe from the definition of VMO($\partial\Omega$) in (1.11) that it suffices to prove Theorem 2 for $\Delta(X^*,t^*,\hat r)$ where $|\hat X - X^*|^2 \le A(\hat t - t^*)$ with $8\hat r^2 < \hat t - t^*$, since any compact set $\subset \{(Y,s): s<\hat t\}$ can be covered by a finite number of caloric cylinders with this property. Also thanks to (1.8), we can assume $\hat r = \hat r(\delta)$ is so small that the hypotheses of Proposition 2.2 hold whenever $(X,t)\in\Delta(X^*,t^*,\hat r)$ and $R\le\hat r$. We proceed under these assumptions, using the same notation as introduced above (3.29). Given $\epsilon, 0<\epsilon\le 10^{-8}$, we show the existence of $\tilde r = \tilde r(\epsilon,A), 0<\tilde r<\hat r$, such that if $(X,t)\in\Delta(X^*,t^*,\hat r), \delta=\delta(\epsilon,A)$, is sufficiently small and $0<r\le\tilde r$, then
$$\frac{\displaystyle\fint_{\Delta(X,t,r)} k(\hat X,\hat t,Y,s)^{\hat p}\, d\sigma(Y,s)}{\left(\displaystyle\fint_{\Delta(X,t,r)} k(\hat X,\hat t,Y,s) d\sigma(Y,s)\right)^{\hat p}} \le 1+\epsilon \tag{4.12}$$
where $\hat p = (1+\tilde p)/2$ and $\tilde p$ is as in (3.29). Theorem 2 follows easily from (4.12) and the same argument as in [KT, Corollary 5.2]. Thus we shall only prove (4.12).

As motivation for the proof of (4.12) we note that if $\Omega\equiv\Omega_+$, then since (1.13), (1.14) and the heat - adjoint heat equations are invariant under translations, rotations in the Y variable, and the mapping $(Y,s)\to(\rho Y,\rho^2 s)$, one sees that it suffices to prove (4.12) when $(\hat X,\hat t)$ is at distance 1 from $\partial\Omega$ and $(X^*,t^*)=(0,0)$. Also using the results in section 3 it follows that it suffices to prove (4.12) when $r=\eta$ where η is a small but fixed positive number. One can now use (4.1), (4.10), (4.11), and a compactness argument letting $\delta\to 0$, to reduce the validity of (4.12) to the case when $\Omega\equiv\{(Y,s)\in\mathbb{R}^{n+1}: y_0>0\}$. Finally for this half space one deduces (4.12) easily for η small enough from (4.11) and the fact that the Green's function of this domain extends to an infinitely differentiable function on the boundary of this domain.

To begin the proof of (4.12) let $\bar c,\gamma$ be as in (3.32). For the moment we allow both δ and $\tilde r$ to vary but shall later fix them at small positive numbers satisfying several conditions. If $\theta^4=\tilde r/\hat r$, we first require that
$$[1-\bar c\,\theta^\gamma]^{-\hat p} \le 1+\epsilon/4 \tag{4.13}$$
If $0<r\le\tilde r$, let $(U_2,t_2)=(U_2^*(X,t,r/\theta), t_2^*(X,t,r/\theta))$. From (3.32) with R,ρ replaced by $r/\theta, r$ as well as $(\tilde X,\tilde t)=(U_2,t_2)$ we see that
$$(1-\bar c\theta^\gamma)^{\hat p}\frac{\displaystyle\fint_{\Delta(X,t,r)} k(\hat X,\hat t,Y,s)^{\hat p}\, d\sigma(Y,s)}{\left(\displaystyle\fint_{\Delta(X,t,r)} k(\hat X,\hat t,Y,s) d\sigma(Y,s)\right)^{\hat p}} \le \frac{\displaystyle\fint_{\Delta(X,t,r)} k(U_2,t_2,Y,s)^{\hat p}\, d\sigma(Y,s)}{\left(\displaystyle\fint_{\Delta(X,t,r)} k(U_2,t_2,Y,s) d\sigma(Y,s)\right)^{\hat p}} \tag{4.14}$$

From (4.13), (4.14) we conclude that it suffices to prove (4.12) with $(\hat{X}, \hat{t})$ replaced by (U_2, t_2) and ϵ by $\epsilon/4$. Next we use Proposition 2.2 with R replaced by r/θ^2 which is permissible from the above remarks. After a rotation if necessary, we see from this proposition and its proof that we may assume $\hat{n}(X, t, r/\theta^2) = e_0$ and for $n > 1$, that there exists $\Omega_+ = \{(Y, s) : y_0 > \psi_+(y, s)\}$ for which

$$\begin{aligned}
&(a) \qquad \psi_+ \text{ satisfies (1.13), (1.14) with } b_1 + b_2 = b \leq c_2\delta, \\
&(b) \qquad \sigma\left(\Delta(X, t, r/\theta^2) \setminus \partial\Omega_+\right) + \sigma\left(\partial\Omega_+ \cap C_{r/\theta^2}(X, t) \setminus \Omega\right) \leq e^{-1/(c_2\delta)}(r/\theta^2)^{n+1}. \\
&(c) \qquad \Delta(X, t, 2r/\theta^2) = D \cup B \text{ where } D \subset \partial\Omega_+ \text{ and } \sigma(B) \leq e^{-1/(c_2\delta)}(r/\theta^2)^{n+1}. \\
&(d) \qquad \text{If } p \text{ denotes the orthogonal projection of } \mathbb{R}^{n+1} \text{ onto } \{0\} \times \mathbb{R}^n, \text{ and } (Y, s) \in B, \text{ then} \\
&\qquad\quad\ |y_0 - \psi(y, s)| \leq c_2\,\delta\, d(\{p(Y, s)\}, p(D)), \\
&(e) \qquad \Omega \cap C_{2r/\theta^2}(X, t) \subset \Omega_+.
\end{aligned} \tag{4.15}$$

If $n = 1$ we replace $\psi_+(y, s)$ by $\psi_+(s)$ in (4.15) and the definition of Ω_+. Let ω_+, k_+, G_+ denote caloric measure, the density of caloric measure with respect to σ_+, and the Green's function for Ω_+, as above. Let $D_1 = D \cap C_r(X, t)$ and suppose that $(Y, s) \in D_1$ with $\sigma_+(D_1 \cap \Delta(Y, s, \rho)) > 0$ where $0 < \rho < r$. We cover $\partial\Omega_+ \cap \{(Y, s) : s = t \pm r^2/\theta^2\}$ by at most $N = N(n)$ caloric cylinders $\{C_i\}$ of radius $r/(8\theta^2)$ with centers in this set. Let Δ_i denote the intersection of C_i with $\partial\Omega_+$. Using (3.9) and Harnack's inequality we deduce for some $c \geq 1$ that

$$c \sum_i \omega_+(\cdot, \Delta_i) \geq 1 \text{ on } \partial C_{r/\theta^2}(X, t) \cap \Omega_+. \tag{4.16}$$

Using (4.16), the definition of caloric measure, the maximum principle for the heat equation, and Lemma 3.7 we deduce that if $\tilde{U}_2 = U_2^*(X, t, r/\theta^2)$, $\tilde{t}_2 = t_2^*(X, t, r/\theta^2)$ are defined as in (3.4), then

$$\begin{aligned}
&\omega(U_2, t_2, D_1 \cap \Delta(Y, s, \rho)) \\
&\qquad \leq \omega_+(U_2, t_2, D_1 \cap \Delta(Y, s, \rho)) + c\omega(\tilde{U}_2, \tilde{t}_2, D_1 \cap \Delta(Y, s, \rho)) \sum_i \omega(U_2, t_2, \Delta_i) \\
&\qquad \leq \omega_+(U_2, t_2, D_1 \cap \Delta(Y, s, \rho)) + \hat{c}\,\theta^\alpha\, \omega(U_2, t_2, D_1 \cap \Delta(Y, s, \rho)).
\end{aligned} \tag{4.17}$$

Dividing by $\sigma[D_1 \cap \Delta(Y, s, \rho)]$, using Lebesgue differentiation theory and taking a limit as $\rho \to 0$ we see for σ a.e $(Y, s) \in D_1$ that

$$k(U_2, t_2, Y, s) \leq (1 - \hat{c}\,\theta^\alpha)^{-1} k_+(U_2, t_2, Y, s).$$

Integrating this inequality we find that if $\Delta_+(X, t, r) = C_r(X, t) \cap \partial\Omega_+$, then

$$\begin{aligned}
I = \fint_{\Delta(X,t,r)} k^{\hat{p}}(U_2, t_2, \cdot)d\sigma &\leq (1 - \hat{c}\,\theta^\alpha)^{-\hat{p}} \fint_{\Delta_+(X,t,r)} k_+^{\hat{p}}(U_2, t_2, \cdot)d\sigma \\
&\quad + \sigma[\Delta(X, t, r)]^{-1} \int_{\Delta(X,t,r)\setminus D} k^{\hat{p}}(U_2, t_2, \cdot)d\sigma.
\end{aligned} \tag{4.18}$$

We can estimate the last integral using (4.15) (b). From Hölder's inequality and (3.29) we have

$$
\begin{aligned}
&\sigma[\Delta(X,t,r)]^{-1}\int_{\Delta(X,t,r)\setminus D} k^{\hat p}(U_2,t_2,\cdot)d\sigma \\
&\qquad\le c\left(\fint_{\Delta(X,t,r)} k^{\tilde p}(U_2,t_2,\cdot)\,d\sigma\right)^{\hat p/\tilde p}\ \left(\tfrac{\sigma(\Delta(X,t,r)\setminus D)}{\sigma(\Delta(X,t,r))}\right)^{1-\hat p/\tilde p} \\
&\qquad\le c\left(\tfrac{\omega(U_2,t_2,\Delta(X,t,r))}{\sigma(\Delta(X,t,r))}\right)^{\hat p}\ \exp\left[(1-\tilde p)/(2c_2\delta)\right]\ \theta^{2(n+1)(\hat p-\tilde p)/\tilde p} \\
&\qquad\le Ic\,\exp\left[(1-\tilde p)/(2c_2\delta)\right]\theta^{2(n+1)(\hat p-\tilde p)/\tilde p}.
\end{aligned}
\tag{4.19}
$$

Thus if θ is so small that

$$(1-\hat c\,\theta^{\alpha})^{-\hat p}\le 1+\epsilon/100 \tag{4.20}$$

where $\hat c$ is as in (4.18), we can then choose δ so small in (4.19) that

$$\fint_{\Delta(X,t,r)} k^{\hat p}(U_2,t_2,\cdot)d\sigma\ \le\ (1+\epsilon/50)\fint_{\Delta_+(X,t,r)} k_+^{\hat p}(U_2,t_2,\cdot)d\sigma_+. \tag{4.21}$$

On the other hand using (4.15) (c), the same argument as in (4.17), and (3.30) with D replaced by B, one deduces that

$$
\begin{aligned}
\omega_+(U_2,t_2,\Delta_+(X,t,r))\ &\le\ \omega(U_2,t_2,\Delta(X,t,r)\cup B) \\
&\quad+\omega_+(\tilde U_2,\tilde t_2,\Delta_+(X,t,r))\sum_i \omega(U_2,t_2,\Delta_i') \\
&\le \omega(U_2,t_2,\Delta(X,t,r))\ + c\,\exp\left[(1-\tilde p)/(c_2\tilde p\delta)\right]\ \theta^{2(n+1)(1-\tilde p)/\tilde p} \\
&\qquad+\ \hat c\,\theta^{\alpha}\omega_+(U_2,t_2,\Delta_+(X,t,r)).
\end{aligned}
\tag{4.22}
$$

Here $\{\Delta_i'\}$ is defined in the same way as $\{\Delta_i\}$ only with Ω replacing Ω_+. From Harnack's inequality and (3.9) we have

$$\omega(U_2,t_2,\Delta(X,t,r))\ge\theta^{\mu} \tag{4.23}$$

for some $\mu\ge 2$. From (4.23), (4.22), (4.20) and (4.15) (b) we see that if $\delta>0$ is small enough, then

$$\left[\frac{\omega_+(U_2,t_2,\Delta_+(X,t,r))}{\sigma_+(\Delta_+(X,t,r))}\right]^{\hat p}\ \le\ (1+\epsilon/50)\left[\frac{\omega(U_2,t_2,\Delta(X,t,r))}{\sigma(\Delta(X,t,r))}\right]^{\hat p}. \tag{4.24}$$

Combining (4.24), (4.21) we get

$$\frac{\fint_{\Delta(X,t,r)} k(U_2,t_2,Y,s)^{\hat p}\,d\sigma(Y,s)}{\left(\fint_{\Delta(X,t,r)} k(U_2,t_2,Y,s)d\sigma(Y,s)\right)^{\hat p}}\ \le\ (1+\epsilon/16)\,\frac{\fint_{\Delta_+(X,t,r)} k_+(U_2,t_2,Y,s)^{\hat p}\,d\sigma(Y,s)}{\left(\fint_{\Delta_+(X,t,r)} k_+(U_2,t_2,Y,s)d\sigma(Y,s)\right)^{\hat p}}. \tag{4.25}$$

Next from (4.11), (4.10), and (4.1) with $(\hat{X}, \hat{t})$ replaced by (U_2, t_2) we see that

$$k_+(U_2, t_2, \cdot) = -2W_{n_\tau}(U_2 - \cdot, t_2 - \cdot) \, + \, 2K(k_+(U_2, t_2, \cdot))(\cdot) \tag{4.26}$$

where K, W are as defined earlier and

$$\|K(k_+(U_2, t_2, \cdot))\|_{\hat{p}} \leq \, \alpha(\hat{p}, \delta) \|k_+(U_2, t_2, \cdot)\|_{\hat{p}}$$

$$\leq \, 2\alpha(\hat{p}, \delta) \, [\|W_{n_\tau}(U_2 - \cdot, t_2 - \cdot)\|_{\hat{p}} \, + \, \|K(k_+(U_2, t_2, \cdot))\|_{\hat{p}} \,].$$

Since $\alpha(\hat{p}, \delta) \to 0$ as $\delta \to 0$ it follows from the above inequality that if $\delta > 0$ is small enough, then

$$\|K(k_+(U_2, t_2, \cdot))\|_{\hat{p}} \, \leq \, 4\alpha(\hat{p}, \delta) \| | \, \nabla_Z W(U_2 - \cdot, t_2 - \cdot)| \, \|_{\hat{p}} \, \leq \, c \, \alpha(\hat{p}, \delta) (r/\theta)^{(n+1)(1-\hat{p})/\hat{p}} \tag{4.27}$$

where the last inequality is obtained by direct calculation. Also from (3.9), an inequality similar to (4.23), and Hölder's inequality we find that

$$\theta^{\mu \hat{p}} \, \leq \, \omega_+[U_2, t_2, \Delta_+(X, t, r)]^{\hat{p}} \, \leq \, c \, \left[\int_{\Delta_+(X,t,r)} k_+(U_2, t_2, \cdot)^{\hat{p}} d\sigma_+ \right] \, r^{(n+1)(\hat{p}-1)}. \tag{4.28}$$

Let χ be the characteristic function of $\Delta_+(X, t, r)$. From (4.27), (4.28), we deduce for fixed $\theta, \hat{p}$ that

$$\|K(k_+(U_2, t_2, \cdot))\|_{\hat{p}} \leq c(\delta) \|k_+(U_2, t_2, \cdot) \, \chi\|_{\hat{p}}$$

where $c(\delta) \to 0$ as $\delta \to 0$. Using (4.26), the triangle inequality and this inequality we find that

$$\|k_+(U_2, t_2, \cdot) \, \chi\|_{\hat{p}} \, \leq \, 2\|W_{n_\tau}(U_2 - \cdot, t_2 - \cdot) \, \chi\|_{\hat{p}} \, + \, 2c(\delta) \|k_+(U_2, t_2, \cdot) \, \chi\|_{\hat{p}}.$$

Solving this inequality for $\|k_+(U_2, t_2, \cdot)\|_{\hat{p}}$ and raising the resulting inequality to the $\hat{p}$ th power we deduce for δ small enough that

$$⨍_{\Delta_+(X,t,r)} k_+(U_2, t_2, \cdot)^{\hat{p}} \, d\sigma_+ \, \leq \, (1 + \epsilon/100) ⨍_{\Delta_+(X,t,r)} |2W_{n_\tau}(U_2 - \cdot, t_2 - \cdot)|^{\hat{p}} d\sigma_+. \tag{4.29}$$

Finally using (4.15) (a) and making simple estimates as in (4.27) using smoothness of W and the definition $n(Z, \tau)$, we see that if $P = \{(Y, s) : y_0 > x_0\}$, then for δ small enough we have

$$⨍_{\Delta_+(X,t,r)} |2W_{n_\tau}(U_2 - \cdot, t_2 - \cdot)|^{\hat{p}} d\sigma_+ \leq (1 + \epsilon/100) ⨍_{C_r(X,t) \cap P} |2\tfrac{\partial}{\partial z_0} W(U_2 - \cdot, t_2 - \cdot)|^{\hat{p}} dz d\tau. \tag{4.30}$$

Combining (4.30) with (4.29) we find

$$⨍_{\Delta_+(X,t,r)} k_+(U_2, t_2, \cdot)^{\hat{p}} d\sigma_+ \, \leq \, (1 + \epsilon/40) \, ⨍_{C_r(X,t) \cap P} |2\tfrac{\partial}{\partial z_0} W(U_2 - \cdot, t_2 - \cdot)|^{\hat{p}} dz d\tau. \tag{4.31}$$

Likewise, from Hölder's inequality and (4.27) we have

$$\begin{aligned} &\int_{\Delta_+(X,t,r)} |K(k_+(U_2, t_2, \cdot)(\cdot)| d\sigma_+ \\ &\leq \, c\|K(k_+(U_2, t_2, \cdot)\|_p \, \sigma_+(\Delta_+(X, t, r))^{1-1/\hat{p}} \leq c \, \alpha(\hat{p}, \delta) \, . \end{aligned} \tag{4.32}$$

From (4.32), (4.26), and (4.23) with $\omega, \Delta(X,t,r)$ replaced by $\omega_+, \Delta_+(X,t,r)$, we see for δ small enough that

$$2 \fint_{\Delta_+(X,t,r)} |W_{n_\tau}(U_2 - \cdot, t_2 - \cdot)| d\sigma_+ \,\le\, (1+\epsilon/100) \frac{\omega_+[U_2, t_2, \Delta_+(X,t,r)]}{\sigma_+[\Delta_+(X,t,r)]}.$$

Moreover, for δ small enough we obtain as in (4.30) that

$$\fint_{C_r(X,t)\cap P} |\tfrac{\partial}{\partial z_0} W(U_2 - \cdot, t_2 - \cdot)|\, dz d\tau \,\le\, (1+\epsilon/100) \fint_{C_r(X,t)\cap P} |W_{n_\tau}(U_2 - \cdot, t_2 - \cdot)| d\sigma_+.$$

Combining these inequalities we find that

$$2 \fint_{C_r(X,t)\cap P} |\tfrac{\partial}{\partial z_0} W((U_2 - \cdot, t_2 - \cdot)| dz d\tau \,\le\, (1+\epsilon/40) \frac{\omega_+[U_2, t_2, \Delta_+(X,t,r)]}{\sigma[\Delta_+(X,t,r)]}. \tag{4.33}$$

From (4.33), (4.29) we get

$$\frac{\fint_{\Delta_+(X,t,r)} k_+(U_2, t_2, \cdot)^{\hat p}\, d\sigma_+}{\left(\fint_{\Delta_+(X,t,r)} k_+(U_2, t_2, \cdot) d\sigma_+\right)^{\hat p}} \,\le\, (1+\epsilon/16)\, \frac{\fint_{C_r(X,t)\cap P} |\frac{\partial}{\partial z_0} W(U_2 - \cdot, t_2 - \cdot)|^{\hat p}\, dz d\tau}{\left(\fint_{C_r(X,t)\cap P} |\frac{\partial}{\partial z_0} W(U_2 - \cdot, t_2 - \cdot)| dz d\tau\right)^{\hat p}}\,. \tag{4.34}$$

We now fix θ to be the largest positive number for which (4.13), (4.20) and the following inequality are valid:

$$\frac{\fint_{C_r(X,t)\cap P} |\frac{\partial}{\partial z_0} W(U_2 - \cdot, t_2 - \cdot)|^{\hat p}\, dz d\tau}{\left(\fint_{C_r(X,t)\cap P} |\frac{\partial}{\partial z_0} W(U_2 - \cdot, t_2 - \cdot)| dz d\tau\right)^{\hat p}} \,\le\, 1 + \epsilon/16. \tag{4.35}$$

As mentioned above (4.13), one can verify (4.35) directly or reduce to the case $r = \theta$ by using an appropriate transformation. (4.35) then follows from smoothness of $|\frac{\partial}{\partial z_0} W|$ and the fact that this function is bounded below on $C_1(X,t) \cap P$. With θ fixed we can retrace our steps and choose δ as above. From (4.14), (4.25), (4.34), (4.35) we conclude first that (4.12) is true whenever $0 < r \le \tilde r$. Second from the remark after (4.12) we get Theorem 2. □

5. Flatness and Asymptotic Doubling. Let $\hat\Omega$ be a domain, $(X,t) \in \partial\hat\Omega$, and $r > 0$. We shall say that $C_r(X,t) \cap \partial\hat\Omega$ is Reifenberg flat with vanishing constant in the parabolic sense, if for each $\epsilon > 0$, there exists $\rho_0 = \rho_0(\epsilon) > 0$ with the following property. If $(\tilde X, \tilde t) \in C_r(X,t) \cap \partial\hat\Omega$ and $0 < \rho \le \rho_0$, then there is a plane $P'(\tilde X, \tilde t, \rho)$ containing a line parallel to the t axis such that (1.1) holds with $R, \delta_0, \hat P$ replaced by ρ, ϵ, P'. The following proposition will play a key role in our proof of Theorem 3.

Proposition 5.1. *Let Ω' be a domain which satisfies (1.1) for some $\delta_0 > 0$. Given $(\hat X, \hat t) \in \Omega'$, let $\omega'(\hat X, \hat t, \cdot)$ be the corresponding caloric measure. There exists $c \ge 1$ such that if $\delta_0 \le c^{-1}$, then $\omega'(\hat X, \hat t, \cdot)$ is asymptotically optimally doubling if and only if $C_r(X,t) \cap \partial\Omega'$ is Reifenberg flat with vanishing constant whenever $(X,t) \in \partial\Omega'$, $\hat t \ge t + 4r^2$.*

Proof: We first prove the if part of Proposition 5.1. The proof is by contradiction. We use notation similar to that introduced above (1.12). Set

$$C_{r_1,r_2}(X,t) = \{(Y,s) : |Y-X| < r_1, |s-t| < r_2^2\}$$

$$\Delta'(X,t,r_1,r_2) = C_{r_1,r_2}(X,t) \cap \Omega'$$

whenever $(X,t) \in \partial\Omega'$. Suppose $\omega'(\hat{X},\hat{t},\cdot)$ is not asymptotically optimally doubling under the above vanishing flatness condition. Then for some $0 < \epsilon, \tau_1, \tau_2 < 1$, there exists sequences $\{(X_k,t_k)\}, (r_k)$, with $(X_k,t_k) \in \partial\Omega', r_k > 0$, and $\lim\limits_{k\to\infty}(X_k,t_k) = (X,t) \in \partial\Omega'$, while $\lim\limits_{k\to\infty} r_k = 0$. Moreover, $t < \hat{t}$ and

$$\frac{\omega'[\hat{X},\hat{t},\Delta'(X_k,t_k,\tau_1 r_k,\tau_2 r_k)]}{\omega'[\hat{X},\hat{t},\Delta'(X_k,t_k,r_k,r_k)]} \geq (1+\epsilon)\tau_2^2\tau_1^{n-1}$$

for $k = 1, 2, \ldots$. We assume as we may that $\hat{t} - t \geq 4r_k^2$, $k = 1, \ldots,$ and shall proceed as in the proof of (4.12) to get a contradiction. We note that (3.34) holds with $\Omega = \Omega'$ provided the lefthand side of this inequality is interpreted as the Radon-Nikodym derivative of $\omega'(\tilde{X},\tilde{t},\cdot)$ with respect to $\omega'(\hat{X},\hat{t},\cdot)$. Using this note we see that Lemma 3.18 can be used as in (4.14) to get that if θ is sufficiently small and $(U_k,t_k) = (U_2^*(X_k,t_k,r_k/\theta), t_2^*(X_k,t_k,r_k/\theta)), k = 1,2,\ldots,$ where (U_2^*,t_2^*) is as in (3.4), then

$$\begin{aligned}\frac{\omega'[U_k,t_k,\Delta'(X_k,t_k,\tau_1 r_k,\tau_2 r_k)]}{\omega'[U_k,t_k,\Delta'(X_k,t_k,r_k,r_k)]} &\geq (1-\epsilon/8)\frac{\omega'[\hat{X},\hat{t},\Delta'(X_k,t_k,\tau_1 r_k,\tau_2 r_k)]}{\omega'[\hat{X},\hat{t},\Delta'(X_k,t_k,r_k,r_k)]} \\ &\geq (1+3\epsilon/4)\tau_2^2\tau_1^{n-1},\end{aligned} \tag{5.2}$$

where we have assumed, as we may, that $0 < \epsilon < 10^{-5}$. Let $\Omega_k = \{(Y,s) : (X_k + (r_k/\theta)Y, t_k + (r_k^2/\theta^2)s) \in \Omega'\}$ for $k = 1, 2, \ldots$, and let ω_k be caloric measure corresponding to Ω_k. Since caloric measure is invariant under translations and dilations of the above form we find from (5.2) that if $(\tilde{U}_k,\tilde{t}_k), \Delta_k, \Lambda_k$ denote the image of $(U_k,t_k), \Delta'(X_k,t_k,r_k,r_k), \Delta'(X_k,t_k,\tau_1 r_k,\tau_2 r_k)$ under the mapping $(Z,\tau)\to((Z-X_k)\theta/r_k, (\tau-t_k)\theta^2/r_k^2)$, then

$$\frac{\omega'[\tilde{U}_k,\tilde{t}_k,\Delta'(X_k,t_k,\tau_1 r_k,\tau_2 r_k)]}{\omega'[U_k,t_k,\Delta'(X_k,t_k,r_k,r_k)]} = \frac{\omega_k[\tilde{U}_k,\tilde{t}_k,\Lambda_k]}{\omega_k[\tilde{U}_k,\tilde{t}_k,\Delta_k]} \geq (1+3\epsilon/4)\tau_2^2\tau_1^{n-1}. \tag{5.3}$$

Recall that the parabolic Hausdorff distance between K_1 and K_2 is defined

$$h(K_1,K_2) = \sup_{(Y,s)\in K_1} d(\{(Y,s)\},K_2) + \sup_{(Y,s)\in K_2} d(\{(Y,s)\},K_1).$$

Now from the vanishing Reifenberg flat assumption it is easily seen that a subsequence of (Ω_k) converges uniformly on compact subsets (in the parabolic Hausdorff metric) to a halfspace, which after a rotation if necessary, we may assume to be $\{(Z,\tau) : z_0 > 0\}$. Also corresponding subsequences of Δ_k, Γ_k converge uniformly in this metric to cylinders, Δ, Γ. Using Lemma 3.6, applied to $\omega_k(\tilde{U}_k,\tilde{t}_k,\cdot), 1-\omega_k(\tilde{U}_k,\tilde{t}_k,\cdot)$ and well known interior estimates for caloric functions we see that subsequences of $\omega_k(\cdot,\Delta_k), \omega_k(\cdot,\Lambda_k)$ converge uniformly on compact subsets of the above half space to $\tilde{\omega}(\cdot,\Delta),\tilde{\omega}(\cdot,\Lambda)$, where $\tilde{\omega}$ is caloric measure for the halfspace. Finally we may suppose that the corresponding subsequence of $\{(U_k,t_k)\}$ has a limit, denoted $(\tilde{U},\tilde{t})$. Then from (5.3) we deduce that

$$\frac{\tilde{\omega}[\tilde{U},\tilde{t},\Lambda]}{\tilde{\omega}[\tilde{U},\tilde{t},\Delta]} \geq (1+3\epsilon/4)\tau_2^2\tau_1^{n-1}\,. \tag{5.4}$$

We observe that Δ,Λ are cylinders with radii $\leq\theta$ in the space variable and height $\leq 2\theta^2$ in the time variable. Moreover $|\Lambda|/|\Delta| = \tau_2^2\tau_1^{n-1}$, where $|\cdot|$ denotes Lebesgue n measure on $\{(Z,\tau): z_0=0\}$. Finally observe that $(\tilde{U},\tilde{t})$ has parabolic distance at least $1/100$ and at most 100 from the boundary of the half space. Using these facts and well known representation formulas for bounded caloric functions in a half space we can argue as in the proof of (4.35) to get a contradiction to (5.4) for θ small enough. From this contradiction we conclude that $\omega'(\hat{X},\hat{t},\cdot)$ is asymptotically optimally doubling if $\partial\Omega'$ is Reifenberg flat with vanishing constant.

We also prove the only if part of Proposition 5.1 by contradiction. Suppose for some $(\bar{X},\bar{t})\in\partial\Omega', r>0$, that $C_r(\bar{X},\bar{t})\cap\partial\Omega'$ is not Reifenberg flat with vanishing constant. Also assume that $\omega'(\hat{X},\hat{t},\cdot)$ is asymptotically optimally doubling, and $\hat{t}-\tilde{t}\geq 4r^2$. Under these assumptions we claim there exists sequences $\{(X_k,t_k)\},(r_k),(\eta_k)$ with $(X_k,t_k)\in C_r(\bar{X},\bar{t})\cap\partial\Omega', r_k,\eta_k>0$, and $\lim\limits_{k\to\infty}(X_k,t_k)=(X,t)\in\bar{C}_r(\bar{X},\bar{t})\cap\partial\Omega'$, while $\lim\limits_{k\to\infty} r_k = \lim\limits_{k\to\infty}\eta_k = 0$. Moreover $0<\eta_k\leq [\ln(1/r_k)]^{-1}, k=1,\dots,$ and for some $\epsilon>0$, we have

$$\inf_P h[P\cap C_{r_k}(X_k,t_k),\Delta'(X_k,t_k,r_k,r_k)] \geq \epsilon\, r_k,\ \text{for } k=1,2,\dots, \tag{5.5}$$

where the infimum is taken over all planes P containing a line parallel to the t axis and h (defined below (5.3)) is the parabolic Hausdorff distance between the above sets. Furthermore, from the definition of optimal doubling we have for $\eta_k\leq\tau_1,\tau_2\leq 1$, and $(Y,s)\in\Delta'(X_k,t_k,r_k/\eta_k,r_k/\eta_k), 0<r\leq r_k/\eta_k$ that

$$(1-\eta_k)\tau_2^2\tau_1^{n-1} \;\leq\; \frac{\omega'[\hat{X},\hat{t},\Delta'(Y,s,\tau_1 r,\tau_2 r)]}{\omega'[\hat{X},\hat{t},\Delta'(Y,s,r,r)]} \leq (1+\eta_k)\tau_2^2\tau_1^{n-1}, \tag{5.6}$$

for $k=1,2,\dots$. To prove (5.6) for given η_k, one divides $[0,1]\times[0,1]$ into N^2 squares of side length N^{-1} and uses (1.12) to choose $\hat{r}_k$ so small that (5.6) holds for some (τ_1,τ_2) in each of the subrectangles with η_k replaced by $\eta_k/2$ whenever $0<r\leq\hat{r}_k$. Using positivity of $\omega(\hat{X},\hat{t},\cdot)$, it then follows that (5.6) is valid for N large enough and $\hat{r}_k$ suitably small. With (η_k) chosen, we can now choose $r_k\leq\eta_k\hat{r}_k$ so small that $\eta_k\leq[\ln(1/r_k)]^{-1}$ and (5.5) holds. To get a contradiction to (5.5), we again use a blowup type argument. Let $\tilde{\Omega}_k$ denote the image of Ω' under the transformation, $(Z,\tau)\to((Z-X_k)/r_k,((\tau-t_k)/r_k^2), k=1,\cdots,$ and for E Borel $\subset\mathbb{R}^{n+1}$, let

$$\tilde{\omega}_k(E) = \frac{\omega'\left(\hat{X},\hat{t},\{(Z,\tau)\in\partial\Omega' : ((Z-X_k)/r_k,(\tau-t_k)/r_k^2)\in E\}\right)}{\omega'[\hat{X},\hat{t},\Delta'(X_k,t_k,r_k,r_k)]}.$$

Clearly $(\tilde{\omega}_k)$ is a sequence of uniformly bounded Borel measures on compact subsets of $\mathbb{R}^{n+1}$. Thus a subsequence of $(\tilde{\omega}_k)$ (also denoted $(\tilde{\omega}_k)$) converges weakly to a measure μ. Using (5.6) and the doubling property of $\omega'(\hat{X},\hat{t},\cdot)$, as in [KT1, Lemma 2.1], it is easily seen that if

$$(Y_k,s_k)\in\ \text{supp}\ \omega_k = \partial\tilde{\Omega}_k\cap\{(Y,s): s\leq(\hat{t}-t_k)/r_k^2\},$$

and $\lim_{k\to\infty}(Y_k, s_k) = (\hat{Y}, \hat{s})$, then $(\hat{Y}, \hat{s}) \in \text{supp}\ \mu$. Moreover every $(\hat{Y}, \hat{s}) \in \text{supp}\ \mu$ is the limit of such a sequence, since (ω_k) converges weakly to μ. Using these facts we see that $\tilde{\Omega}_k, \partial\tilde{\Omega}_k$ converge uniformly on compact subsets in the parabolic Hausdorff metric to $\Omega^*, \partial\Omega^*$ with supp $\mu = \partial\Omega^*$. Also $\partial\Omega^*$ satisfies (1.1) with the same δ_0 as Ω'. Using (5.6) and weak convergence, once again, we find that

$$\mu[C_{r_1,r_2}(Y,s) \cap \partial\Omega^*] = r_2^2\, r_1^{n-1} \tag{5.7}$$

whenever $(Y,s) \in \partial\Omega^*$ and $r_1, r_2 > 0$. If $n = 1$ and $(Y,s) \in \partial\Omega^*$, then (5.7) easily implies that $\partial\Omega^* \equiv \{(Y,\tau) : -\infty < \tau < \infty\}$ (so Ω^* is a halfspace). Indeed letting $r_1 \to 0$ in (5.7) and using basic measure theory we deduce

$$\mu\left(\{(Y,\tau) \in \partial\Omega^* : |\tau - s| < r_2^2\}\right) = r_2^2$$

which in view of (5.7), (1.1), implies that

$$\partial\Omega^* \cap (I\!\!R \times (s - r_2^2, s + r_2^2)) = \{(Y,\tau) : |\tau - s| < r_2^2\}.$$

Letting $r_2 \to \infty$ we deduce our implication.

If $n > 1$ we need a more complicated argument to obtain essentially the same conclusion. For this purpose assume $n > 1$ and let $B(Y,s,r) = \{(Z,s) : |Z - Y| < r\}$ be an n dimensional ball in the plane $I\!\!R^n \times \{s\}$. For fixed s, and E Borel $\subset I\!\!R^n \times \{s\}$ set

$$\nu_m(E) = m\mu[E \times (s - 1/m, s + 1/m)].$$

Observe from (5.7) that (ν_m) converges weakly to a measure ν on $I\!\!R^n \times \{s\}$ with

$$\nu(B(Y,s,r)) = r^{n-1} \text{ whenever } (Y,s) \in \partial\Omega^* \cap (I\!\!R^n \times \{s\}). \tag{5.8}$$

From (5.8),(1.1), and a theorem in [KP] (see also [P]), it follows for δ_0 small enough (as in [KT1], Theorem 2.1]) that ν is a constant multiple of $n-1$ dimensional Lebesgue measure on an $n-1$ plane. We assume as we may that this plane contains the origin and has normal $e_0 = (1, 0, \dots, 0)$ since otherwise we change coordinate systems. By the same argument, we see that $\partial\Omega^* \cap (I\!\!R^n \times \{t\})$ is an $n-1$ plane for $-\infty < t < \infty$. We show in fact that all these planes must have normal e_0. Indeed for fixed, s, t we claim for δ_0 small enough that either $\{(X,t) : x_0 > 2|s-t|^{1/2}\}$ or $\{(X,t) : x_0 < -2|s-t|^{1/2}\} \subset \Omega^*$ while the other set is contained in $I\!\!R^{n+1} \setminus \bar{\Omega}^*$. To prove this claim we note from (1.1) with $\Omega = \Omega^*$ and (X,t) replaced by $(0,x,s)$ that the normal to $\hat{P}(0,x,s,r)$ can make an angle at most $c\delta_0$ with e_0 whenever $r > 0$. Using this note we deduce that if our claim is false, then there exists $(Z,t) \in \partial\Omega^*$ with $|z_0| > 2|s-t|^{1/2}$. Also from our note and (1.1) with $r = 2|z_0|$ and (X,t) replaced by $(0,z,s)$, we obtain for δ_0 small enough that either $(Z,t) \in \Omega^*$ or $(Z,t) \in I\!\!R^{n+1} \setminus \Omega^*$ which is a contradiction. Thus our claim is true. From our claim we easily find that the $n-1$ plane which equals, $\partial\Omega^* \cap (I\!\!R^n \times \{t\})$ must have e_0 for a normal. Indeed otherwise this plane would contain points (Z,t) with z_0 arbitrarily large both positive and negative so by the definition of a boundary point it would follow that $\Omega^* \cap (I\!\!R^n \times \{t\})$ has the same property. Hence,

$$\partial\Omega^* = \{(X,t) : x_0 = f(t), (x,t) \in I\!\!R^n\}. \tag{5.9}$$

Moreover since the normal $\hat{n}(X,t,R)$ to $\hat{P}(X,t,R)$ in (1.1) can make at most an angle $c\delta_0$ with e_0 we see from (1.1) that

$$|f(t) - f(\tau)| \le c\delta_0|\tau - t|^{1/2},\ \tau, t \in I\!\!R. \tag{5.10}$$

Let α_{n-1} denote the volume of the unit ball in $I\!R^{n-1}$. From the definition of ν_m, (5.8), (5.9), and a Radon-Nikodym type argument we find for $(Z,\tau) \in I\!R^{n+1}$ that

$$\mu[C_{r_1,r_2}(Z,\tau)] = (2\alpha_{n-1})^{-1} \int_{\tau-r_2^2}^{\tau+r_2^2} H^{n-1}(B(Z,t,r_1) \cap \partial\Omega^*)\, dt.$$

where as usual H^{n-1} denotes $n-1$ dimensional Hausdorff measure. Using this equality with $r_1 = r_2 = r, (Z,\tau) = (f(\tau), z, \tau) \in \partial\Omega^*$, (5.7), (5.9), (5.10) we get

$$r^{n+1} = \mu[C_r(Z,\tau)] = (1/2) \int_{\tau-r^2}^{\tau+r^2} \max\{r^2 - |f(\tau) - f(t)|^2, 0\}^{(n-1)/2}\, dt. \tag{5.11}$$

Clearly, (5.10), (5.11) imply that $f(t) \equiv f(\tau)$ for $t \in (\tau - r^2, \tau + r^2)$. Letting $r \to \infty$ we conclude that $f \equiv$ constant $= f(s) = 0$. Hence $\partial\Omega^*$ is a n dimensional plane containing a line parallel to the t axis. To get a contradiction to our assumption that $C_r(\bar{X},\bar{t}) \cap \partial\Omega'$ is not Reifenberg flat with vanishing constant we note that (5.5) implies

$$\inf_P h[P \cap C_1(0,0), \partial\tilde{\Omega}_k \cap C_1(0,0)] \geq \epsilon, \ k = 1, 2, \ldots.$$

Since $(\partial\tilde{\Omega}_k)$ converges uniformly in the parabolic Hausdorff metric to $\partial\Omega^*$ = a plane containing a line parallel to the t axis, we have reached a contradiction. Hence Proposition 5.1 is true. □.

Next we consider the possibility of replacing the asymptotic optimal doubling assumption in Proposition 5.1 by a weaker assumption. Let $\Omega', \omega', \Delta'(X,t,r_1,r_2)$ be as in the proof of Proposition 5.1 and put $\Delta'(X,t,r) = \Delta'(X,t,r)$. If $(\hat{X},\hat{t}) \in \Omega'$, we say that $\omega'(\hat{X},\hat{t},\cdot)$ is weakly asymptotically optimally doubling if whenever $K \subset \partial\Omega' \cap \{(Y,s) : s < \hat{t}\}$ is compact, and $0 < \tau < 1$ we have

$$\lim_{r \to 0} \sup_{(X,t)\in K} \frac{\omega'(\hat{X},\hat{t},\Delta'(X,t,\tau r))}{\omega'(\hat{X},\hat{t},\Delta'(X,t,r))} = \lim_{r \to 0} \inf_{(X,t)\in K} \frac{\omega'(\hat{X},\hat{t},\Delta'(X,t,\tau r))}{\omega'(\hat{X},\hat{t},\Delta'(X,t,r))} = \tau^{n+1}. \tag{5.12}$$

We would like to know whether Proposition 5.1 is still valid if the asymptotically optimally doubling assumption in this proposition is replaced by (5.12). As a modest step in this direction, we prove

Proposition 5.13. *If $\Omega' \subset I\!R^2$ is a parabolic regular domain, $\omega'(\hat{X},\hat{t},\cdot)$ is weakly asymptotically optimally doubling, and δ_0 is sufficiently small, then $C_r(X,t) \cap \partial\Omega'$ is Reifenberg flat with vanishing constant whenever $(X,t) \in \partial\Omega'$ and $\hat{t} \geq t + 4r^2$.*

Proof: We use the same notation as in Proposition 5.1 and in fact just copy the only if part of the proof of this proposition with $r = r_1 = r_2$ up to (5.7), which is permissible thanks to (5.12). Thus we obtain a parabolic regular domain Ω^* with the same constants as Ω' and a measure μ satisfying (5.7) with $n = 1, r = r_1 = r_2$. From (1.1) we deduce that $\partial\Omega^* = \{(t, f(t)) : t \in I\!R\}$ where f satisfies (5.10). We assume as we may that $\Omega^* = \{(x,t) : x > f(t), t \in I\!R\}$. Also since Ω^* is a parabolic regular domain we claim that

$$D^t_{1/2} f \in \text{ BMO } (I\!R). \tag{5.14}$$

Indeed the proof of this is given in section 2. We shall show in fact that $f \equiv$ constant. To do this we temporarily write x, z for X, Z. Let

$$u_k(z,\tau) = \frac{r_k\, G'(\hat{x},\hat{t},x_k + r_k z, t_k + r_k^2 \tau)}{\omega'[\hat{x},\hat{t},\Delta'(x_k,t_k,r_k)]}$$

whenever $(z,\tau) \in \tilde{\Omega}_k$ where G' denotes Green's function for the adjoint heat equation in Ω' with pole at $(\hat{x},\hat{t})$. Let $u_k \equiv 0$ in the complement of $\tilde{\Omega}_k$. Using interior properties of caloric functions, Lemmas 3.6, 3.10, 3.11, and 3.17, it follows that a subsequence of (u_k) converges uniformly on compact subsets of $\mathbb{R}^2$ to u adjoint caloric in Ω^* with u vanishing continuously on $\mathbb{R}^2 \setminus \Omega^*$. Then u is adjoint subcaloric in $\mathbb{R}^2$ and we can also choose the subsequence so that μ is the Riesz measure corresponding to u in the Riesz representation formula for adjoint subcaloric functions (see [Do]). Next we note from Lemmas 3.10, 3.17 and (5.6) with $\tau_1 = \tau_2$ that

$$u(z,\tau) \leq c\, d(\{(z,\tau)\}, \partial\Omega^*) \text{ whenever } (z,\tau) \in \Omega^*. \tag{5.15}$$

Using interior estimates for the adjoint heat equation we conclude from (5.15) that

$$|u_x(x,t)| \leq c,\ (x,t) \text{ in } \Omega^*. \tag{5.16}$$

We claim for almost every $t \in \mathbb{R}$ with respect to one dimensional Lebesgue measure that

$$\lim_{\lambda \longrightarrow 0^+} u_x(f(t)+\lambda, t) = 1/2. \tag{5.17}$$

To prove this equality, let $\phi \geq 0 \in C_0^\infty[-1,1]$ be an even function with $\int_{\mathbb{R}} \phi = 1$ and put $f_\epsilon = f * \phi_\epsilon$ for $\epsilon > 0$ where $*$ denotes convolution on $\mathbb{R}$ and $\phi_\epsilon(\cdot) = \epsilon^{-1}\phi(\cdot/\epsilon)$. Set

$$u_\epsilon(x,t) \;=\; \epsilon^{-3} \int_{\mathbb{R}^2} u(y,s)\phi[(t-s)/\epsilon^2]\, \phi[(x-y)/\epsilon]\, dyds \text{ for } (x,t) \in \mathbb{R}^2.$$

Using the fact that u is adjoint subcaloric and weak formulations of adjoint subcaloricity similar to (3.8), one deduces that if $d\mu_\epsilon = (u_\epsilon)_{xx} + (u_\epsilon)_t$ then the sequence (μ_ϵ) converges weakly as measures to μ. From (5.10) we note that if $f_\pm(t) = f_\epsilon(t) \pm c\epsilon^{1/2}$, then for c large enough, $\{(f_+(t),t) : t \in \mathbb{R}\} \subset \Omega^*$ while $u_\epsilon \equiv 0$ on $\{(f_-(t),t) : t \in \mathbb{R}\} \subset \mathbb{R}^2 \setminus \Omega^*$. Let $\tau \in \mathbb{R}$ and put

$$T = \{(r,t) : f_-(t) < r < f_+(t), |t-\tau| < \rho\}.$$

Using the above facts and integrating by parts we see that

$$\begin{aligned} \mu_\epsilon(T) = \int_T [(u_\epsilon)_{rr} + (u_\epsilon)_t]\, drdt \;=\; \int_{\tau-\rho}^{\tau+\rho} (u_\epsilon)_r(f_+(t),t)dt \\ - \int_{\tau-\rho}^{\tau+\rho} \tfrac{\partial f_\epsilon}{\partial t}\, u_\epsilon(f_+(t),t)\, dt \;+\; \int_{f_-(\rho+\tau)}^{f_+(\rho+\tau)} u(r,\tau+\rho)dr \\ - \int_{f_-(\rho-\tau)}^{f_+(\rho-\tau)} u(r,\tau-\rho)dr. \end{aligned} \tag{5.18}$$

From (5.15) we deduce that $u_\epsilon \leq c\epsilon^{1/2}$ on ∂T. Also as mentioned above, (5.14) implies (2.8) from which it follows easily (see [HL, Lemma 2.8]) that

$$\epsilon^{1/2} |\tfrac{\partial}{\partial t} f_\epsilon(t)| \, \leq \, c \| D^t_{1/2} f \|_*$$

and

$$\lim_{\epsilon \longrightarrow 0} \epsilon^{1/2} \tfrac{\partial}{\partial t} f_\epsilon(t) = 0$$

for almost every $t \in I\!\!R$. Using these estimates and bounded convergence we see that the last three integrals in (5.18) approach zero as $\epsilon \longrightarrow 0$. Using this fact, (5.7) with $r_1 = r_2 = \rho$, (5.10), weak convergence and (5.18) we find for δ_0 small enough that

$$\begin{aligned} \rho \, &= \, \mu[\partial\Omega^* \cap (I\!\!R \times (\tau - \rho, \tau + \rho))] \\ &= \, \lim_{\epsilon \longrightarrow 0} \mu_\epsilon(T) \, = \, \lim_{\epsilon \longrightarrow 0} \int_{\tau-\rho}^{\tau+\rho} (u_\epsilon)_x (f_+(t), t) dt. \end{aligned} \tag{5.19}$$

Now since u is adjoint subcaloric in $C_{\sqrt{2\rho}}(f(\tau), \tau) \subset I\!\!R^2$, it follows from the Riesz representation formula for adjoint subcaloric functions and the top line in (5.19) (using arbitrariness of ρ, τ) that for $(x, t) \in \Omega \cap C_{\sqrt{2\rho}}(f(\tau), \tau)$ we have

$$u(x, t) = h(x, t) + (1/2) \int_t^{\tau+2\rho} W(x - f(s), s - t) ds$$

where W is the fundamental solution to the heat equation in $I\!\!R^2$ defined at the beginning of section 4 and h is adjoint caloric in $C_{\sqrt{2\rho}}(f(\tau), \tau)$. Differentiating this equality with respect to x and using (4.5), (4.6) we deduce that

$$u_x(f(t), t) = \lim_{(x,s) \longrightarrow (f(t),t)} u_x(x, s) \text{ exists nontangentially for a.e } t \in (\tau - \rho, \tau + \rho). \tag{5.20}$$

Observe from (5.10) that $(u_\epsilon)_x(f_+(t), t)$ represents an average of u_x evaluated at points in the interior of a certain parabolic cone $\subset \Omega^*$, except for its vertex at $(f(t), t)$. Using this fact, (5.19), (5.20), it follows that

$$\int_{\tau-\rho}^{\tau+\rho} u_x(f(t), t) dt = \rho.$$

(5.17) follows from this equality and the Lebesgue differentiation theorem. Next we show that

$$u_x \equiv 1/2 \text{ in } \Omega^*. \tag{5.21}$$

To prove (5.21) we note from (5.14) and [LM, chapter 3] that if $\omega^*(x, t, \cdot)$ denotes adjoint caloric measure for given $(x, t) \in \Omega^*$, then

$$d\omega^*(x, t, f(\tau), \tau))/d\tau = L(x, t, \tau) \text{ for a.e } \tau \in I\!\!R,$$

where $L(x, t, \cdot)$ is positive with integral 1. Using this fact, the fact that the continuous Dirichlet problem has a solution, (5.16), (5.17), and dominated convergence it follows that

$$u_x(x, t) - 1/2 = \lim_{\epsilon \longrightarrow 0} \int_{I\!\!R} (u_x(f(\tau) + \epsilon, \tau) - 1/2) \, L(x, t, \tau) \, d\tau \, = \, 0$$

which proves (5.21). From (5.21) we deduce for $x > f(t)$ that

$$u(x,t) = \int_{f(t)}^{x} u_r(r,t)dr = (x - f(t))/2.$$

Since u is a solution to the adjoint heat equation in Ω^* we conclude that f' exists $\equiv 0$ and so $f \equiv$ constant. Hence Ω^* is a half plane which contradicts (5.5) in exactly the same way as in the proof of the only if part of Proposition 5.1. The proof of Proposition 5.13 is now complete. $\Box$.

Remark: We note that caloric analogues of the theorem in [KP] (alluded to below (5.8)) using (5.7) with $n = 1$ and $r_1 = r = r_2$ are false. To get a counterexample take $E = \{f(t), t) : t \in \mathbb{R}\}$ where f is any function defined on $\mathbb{R}$ with Lip 1/2 norm $\delta < 1$ and take $\mu = \sigma$ defined relative to E as in (1.1) with $n = 1$. Then

$$\mu(C_r(f(t),t) \cap E) = r^2 \text{ for } r > 0, \ -\infty < t < \infty.$$

Thus to prove the last statement in Proposition 5.13 we were forced to use in addition to (5.7) (with $r_1 = r_2$), the information garnered from using the subsolution associated with μ and the adjoint heat equation by way of the Riesz representation theorem. Also we would have preferred to prove Proposition 5.13 with our parabolic regular domain assumption replaced by just (1.1). The problem here is that with this weaker assumption we were not able to show that boundedness of u_x implied adjoint caloric measure was absolutely continuous with respect to Lebesgue measure and then that $u_x \to 1$ nontangentially for μ almost every boundary point. One can reduce this statement to another problem which is a one variable analogue of the Riesz transform problem for uniformly rectifiable sets (see [DS, ch 21], [MMV]). In fact differentiating the Riesz representation formula for u in the display below (5.19) and using boundedness of u_x as well as standard estimates for the heat kernel it follows first that

$$\sup_{\rho > \epsilon > 0} \left| \int_{t+\epsilon}^{t+\rho} \frac{f(t) - f(s)}{(s-t)^{3/2}} \exp[-\frac{(f(t) - f(s))^2}{4(s-t)}]\, ds \right| \leq c \text{ for all } t \in \mathbb{R} \tag{5.22}$$

and thereupon that the operator say $T_\epsilon(\chi_I)$ which one gets from integrating the above integrand against characteristic functions of intervals I (denoted χ_I) is bounded on $L^2(\mathbb{R})$ with constants independent of I and ϵ. So one possible way to prove the proposed stronger version of Proposition 5.13 would be to show that (5.22) and L^2 boundendness as above imply (5.14).

Finally we note that one can also ask whether Proposition 5.13 and the above problems have a solution in $\mathbb{R}^{n+1}, n > 1$. To see the added difficulty in proving Proposition 5.13 we note that the argument in Proposition 5.13 can be repeated to get μ, u, Ω^* as in this proposition. One may even be able to show that (5.7) with $n > 1, r_1 = r_2$ implies that

$$\lim_{(X,t) \to (Y,s)} |\nabla u(X,t)| = 1/2 \text{ nontangentially, for } \mu \text{ a.e } (Y,s) \text{ in } \partial\Omega^*.$$

However we do not know how to get from the the above equality that u is linear or equivalently that Ω^* is a half space. One can also state a higher dimensional analogue of (5.22) and ask if this analogue implies Ω^* is a parabolic regular domain. Finally we note that similar blow up type arguments were used in [KT2] to remove (a) in Theorem 3 for Laplace's equation. In this case however the authors were able to use an important result in [AC] to deduce that the above display implies u is linear. These ‘ free boundary ’ type problems appear harder in the caloric case. Similar problems have been considered in [ACS], [ACS1], where stronger assumptions are made on $\partial\Omega^*$.

6. Preliminary Reductions for Theorem 3. We now begin the proof of Theorem 3. For notational purposes, we prove this theorem only for $n > 1$. Let $\Omega, (\hat{X}, \hat{t}) \in \Omega, \omega(\hat{X}, \hat{t}, \cdot)$, and $k(\hat{X}, \hat{t}, \cdot) = d\omega(\hat{X}, \hat{t}, \cdot)/d\sigma$ be as in the statement of Theorem 3. Throughout this section we write c for a constant which ‘ depends on the data.’ By this statement we mean that after certain constants such as δ_0 in (1.1) are chosen small enough and fixed, then c depends only on n. Recall that Ω is a parabolic regular domain and

$$\begin{aligned}
&(a) \quad \omega(\hat{X}, \hat{t}, \cdot) \text{ is asymptotically optimal doubling,} \\
&(b) \quad \log k(\hat{X}, \hat{t}, \cdot) \in VMO(\partial\Omega), \\
&(c) \quad \|\nu\|_+ \leq \delta' \leq 10^{-10}.
\end{aligned} \tag{6.1}$$

We assume that $\delta_0, \delta' > 0$ are so small that Propositions 2.1, 2.2, 5.1 are valid for Ω. From Proposition 5.1 we see that $\partial\Omega$ is locally Reifenberg flat with vanishing constant as defined at the beginning of section 5. With δ_0 now fixed subject to the above requirements and ν as in (1.6), let $\|\nu\|(\Delta(X, t, r) \times [0, r])$ be the Carleson norm of ν on $\Delta(X, t, r) \times [0, r]$ when $(X, t) \in \partial\Omega$ and $r > 0$. As in [KT1] we prove the following proposition.

Proposition 6.2. *For Ω as above there exists $c \geq 1$ such that if $c^{-1} = \delta'' \leq \delta'$, then for $0 < \delta \leq \delta''$, the following statement is true. If $r_0 > 0, F$ is a compact subset of $\partial\Omega \cap \{(Y, s) : s < \hat{t} - 4r_0^2\}$ and*

$$\sup_{(X,t)\in F} \|\nu\|(\Delta(X, t, r_0) \times [0, r_0]) \leq \delta \leq \delta'',$$

then there exists $\hat{r} < r_0$ such that

$$\sup_{(X,t)\in F} \|\nu\|(\Delta(X, t, \hat{r}) \times [0, \hat{r}]) \leq \delta/2.$$

Proof: Theorem 3 follows easily from Proposition 6.2 and iteration. Thus to complete the proof of Theorem 3 we need only prove Proposition 6.2. To accomplish this we first observe from (6.1)(b) as in [KT1, Lemma 5.6] that for each $\epsilon > 0$ there exists $r_1, 0 < r_1 < r_0$, for which the following statement is true. If $\rho < r_1$ and $(X, t) \in \Delta(Y, s, r_1)$ for some $(Y, s) \in F$, then there exists $F_1 = F_1(X, t, \rho)$, a closed subset of $\Delta(X, t, \rho)$ with

$$1 - \epsilon \leq \frac{k(\hat{X}, \hat{t}, Z, \tau)}{\displaystyle ⨍_{\Delta(X,t,\rho)} k(\hat{X}, \hat{t}, Y, s) d\sigma(Y, s)} \leq 1 + \epsilon \text{ whenever } (Z, \tau) \in F_1(X, t, \rho). \tag{6.3}$$

Moreover, (see [KT1, (5.56)])

$$\int_{\Delta(X,t,\rho)\setminus F_1(X,t,\rho)} k(\hat{X}, \hat{t}, Y, s)^2 \, d\sigma(Y, s) \leq \epsilon \int_{\Delta(X,t,\rho)} k(\hat{X}, \hat{t}, Y, s)^2 \, d\sigma(Y, s). \tag{6.4}$$

From (6.3), (6.4) we deduce for ϵ small enough that

$$\frac{\displaystyle\fint_{\Delta(X,t,\rho)} k(\hat{X},\hat{t},Y,s)^2\,d\sigma(Y,s)}{\left(\displaystyle\fint_{\Delta(X,t,\rho)} k(\hat{X},\hat{t},Y,s)d\sigma(Y,s)\right)^2} \leq 1+4\epsilon \tag{6.5}$$

whenever $(X,t)\in\Delta(Y,s,r_1)$ for some $(Y,s)\in F$ and $0<\rho\leq r_1$. Also from Lemma 5.1 we see that if r_1 is small enough, then for each $(X,t),\rho$ as above there exists a plane $P=P(X,t,\rho)$ containing a line parallel to the t axis for which

$$h(P\cap C_\rho(X,t),\Delta(X,t,\rho))\leq \epsilon\,\rho \tag{6.6}$$

where h as usual denotes the parabolic Hausdorff distance between the above two sets. Let $\beta=\max\{20\gamma^{-1},20\alpha^{-1},1000\}$ where γ,α are as in Lemmas 3.19, 3.6, respectively. Fix $(X,t)\in\Delta(Y,s,\delta^\beta r_1)$ for some $(Y,s)\in F$ and $r\leq\delta^\beta r_1$. Put $\theta=\delta^{\beta/4}$ and let $(U_2,t_2)=(U_2^*(X,t,r/\theta),$ $t_2^*(X,t,r/\theta))$ be as defined in (3.4). We suppose ϵ is so small that

$$\epsilon\leq\exp(-\delta^{-\beta}). \tag{6.7}$$

Then as in (4.13), (4.14) we deduce from Lemma 3.19, (6.5), (6.7) for some $c\geq 1$ that

$$\begin{aligned}&\frac{\displaystyle\fint_{\Delta(X,t,r)} k(U_2,t_2,Y,s)^2\,d\sigma(Y,s)}{\left(\displaystyle\fint_{\Delta(X,t,r)} k(U_2,t_2,Y,s)d\sigma(Y,s)\right)^2}\\ &\qquad\leq (1+c\delta^5)\frac{\displaystyle\fint_{\Delta(X,t,r)} k(\hat{X},\hat{t},Y,s)^2\,d\sigma(Y,s)}{\left(\displaystyle\fint_{\Delta(X,t,r)} k(\hat{X},\hat{t},Y,s)d\sigma(Y,s)\right)^2}\leq 1+c\delta^5.\end{aligned} \tag{6.8}$$

Next we apply Proposition 2.2 with $R=r/\theta^2$. We assume as we may that $\hat{P}(X,t,\rho)$ in (1.1) is equal to $P(X,t,\rho)$ in (6.6) for $0<\rho\leq r_1$. Also after a rotation if necessary we may assume (as in Proposition 2.2) that $\hat{n}(X,t,r/\theta^2)=e_0=(1,0,\dots,0)$. Using Proposition 2.2 we get $\Omega_-=\{(Y,s): y_0>\psi_-(y,s)\}$ for which

(a) ψ_- satisfies (1.13), (1.14) with $b_1+b_2=b\leq c_2\delta^a$, where $a=\kappa^{-1}$,

(b) $\sigma\left(\Delta(X,t,r/\theta^2)\setminus\partial\Omega_-\right)+\sigma\left(\partial\Omega_-\cap C_{r/\theta^2}(X,t)\setminus\Omega\right)\leq e^{-1/(c_2\delta^a)}(r/\theta^2)^{n+1}$.

(c) $\Delta(X,t,2r/\theta^2)=G\cup B$ where $G\subset\partial\Omega_-$ and $\sigma(B)\leq e^{-1/(c_2\delta^a)}\,(r/\theta^2)^{n+1}$.

(d) If p denotes the orthogonal projection of $\mathbb{R}^{n+1}$ on $\{0\}\times\mathbb{R}^n$, and $(Y,s)\in B$, then $|y_0-\psi(y,s)|\leq c_2\,\delta\,d(\{p(Y,s)\},p(G))$,

(e) $\Omega_-\subset\Omega\cap C_{2r/\theta^2}(X,t)$

(6.9)

Let ω_- denote caloric measure for the heat equation in Ω_- and put

$$\Delta_-(Z,\tau,r') = C_{r'}(Z,\tau)\cap\partial\Omega_-,$$

$$d\sigma_-(Z,\tau) \;=\; \sqrt{1+|\nabla\psi_-|^2}(z,\tau)dzd\tau$$

$$k_-(Z,\tau,\cdot) = d\omega_-(Z,\tau,\cdot)/d\sigma_-.$$

We note that if δ' in (6.1) (c) is small enough, then it follows from (4.1),(4.9),(4.10) and (3.19) as in (4.31) - (4.34) that

$$\frac{\displaystyle\fint_{\Delta_-(X,t,r)} k_-(U_2,t_2,Y,s)^4\, d\sigma_-(Y,s)}{\left(\displaystyle\fint_{\Delta_-(X,t,r)} k_-(U_2,t_2,Y,s)d\sigma_-(Y,s)\right)^4} \;\le\; 100 \tag{6.10}$$

Next we claim for some $c\ge 1$ that if $E = p(\Delta_-(X,t,r))$, $r_\pm = [\,1\pm e^{-1/(c\delta^a)}\,]\,r$ and if,

$$C_\rho(z,\tau) = \{(y,s) : |y-z|<\rho, |s-\tau|<\rho^2\,\},$$

then

$$C_{r_-}(x,t)\subset E\subset C_{r_+}(x,t). \tag{6.11}$$

In fact from (6.6) we see that

$$|\psi_(y,s) - x_0| \le c\epsilon r/\theta^2$$

whenever $(\psi_-(y,s),y,s)\in\partial\Omega_-\cap C_{r/\theta^2}(X,t)\cap\partial\Omega$. Indeed each point of $\partial\Omega$ must lie within $c\epsilon r/\theta^2$ of $\hat P(X,t,r/\theta^2)$ as we see from (6.6). Now if $(\psi_-(y,s),y,s)\in\partial\Omega_-\cap C_{r/2\theta^2}(X,t)\setminus\partial\Omega$, then for some $c\ge 1$, this point must lie at most $e^{-1/(c\delta^a)}r$ in parabolic distance from a point in $\partial\Omega_-\cap C_{r/\theta^2}(X,t)\cap\partial\Omega$ thanks to (b) of (6.9). Using this fact, (6.7), and (6.9)(a) we get

$$|x_0-\psi_-(y,s)| \;\le\; e^{-1/(c\delta^a)}\,r \tag{6.12}$$

whenever $(Y,s)\in C_{r/2\theta^2}(X,t)\cap\partial\Omega_-$. Clearly (6.12) implies (6.11) for δ' small enough. We also set

$$Q_\rho(z,\tau) = \{(y,s)\in I\!\!R^n : |y_i - z_i|<\rho,\ 1\le i\le n-1,\ |s-\tau|<\rho^2\}\text{ whenever }(z,\tau)\in I\!\!R^n, \rho>0.$$

Let $r^* = (1-100n\theta)r$. We cover $C_{r^*}(x,t)$ by a grid of parabolic rectangles (as in the above display) with disjoint interiors and sidelength $\theta^2 r$ in the space direction. Let Λ denote the family of all such rectangles which have a nonempty intersection with $C_{r^*}(x,t)$. Using (6.4)-(6.10) we can now repeat the argument in (4.17) - (4.25) with $\Delta(X,t,R)$ replaced by $p^{-1}(Q)\cap\partial\Omega$ whenever $Q\in\Lambda$. Also $\hat p = 2$ and k plays the role of k_+ while k_- plays the role of k in these inequalities. Arguing as in

(4.17) - (4.25) we find for δ' sufficiently small that

$$\frac{\displaystyle \fint_{p^{-1}(Q)\cap\partial\Omega_-} k_-(U_2,t_2,Y,s)^2\, d\sigma_-(Y,s)}{\left(\displaystyle \fint_{p^{-1}(Q)\cap\partial\Omega_-} k_-(U_2,t_2,Y,s) d\sigma_-(Y,s)\right)^2}$$

$$\leq\ (1+c\delta^5)\ \frac{\displaystyle \fint_{p^{-1}(Q)\cap\partial\Omega} k(U_2,t_2,Y,s)^2\, d\sigma(Y,s)}{\left(\displaystyle \fint_{p^{-1}(Q)\cap\partial\Omega} k(U_2,t_2,Y,s) d\sigma(Y,s)\right)^2} \leq\ 1+c\delta^5 \tag{6.13}$$

and

$$1-c\delta^5\ \leq\ \frac{\omega_-(U_2,t_2,H_-)}{\omega(U_2,t_2,H)}\ \leq\ 1+c\delta^5 \tag{ 6.14}$$

whenever either $H=p^{-1}(Q)\cap\partial\Omega$, $H_-=p^{-1}(Q)\cap\partial\Omega_-$ or $H=\Delta(X,t,r), H_-=\Delta_-(X,t,r)$.

Next put $\bar r = r(1+\theta^{2n+2})$ and set

$$\tilde\psi(y,s) = (\psi_-(y,s)-\psi_-(x,t))\phi(y,s)\ +\ \psi_-(x,t),$$

where $\phi\in C_0^\infty[C_{\bar r}(x,t)]$ with $0\leq\phi\leq 1, \phi\equiv 1$ on $C_r(x,t)$, and

$$(\theta^{2n+2}r)^2\,|\tfrac{\partial\phi}{\partial s}| + \theta^{2n+2}r\sum_{i=0}^{n-1}|\tfrac{\partial\phi}{\partial y_i}|\ \leq\ c$$

at each $(y,s)\in I\!\!R^n$. We claim that

$$\tilde\psi \text{ satisfies (1.13), (1.14) with } b_1+b_2\leq c\,\delta^a. \tag{6.15}$$

In fact the desired estimate in (1.13) follows easily from the corresponding estimate for ψ_- in (6.9) (a) and (6.12). To get the desired estimate in (1.14) we first observe that (1.14) in (6.15) is equivalent to showing (see [St])

$$\int_{C_\rho(z,\tau)}\int_{\tau-\rho^2}^{\tau+\rho^2}\left(\frac{\tilde\psi(y,t)-\tilde\psi(y,s)}{t-s}\right)^2 dt\,dyds\ \leq\ c\delta^{2a}\rho^{n+1} \tag{6.16}$$

whenever $(z,\tau)\in I\!\!R^n$, $\rho>0$. Moreover, (6.16) is easily checked using the corresponding inequality for ψ_- and (6.12). Let $\tilde\Omega=\{(Y,s): y_0>\tilde\psi(y,s)\}$, and put

$$\tilde\Delta(Z,\tau,r') = C_{r'}(Z,\tau)\cap\partial\tilde\Omega,$$

$$d\tilde\sigma(Z,\tau)\ =\ \sqrt{1+|\nabla\tilde\psi|^2}(z,\tau)dzd\tau$$

$$\tilde k(Z,\tau,\cdot) = d\tilde\omega(Z,\tau,\cdot)/d\tilde\sigma.$$

From the above claim, the fact that $\Delta_-(X,t,r) = \tilde{\Delta}(X,t,r)$ and Lemma 3.19 with u, v the adjoint Green functions for $\Omega, \tilde{\Omega}$ with pole at (U_2, t_2), we see that (6.10), (6.13) are valid with $\partial\Omega_-, \psi_-, \omega_-, k_-,$ replaced by $\partial\tilde{\Omega}, \tilde{\psi}, \tilde{\omega}, \tilde{k}$. Also (6.14) is valid with the above replacements for $H = p^{-1}(Q) \cap \partial\Omega$, and H_- replaced by $p^{-1}(Q) \cap \partial\tilde{\Omega}$. Using this version of (6.13), (6.14), and (6.8), we deduce that

$$\begin{aligned}
&\sum_{Q\in\Lambda} \int_{p^{-1}(Q)\cap\partial\tilde{\Omega}} \tilde{k}(U_2, t_2, Y, s)^2 \, d\tilde{\sigma}(Y, s) \\
&\leq (1 + c\delta^5) \sum_{Q\in\Lambda} \tilde{\sigma}(p^{-1}(Q) \cap \partial\tilde{\Omega})^{-1} \left(\int_{p^{-1}(Q)\cap\partial\tilde{\Omega}} \tilde{k}(U_2, t_2, Y, s) d\tilde{\sigma}(Y, s) \right)^2 \\
&\leq (1 + c\delta^5) \sum_{Q\in\Lambda} \sigma(p^{-1}(Q) \cap \partial\Omega)^{-1} \left(\int_{p^{-1}(Q)\cap\partial\Omega} k(U_2, t_2, Y, s) \, d\sigma(Y, s) \right)^2 \\
&\leq \Theta(1 + c\delta^5) \sum_{Q\in\Lambda} \sigma(p^{-1}(Q) \cap \partial\tilde{\Omega})^{-1} \left(\int_{p^{-1}(Q)\cap\partial\Omega} k(\hat{X}, \hat{t}, Y, s) \, d\sigma(Y, s) \right)^2
\end{aligned} \tag{6.17}$$

where

$$\Theta = \left(\frac{\omega(U_2, t_2, \Delta(X, t, r))}{\omega(\hat{X}, \hat{t}, \Delta(X, t, r))} \right)^2 = \left(\frac{\fint_{\Delta(X,t,r)} k(U_2, t_2, Y, s) \, d\sigma(Y, s)}{\fint_{\Delta(X,t,r)} k(\hat{X}, \hat{t}, Y, s) d\sigma(Y, s)} \right)^2$$

From (6.3), (6.4) we find as in (6.5) that

$$\fint_{p^{-1}(Q)\cap\partial\Omega} k(\hat{X}, \hat{t}, Y, s) \, d\sigma(Y, s) \leq (1 + c\epsilon\theta^{-(2n+2)}) \fint_{\Delta(X,t,r)} k(\hat{X}, \hat{t}, Y, s) d\sigma(Y, s).$$

From (6.17), the above inequality, and the fact that $c\epsilon\theta^{-(2n+2)} \leq c\delta^5$ for δ' small enough (see (6.7)), we conclude that

$$\begin{aligned}
&\sum_{Q\in\Lambda} \int_{p^{-1}(Q)\cap\partial\tilde{\Omega}} \tilde{k}(U_2, t_2, Y, s)^2 \, d\tilde{\sigma}(Y, s) \\
&\leq (1 + c\delta^5)\sigma(\Delta(X, t, r)) \left(\fint_{\Delta(X,t,r)} k(U_2, t_2, Y, s) \, d\sigma(Y, s) \right)^2.
\end{aligned} \tag{6.18}$$

Observe from (6.11) and the definition of Λ that if $\tilde{\Gamma} = \bigcup_{Q\in\Lambda} p^{-1}(Q \cap \partial\tilde{\Omega})$ then $\sigma(\Delta(X, t, r) \setminus \tilde{\Gamma}) \leq c\theta\, r^{n+1}$ so from Hölder's inequality and (6.10) for $\tilde{k}$ we get

$$\begin{aligned}
&\int_{\tilde{\Delta}(X,t,r)\setminus\tilde{\Gamma}} \tilde{k}(U_2, t_2, Y, s)^2 \, d\tilde{\sigma}(Y, s) \leq c \left(\int_{\tilde{\Delta}(X,t,r)} \tilde{k}(U_2, t_2, Y, s)^4 \, d\tilde{\sigma}(Y, s) \right)^{1/2} \theta^{1/2}\, r^{(n+1)/2} \\
&\leq c\delta^5 \tilde{\sigma}(\tilde{\Delta}(X, t, r))^{-1} \left(\int_{\tilde{\Delta}(X,t,r)} \tilde{k}(U_2, t_2, Y, s) \, d\tilde{\sigma}(Y, s) \right)^2.
\end{aligned} \tag{6.19}$$

Similarly from (6.8) and Hölder's inequality we get for $\Gamma = \bigcup_{Q\in\Lambda} p^{-1}(Q\cap\partial\Omega)$ that

$$\int_{\Delta(X,t,r)\setminus\Gamma} k(U_2,t_2,Y,s)\,d\sigma(Y,s) \;\le\; c\delta^5 \int_{\Delta(X,t,r)} k(U_2,t_2,Y,s)\,d\sigma(Y,s)$$

From this inequality and (6.14) with $H = p^{-1}(Q)\cap\partial\Omega$ we find after summing that

$$\begin{aligned}\int_{\Delta(X,t,r)} k(U_2,t_2,Y,s)\,d\sigma(Y,s) \;&\le\; (1+c\delta^5)\int_{\Gamma} k(U_2,t_2,Y,s)\,d\sigma(Y,s)\\ &\le\; (1+c\delta^5)\int_{\tilde\Delta(X,t,r)} \tilde k(U_2,t_2,Y,s)\,d\tilde\sigma(Y,s).\end{aligned} \tag{6.20}$$

Using (6.19), (6.20) in (6.18) and (6.9) (b), (6.12) once again we deduce that

$$\frac{\fint_{\tilde\Delta(X,t,r)} \tilde k(U_2,t_2,Y,s)^2\,d\tilde\sigma(Y,s)}{\left(\fint_{\tilde\Delta(X,t,r)} \tilde k(U_2,t_2,Y,s)d\tilde\sigma(Y,s)\right)^2} \;\le\; 1+c\delta^5. \tag{6.21}$$

If $f:\tilde\Omega\to\mathbb{R}$ is integrable on $\tilde\Delta(X,t,r)$ let

$$f_{\tilde\Delta} \;=\; \fint_{\tilde\Delta(X,t,r)} f\,d\tilde\sigma(Y,s)$$

and note from (6.21) that

$$\int_{\tilde\Delta(X,t,r)} (\tilde k(U_2,t_2,Y,s) - \tilde k_{\tilde\Delta})^2\,d\tilde\sigma(Y,s) \;\le\; c\delta^5|\tilde k_{\tilde\Delta}|^2 r^{n+1} \tag{6.22}$$

where $\tilde k_{\tilde\Delta} = \tilde k(U_2,t_2,\cdot)_{\tilde\Delta}$. To continue the argument observe from (4.9)-(4.11) with $(\hat X,\hat t), k_+$ replaced by $(U_2,t_2), \tilde k$, that

$$(1/2\;-\;K)\,\tilde k(U_2,t_2,\cdot) = -\tfrac{\partial}{\partial n_\tau} W(U_2-\cdot,t_2-\cdot)$$

Where K is the singular integral operator on $L^2(\partial\tilde\Omega)$ defined relative to $\tilde\psi$ (see the display above (4.1)). From (4.1) and (6.15) we see that

$$\|Kf\|_2 \;\le\; c(\delta)\,\|f\|_2 \;\text{ where } c(\delta)\to 0, \text{ as } \delta\to 0. \tag{6.23}$$

If δ' in (6.1) is small enough, then from (6.23) we deduce that

$$\tilde k(U_2,t_2,\cdot) \;=\; 2\sum_{l=0}^{\infty}(2K)^l\,[-\tfrac{\partial}{\partial n_\tau}W(U_2-\cdot,t_2-\cdot)] \tag{6.24}$$

in $L^2(\partial\tilde\Omega)$, where $(2K)^l$ denotes composition of the operator $2K$ with itself l times. Next let χ denote the characteristic function of $\tilde\Delta(X,t,r)$ and set $h(Z,\tau) = -\frac{\partial}{\partial n_\tau}W(U_2-Z,t_2-\tau)$. Using

(6.22) -(6.24), we get for δ' small enough that

$$
\begin{aligned}
&\| [Kh - (Kh)_{\tilde\Delta}]\,\chi\,\|_2 \\
&\leq c\left[\sum_{l=2}^{\infty}\|K^l h\|_2 + \|(h-h_{\tilde\Delta})\,\chi\,\|_2 + \|[\tilde k(U_2,t_2,\cdot)-\tilde k_{\tilde\Delta}]\chi\|_2\right] \qquad (6.25)\\
&\leq c(\delta)\|Kh\|_2 + c\|(h-h_{\tilde\Delta})\chi\|_2 + c\delta^{5/2}\,|\tilde k_{\tilde\Delta}|\,r^{(n+1)/2},
\end{aligned}
$$

where as in section 3, $\|\cdot\|_2$ denotes the L^2 norm of functions square integrable with respect to $\tilde\sigma$. We use (6.25) to show that

$$\Phi = \int_{\mathbb{R}^n} (|D^t_{1/2}\tilde\psi|^2 + |\nabla\tilde\psi|^2)\,dyds \leq c\,\delta^5\,r^{n+1}. \qquad (6.26)$$

Unfortunately to get (6.26) we shall need a number of error estimates. Our work is complicated by the fact that theorems in [H] and [LM] only give an estimate of the form $\|Kh\|_2 \leq c\delta^a r^{(n+1)/2}$ which is not good enough for our purposes. Let $(Z,\tau) = (\tilde\psi(z,\tau),z,\tau) \in \tilde\Delta(X,t,4r)$, $U_2 = (u_0,\dots,u_{n-1}) = (u_0,u)$ and set

$$
\begin{aligned}
&(a)\quad (1+|\nabla\tilde\psi|^2)^{-1/2}(z,\tau) = 1-(1/2)|\nabla\tilde\psi(z,\tau)|^2 + E_1(Z,\tau)\\
&(b)\quad \frac{u_0-\tilde\psi(z,\tau)+\langle\nabla\tilde\psi(z,\tau),z-u\rangle}{(t_2-\tau)^{n/2+1}}\exp\left[-\frac{|Z-U_2|^2}{4(t_2-\tau)}\right] \qquad (6.27)\\
&\qquad = e^{-1/4}(\theta/r)^{n+1} + E_2(Z,\tau).
\end{aligned}
$$

Using (6.15), the definition of $(U_2,t_2),\theta$ and (6.12) we deduce for $(Z,\tau)\in\tilde\Delta(X,t,4r)$,

$$
\begin{aligned}
&|E_1|(Z,\tau) \leq c\delta^{2a}|\nabla\tilde\psi|^2(Z,\tau)\\
&|E_2|(Z,\tau) \leq c\theta(\theta/r)^{n+1} \leq c\delta^5(\theta/r)^{n+1}.
\end{aligned} \qquad (6.28)
$$

Also we note that

$$
\begin{aligned}
&h(Z,\tau) = 2^{-1}(4\pi)^{-n/2}(1+|\nabla\tilde\psi|^2)^{-1/2}(z,\tau)\\
&\left[\frac{u_0-\tilde\psi(z,\tau)+\langle\nabla\tilde\psi(z,\tau),z-u\rangle}{(t_2-\tau)^{n/2+1}}\right]\exp\left[-\frac{|Z-U_2|^2}{4(t_2-\tau)}\right].
\end{aligned}
$$

Using this note we see that

$$|h(Z,\tau)| \leq c(\theta/r)^{n+1},\ (Z,\tau)\in\partial\tilde\Omega. \qquad (6.29)$$

Moreover from the definition of h and (6.27)-(6.29) we deduce first that

$$h_{\tilde\Delta} = \kappa_1\,(\theta/r)^{n+1}\left[1-(1/2)\fint_{\tilde\Delta(X,t,r)}|\nabla\tilde\psi|^2\,d\tilde\sigma\right] + E_3 \qquad (6.30)$$

where $\kappa_1 = 2^{-1}\,(4\pi)^{-n/2}\,e^{-1/4}$ and

$$|E_3| \leq (\theta/r)^{n+1}\,[\,c\delta^5 \,+\, c\delta^{2a} \fint_{\tilde{\Delta}(X,t,r)} |\nabla\tilde{\psi}|^2\,d\tilde{\sigma}\,]. \tag{6.31}$$

Second we get

$$\begin{aligned}\int_{\tilde{\Delta}(X,t,r)} (h - h_{\tilde{\Delta}})^2\,d\tilde{\sigma} &= \int_{\tilde{\Delta}(X,t,r)} (h^2 - h^2_{\tilde{\Delta}})\,d\tilde{\sigma} \\ &\leq\, c\,[\,(\theta/r)^{2n+2}\,\delta^5\,r^{n+1} \,+\, \delta^{2a}\Phi\,].\end{aligned} \tag{6.32}$$

Also using (6.30), (6.23), and (6.24) we find that

$$\begin{aligned}|\tilde{k}_{\tilde{\Delta}}| \,&\leq\, 2|h_{\tilde{\Delta}}| \,+\, c\,\|Kh\|_2\;\tilde{\sigma}[\tilde{\Delta}(X,t,r)]^{-1/2} \\ &\leq\, c(\theta/r)^{n+1} + c\,\|Kh\|_2\,r^{-(n+1)/2}.\end{aligned} \tag{6.33}$$

Using (6.33) and (6.32) in (6.25) we conclude that

$$\begin{aligned}&\|[Kh - (Kh)_{\tilde{\Delta}}]\chi\|_2 \\ &\leq\, c(\delta)\,\|Kh\|_2 \,+\, c\,(\theta/r)^{n+1}\,[\delta^{5/2}\,r^{(n+1)/2} \,+\, \delta^a\,\Phi^{1/2}].\end{aligned} \tag{6.34}$$

We now regard h, Kh as being defined on $\mathbb{R}^n$ and shall write $h(z,\tau), Kh(z,\tau)$ for $h(Z,\tau), Kh(Z,\tau)$ when $(Z,\tau) = (\tilde{\psi}(z,\tau), z, \tau)$. We shall also write $\|\cdot\|_p$ for the norm in $L^p(\mathbb{R}^n), 1 \leq p \leq \infty$. Finally we write $f_{\tilde{C}}$ for the average of f with respect to Lebesgue n measure on $\tilde{C} = C_{\tilde{r}}(x,t)$. To continue the proof of (6.26), we prove

Lemma 6.35. *With the above notation and for δ' small enough, we have*

$$\|\tilde{\chi}\,Kh\|_2 \,\leq\, c\,|(\sqrt{1+|\nabla\tilde{\psi}|^2}Kh)_{\tilde{C}}|\,r^{(n+1)/2} \,+\, (\theta/r)^{n+1}\,[\delta^{5/2}\,r^{(n+1)/2} \,+\, \delta^a\,\Phi^{1/2}\,].$$

Proof: To begin the proof of Lemma 6.35 we note from the display preceding (4.1) that if $(Z,\tau) \in \partial\tilde{\Omega}$ and

$$\hat{h} = (1/2)(4\pi)^{-n/2}\sqrt{1+|\nabla\tilde{\psi}|^2}\,h,$$

then

$$Kh(z,\tau) \,=\, \lim_{\epsilon\longrightarrow 0} K_\epsilon h \,=\, \text{PV}\int_{\mathbb{R}^n} \hat{K}(z,\tau,y,s)\,\hat{h}(y,s)\,dyds,$$

where if $s > \tau$

$$\begin{aligned}\hat{K}(z,\tau,y,s) \,&=\, (1+|\nabla\tilde{\psi}|^2)^{-1/2}(z,\tau)\,\left[\frac{\tilde{\psi}(z,\tau) - \tilde{\psi}(y,s) + \langle\nabla\tilde{\psi}(z,\tau), y-z\rangle}{(s-\tau)^{n/2+1}}\right] \\ &\cdot \exp\left[-\frac{|y-z|^2 + |\tilde{\psi}(y,s) - \tilde{\psi}(z,\tau)|^2}{4(s-\tau)}\right]\end{aligned} \tag{6.36}$$

and $\hat{K}(z,\tau,y,s) \equiv 0$ when $s < \tau$. From now on all integrals will be in the principal value sense, defined before (4.1) and we shall suppress the PV symbol. Recall from the definition of $\tilde{\psi}$ above (6.15) that $\tilde{\psi} \equiv \tilde{\psi}(x,t)$ on $\mathbb{R}^n \setminus C_{\tilde{r}}(x,t)$. thus if $A^+ = \max\{A, 0\}$, then for $(z,\tau) \in \mathbb{R}^n \setminus C_{\tilde{r}}(x,t)$,

$$\sqrt{1+|\nabla\tilde{\psi}|^2}\, Kh(z,\tau) = \int_{C_{\tilde{r}}(x,t)} \frac{\tilde{\psi}(x,t)-\tilde{\psi}(y,s)}{[(s-\tau)^+]^{n/2+1}}$$

$$\cdot \exp\left[-\frac{|y-z|^2+|\tilde{\psi}(y,s)-\tilde{\psi}(x,t)|^2}{4(s-\tau)^+}\right] \hat{h}(y,s)dyds \tag{6.37}$$

$$\leq c\, r\, e^{-1/(c\delta^a)} \int_{C_{\tilde{r}}(x,t)} [(s-\tau)^+]^{-n/2-1} \exp\left[-\frac{|y-z|^2}{4(s-\tau)^+}\right]\ |h|(y,s)dyds,$$

thanks to (6.12). Here the integrand is understood to be 0 when $s < \tau$. Integrating (6.37) and using (6.29) we find that if $r' = \tilde{r} + \theta^{2n+2} r$, then for some $c \geq 1$,

$$\int_{\mathbb{R}^n \setminus C_{r'}(x,t)} |Kh|^2(z,\tau)\ dzd\tau\ \leq\ c\, e^{-1/(c\delta^a)}\, r^{-(n+1)}. \tag{6.38}$$

On the other hand since K is bounded on $L^4(\mathbb{R}^n)$, it follows from Hölder's inequality, the definition of h, and (6.29) that

$$\begin{aligned}\int_{C_{r'}(x,t)\setminus C_{\tilde{r}}(x,t)} |Kh|^2\, dzd\tau\ &\leq\ c\theta^{n+1} r^{(n+1)/2} \|Kh\|_4^2 \\ &\leq c\theta^{n+1} r^{(n+1)/2} \|h\|_4^2 \leq\ c\theta^{5\,(n+1)/2}\ r^{-(n+1)}.\end{aligned} \tag{6.39}$$

Let $\tilde{\chi}$ denote the characteristic function of $\tilde{C} = C_{\tilde{r}}(x,t)$. Arguing as above and using (6.11) we first see that

$$\int_{C_{\tilde{r}}(x,t)} (Kh)^2\, dzd\tau\ \leq\ c \int_{\tilde{\Delta}(X,t,r)} (Kh)^2\, d\sigma\ +\ c\theta^{5(n+1)/2}\, r^{-(n+1)} \tag{6.40}$$

and second that

$$|(Kh)_{\tilde{\Delta}}| \leq c\,|(\sqrt{1+|\nabla\tilde{\psi}|^2} Kh)_{\tilde{C}}| + c\theta^{5(n+1)/4} r^{-(n+1)/2}. \tag{6.41}$$

Using (6.38)-(6.41) and (6.34) we find

$$\|\tilde{\chi}\, Kh\|_2\ \leq\ c\|\chi\, Kh\|_2 + c\delta^{5/2}(\theta^2/r)^{(n+1)/2}$$

$$\leq\ c\|[Kh - (Kh)_{\tilde{\Delta}}]\,\chi\|_2 + cr^{(n+1)/2}\,|(\sqrt{1+|\nabla\tilde{\psi}|^2} Kh)_{\tilde{C}}|\ +\ c\delta^{5/2}\,(\theta^2/r)^{(n+1)/2}$$

$$\leq\ c(\delta)\|\tilde{\chi} Kh\|_2\ +\ c\,|(\sqrt{1+|\nabla\tilde{\psi}|^2} Kh)_{\tilde{C}}|\, r^{(n+1)/2}\ +\ (\theta/r)^{n+1}\,[\delta^{5/2}\, r^{(n+1)/2}\ +\ \delta^a\, \Phi^{1/2}],$$

where the norms involving χ are in $L^2(\partial\tilde{\Omega})$. Hence for δ' small enough we have

$$\|\tilde{\chi}\, Kh\|_2\ \leq\ c\,|(\sqrt{1+|\nabla\tilde{\psi}|^2} Kh)_{\tilde{C}}|\, r^{(n+1)/2}\ +\ (\theta/r)^{n+1}\,[\delta^{5/2}\, r^{(n+1)/2}\ +\ \delta^a\, \Phi^{1/2}\,]. \tag{6.42}$$

Thus Lemma 6.35 is true. □

To outline the rest of the proof of (6.26) we shall now decompose K into successive operators, say Λ, each which will be simpler than the previous one and for which Lemma 6.35 is valid with K replaced by Λ and h by $\hat{h}$. Finally in section 7 we will obtain an operator K_{11} for which (6.42) is valid with K replaced by K_{11}. Also the average of K_{11} over $\tilde{C}$ will have enough cancellation so that it can be absorbed into the second term on the right hand side of (6.42). Finally we will use Plancherel's theorem to obtain $\Phi \leq c\|K_{11}\tilde{\chi}\|_2$ which in view of the revised form of (6.42) implies (6.26). Each of our operators can be estimated using the argument in either [H] or [LM, ch 1]. For precise references we shall refer to [LM]. We write $Kh = K_1\hat{h} + K_2\hat{h}$ where the kernels of K_1, K_2 are given by

$$\begin{aligned} \hat{K}_1(z,\tau,y,s) \;&=\; (1+|\nabla\tilde{\psi}|^2)^{-1/2}(z,\tau)\;\; \frac{\tilde{\psi}(z,\tau)-\tilde{\psi}(y,s)}{(s-\tau)^{n/2+1}} \\ &\quad\cdot \exp\left[-\frac{|y-z|^2+|\tilde{\psi}(y,s)-\tilde{\psi}(z,\tau)|^2}{4(s-\tau)^+}\right] \end{aligned} \tag{6.43}$$

and

$$\begin{aligned} \hat{K}_2(z,\tau,y,s) \;&=\; (1+|\nabla\tilde{\psi}|^2\,)^{-1/2}(z,\tau)\;\; \frac{\langle\nabla\tilde{\psi}(z,\tau), y-z\rangle}{(s-\tau)^{n/2+1}} \\ &\quad \exp\left[-\frac{|y-z|^2+|\tilde{\psi}(y,s)-\tilde{\psi}(z,\tau)|^2}{4(s-\tau)^+}\right]. \end{aligned} \tag{6.44}$$

We end this section by proving

Lemma 6.45. *With the above notation and for δ' small enough, we have*

$$\|\tilde{\chi}K_1\hat{h}\|_2 \;\leq\; c\,|(\sqrt{1+|\nabla\tilde{\psi}|^2}K_1\hat{h})_{\tilde{C}}|r^{(n+1)/2} \;+\; (\theta/r)^{n+1}\,[\delta^{5/2}\,r^{(n+1)/2} \;+\; \delta^a\,\Phi^{1/2}\,].$$

Proof: To prove Lemma 6.45, we first consider $K_2\hat{h}$. We put $K_2\hat{h} = K_3\hat{h} + K_4\hat{h}$ where

$$\begin{aligned} \hat{K}_3(z,\tau,y,s) \;&=\; (1+|\nabla\tilde{\psi}|^2\,)^{-1/2}(z,\tau)\;\; \frac{\langle\nabla\tilde{\psi}(z,\tau), y-z\rangle}{(s-\tau)^{n/2+1}} \\ &\quad\cdot \exp\left[-\frac{|y-z|^2}{4(s-\tau)}\right]\left\{\exp\left[-\frac{|\tilde{\psi}(y,s)-\tilde{\psi}(z,\tau)|^2}{4(s-\tau)}\right] \;-\; 1\right\} \end{aligned} \tag{6.46}$$

when $s > \tau$ and is 0 otherwise. Observe that

$$\exp\left[-\frac{|\tilde{\psi}(y,s)-\tilde{\psi}(z,\tau)|^2}{4(s-\tau)}\right] \;-\; 1 \;\leq\; c\frac{|\tilde{\psi}(y,s)-\tilde{\psi}(z,\tau)|^2}{s-\tau} \;\leq\; e^{-1/(c\delta^a)}\,(r^2/|s-\tau|).$$

Let $g \in C_0^\infty[C_{2\tilde{r}}(x,t)]$ with $g \equiv 1$ on $C_{3\tilde{r}/2}(x,t)$ and $r^{-1}|\nabla g| \,+\, r^{-2}\,|\frac{\partial}{\partial t}g| \,\leq\, c$. Using the above observation and (6.29) we see for $(z,\tau)\in\tilde{C}$ that

$$|K_3\hat{h} - K_3(\hat{h}g)|(z,\tau) \leq ce^{-1/(c\delta^a)}\,r^{-(n+1)}. \tag{6.47}$$

Next set

$$K_3\hat{h}(z,\tau) = \sum_{i=0}^{n-1} \tfrac{\partial}{\partial z_i}\tilde{\psi}(z,\tau)\ K_{3,i}\hat{h}(z,\tau),$$

where

$$\hat{K}_{3,i}(z,\tau,y,s) = (y_i - z_i)\,\langle \nabla\tilde{\psi}(z,\tau), y-z\rangle^{-1}\,\hat{K}_3(z,\tau,y,s) \text{ for } 1 \leq i \leq n-1$$

when $\langle \cdot,\cdot\rangle \neq 0$ and this function is 0 otherwise. Expanding the exponential term involving $\tilde{\psi}$ in a series and using [LM, Theorem 1, ch 1], as well as a lemma of Peetre - Stein (see [Ste, ch 4]) we find that $K_{3,i}\,\hat{h}$ is in BMO $(\mathbb{R}^n)$ (defined with respect to heat cylinders) and

$$\|K_{3,i}\hat{h}\|_* \leq c\delta^{2a}(\theta/r)^{n+1} \tag{6.48}$$

when $1 \leq i \leq n-1$. Also from (6.47), Hölder's inequality, the cited theorem and (6.29) we find

$$\begin{aligned} |(K_{3,i}\,\hat{h})_{\tilde{C}}| &\leq c|K_{3,i}(g\,\hat{h})_{\tilde{C}}| + e^{-1/(c\delta^a)}r^{-(n+1)} \\ &\leq c\|K_{3,i}(g\,\hat{h})\|_2\, r^{-(n+1)/2} + e^{-1/(c\delta^a)}r^{-(n+1)} \\ &\leq c\delta^{2a}\,\|\tilde{\chi}\,\hat{h}\|_2\, r^{-(n+1)/2} + e^{-1/(c\delta^a)}r^{-(n+1)} \leq c\delta^{2a}(\theta/r)^{(n+1)}. \end{aligned} \tag{6.49}$$

It is well known that (6.48), (6.49) imply for $1 \leq i \leq n-1$

$$|K_{3,i}\hat{h}|(z,\tau) \leq \delta^a(\theta/r)^{n+1},\ (z,\tau) \in \tilde{E}, \tag{6.50}$$

where $\tilde{C} \setminus \tilde{E}$ has Lebesgue n measure at most $e^{-1/c\delta^a} r^{n+1}$ for some $c \geq 1$. Using (6.48)-(6.50) we get for $1 \leq i \leq n-1$,

$$\begin{aligned} \|(\tfrac{\partial}{\partial x_i}\tilde{\psi})\,\tilde{\chi}\,K_{3,i}\hat{h}\|_2^2 &\leq c\int_{C_{\tilde{r}}(x,t)\setminus\tilde{E}} [K_{3,i}\hat{h}]^2\,dyds + c\delta^{2a}\,(\theta/r)^{2n+2}\int_{\tilde{E}} |\nabla\tilde{\psi}|^2 dyds \\ &\leq ce^{-1/(c\delta^a)}\, r^{(n+1)/2}\,\|K_{3,i}h\|_4^2 + c\delta^{2a}(\theta/r)^{2n+2}\ \Phi \\ &\leq c\delta^5 r^{(n+1)}\,(\theta/r)^{2n+2} + c\delta^{2a}(\theta/r)^{2n+2}\ \Phi. \end{aligned} \tag{6.51}$$

From (6.51) and Hölder's inequality one also gets an estimate for the average of the above function relative to $\tilde{C}$. Using the resulting estimate, (6.51), and summing, we conclude from (6.42) that

$$\begin{aligned} &\|\tilde{\chi}(K_1+K_4)\hat{h}\|_2 \leq \|\tilde{\chi}Kh\|_2 + \|\tilde{\chi}K_3\hat{h}\|_2 \\ &\leq c\,|\,[\sqrt{1+|\nabla\tilde{\psi}|^2}(K_1+K_4)\,\hat{h}]_{\tilde{C}}\,|r^{(n+1)/2} + c(\theta/r)^{n+1}\,[\delta^{5/2}\,r^{(n+1)/2} + \delta^a\,\Phi^{1/2}\,]. \end{aligned} \tag{6.52}$$

Next we consider $K_4\hat{h}$. If $(y,s) \in \mathbb{R}^n$, $s < t_2$, let

$$h_1(y,s) = 4^{-1}\,(4\pi)^{-n}\,\frac{u_0 - \tilde{\psi}(x,t)}{(t_2-s)^{n/2+1}}\ \exp\left[-\frac{|y-u|^2 + (u_0-\tilde{\psi}(x,t))^2}{4(t_2-s)}\right].$$

Set $h_1 \equiv 0$ elsewhere in $\mathbb{R}^n$. We note from the definition of $h, \hat{h}$, (6.12), and (6.15) that

$$|\hat{h} - h_1|(y,s) \leq c\theta\, \tilde{\chi}(y,s)\, (\theta/r)^{(n+1)}. \tag{6.53}$$

Also, using the Poisson integral formula for the heat equation in a half space and properties of parabolic Riesz transforms we see for $(z,\tau) \in \tilde{C}$ that

$$\begin{aligned} |K_4 h_1(z,\tau)| &= 4^{-1}\, (4\pi)^{-n}\, (1 + |\nabla\tilde{\psi}(z,\tau)|^2\,)^{-1/2} \left| \int_\tau^{t_2} \int_{\mathbb{R}^{n-1}} \frac{\langle \nabla\tilde{\psi}(z,\tau), y - z\rangle}{(s-\tau)^{n/2+1}} \right. \\ &\quad \left. \cdot \exp\left[-\frac{|y-z|^2}{4(s-\tau)}\right] \frac{u_0 - \tilde{\psi}(x,t)}{(t_2 - s)^{n/2+1}} \exp\left[-\frac{|y-u|^2 + (u_0 - \tilde{\psi}(x,t))^2}{4(t_2 - s)}\right] dyds \right| \\ &= (1/2)\,(4\pi)^{-n/2}\,(1 + |\nabla\tilde{\psi}(z,\tau)|^2)^{-1/2}\,(t_2-\tau)^{-(n/2+1)}\,|\,\langle\nabla\tilde{\psi}(z,\tau), u - z\rangle\,| \\ &\quad \cdot \exp\left[-\frac{|z-u|^2 + (u_0 - \tilde{\psi}(x,t))^2}{4(t_2-\tau)}\right] \leq\; c\theta(\theta/r)^{n+1}. \end{aligned} \tag{6.54}$$

Using (6.53), (6.54) and standard Calderón-Zygmund singular integral estimates we obtain first

$$\|\tilde{\chi} K_4 \hat{h}\|_2 \,\leq \|\tilde{\chi} K_4(\hat{h} - h_1)\|_2 \,+ \|\tilde{\chi} K_4 h_1\|_2 \,\leq c\delta^{5/2}\,(\theta^2/r)^{(n+1)/2} \tag{6.55}$$

and thereupon from (6.52), (6.55), that

$$\|\tilde{\chi} K_1 \hat{h}\|_2 \,\leq\, c\,|(\sqrt{1+|\nabla\tilde{\psi}|^2} K_1\hat{h})_{\tilde{C}}| r^{(n+1)/2} \,+\, (\theta/r)^{n+1}\,[\delta^{5/2}\, r^{(n+1)/2} \,+\, \delta^a\,\Phi^{1/2}\,] \tag{6.56}$$

which is the conclusion of Lemma 6.45. □

7. Proof of Theorem 3. In this section we finish the proof of Proposition 6.2 and thus obtain Theorem 3. We use the same notation as in section 6. Recall that we are in the process of proving (6.26) by decomposing K into simpler and simpler operators which satisfy inequalities similar to the ones in Lemmas 6.35 and 6.45 for K, K_1. To analyze, $K_1\hat{h}$ we write $K_1\hat{h} = K_{11}\hat{h} + K_{12}\hat{h}$ where

$$\begin{aligned} \hat{K}_{12}(z,\tau,y,s) \;&=\; (1 + |\nabla\tilde{\psi}|^2(z,\tau))^{-1/2}\; \frac{\tilde{\psi}(z,\tau) - \tilde{\psi}(y,s)}{(s-\tau)^{n/2+1}} \\ &\quad \cdot \exp\left[-\frac{|y-z|^2}{4(s-\tau)}\right] \left(\exp\left[-\frac{|\tilde{\psi}(y,s) - \tilde{\psi}(z,\tau)|^2}{4(s-\tau)}\right] - 1\right) \end{aligned} \tag{7.1}$$

for $s > \tau$ and is 0 otherwise.

Lemma 7.2. *For δ' small enough*

$$|(\sqrt{1+|\nabla\tilde{\psi}|^2} K_{11}\hat{h})_{\tilde{C}}| \;\leq c(\theta/r)^{n+1}|(\sqrt{1+|\nabla\tilde{\psi}|^2} K_{11}\tilde{\chi})_{\tilde{C}}| \,+\, c\delta^{5/2}\,(\theta/r)^{n+1}.$$

Proof: To prove Lemma 7.2 let g be the function defined above (6.47). Arguing as in the proof of (6.47) we deduce that

$$\|\tilde{\chi} K_{12} \hat{h}\|_2 \,\leq\, c \|\tilde{\chi} K_{12}(g\hat{h})\|_2 \,+\, c\delta^{5/2}\,(\theta^2/r)^{(n+1)/2}. \tag{7.3}$$

To simplify our writing we put

$$v = v(z,y,s) = \tilde{\psi}(z,s) - \tilde{\psi}(y,s)$$

$$w = w(z,\tau,s) = \tilde{\psi}(z,\tau) - \tilde{\psi}(z,s)$$

and note that $\tilde{\psi}(z,\tau) - \tilde{\psi}(y,s) = v + w$. Using this notation we see that

$$K_{12}(g\hat{h})(z,\tau) = (J_1 + J_2 + J_3)(g\hat{h})(z,\tau)$$

where for $s > \tau$,

$$\begin{aligned} \hat{J}_1(z,\tau,y,s) \;&=\; (1 + |\nabla \tilde{\psi}|^2(z,\tau))^{-1/2} \;\; \exp\left[-\frac{|y-z|^2}{4(s-\tau)}\right] \\ &(s-\tau)^{-n/2-1}\left(\left(1 - \frac{v^2}{2(s-\tau)}\right) w \,\exp\left[-\frac{v^2}{4(s-\tau)}\right] \; - w\right). \end{aligned} \tag{7.4}$$

Moreover,

$$\begin{aligned} \hat{J}_2(z,\tau,y,s) \;&=\; (1 + |\nabla \tilde{\psi}|^2(z,\tau))^{-1/2} \;\; \exp\left[-\frac{|y-z|^2}{4(s-\tau)}\right] \\ &(s-\tau)^{-n/2-1}\left(v \,\exp\left[-\frac{v^2}{4(s-\tau)}\right] \; - v\right). \end{aligned} \tag{7.5}$$

Finally,

$$\begin{aligned} \hat{J}_3(z,\tau,y,s) \;&=\; \frac{1}{(s-\tau)^{n/2+1}\sqrt{1 + |\nabla \tilde{\psi}|^2(z,\tau)}} \exp\left[-\frac{|y-z|^2 + v^2}{4(s-\tau)}\right] \\ &\cdot \left((v+w)\left\{\exp\left[-\frac{2vw + w^2}{4(s-\tau)}\right] \,+\, \frac{2vw + w^2}{4(s-\tau)} - 1\right\} - \frac{3vw^2 + w^3}{4(s-\tau)}\right). \end{aligned} \tag{7.6}$$

If $s < \tau$ all the $J's$ are defined to be 0 at (z,τ,y,s). Using Taylor's formula and (6.15) we see for $s > \tau$ that

$$|\hat{J}_3(z,\tau,y,s)| \,\leq\, c\delta^a\, w^2\, \exp\left[-\frac{|y-z|^2}{4(s-\tau)}\right]\left(\frac{|z-y|^3 + |s-\tau|^{3/2}}{|s-\tau|^2}\right). \tag{7.7}$$

Using (7.7), (6.29) and integrating in y we obtain for $(z,\tau) \in \mathbb{R}^n$ that

$$J_3(g\hat{h})(z,\tau) \,\leq\, c\,\delta^a\,(\theta/r)^{n+1} \int_\tau^\infty \frac{(\tilde{\psi}(z,\tau) - \tilde{\psi}(z,s))^2}{(s-\tau)^2} ds \,=\, c\delta^a(\theta/r)^{n+1} f(z,\tau). \tag{7.8}$$

We claim that $f \in$ BMO $(\mathbb{R}^n)$ defined with respect to heat cylinders and

$$\|f\|_* \,\leq\, c\delta^{2a}. \tag{7.9}$$

To prove this claim let C be a heat cylinder $\subset \mathbb{R}^n$ with radius r in the space direction and height r^2 in the time direction. Write $f = f_1 + f_2$ where $f_1 = f\,\chi_{2C}$ and $2C$ is the cylinder with the same center as C and of radius, $2r$, height $4r^2$. Then from (6.15) and basic estimates it follows easily that if $(y,s) \in C$ then

$$|f_2(z,\tau) - f_2(y,s)| \leq c\delta^{2a}, \text{ for all } (z,\tau) \in C.$$

Using this fact, as well as (6.16), we deduce for $A = f_2(y,s)$ that

$$\begin{aligned} \int_C |f - A|dxdt \;&\leq\; \int_{2C} |f_1|dxdt \\ &+ \int_C |f_2 - A|dxdt \;\leq\; c\delta^{2a}\, r^{n+1}. \end{aligned} \tag{7.10}$$

Thus claim (7.9) is true.

Next from Plancherel's theorem it follows that

$$\begin{aligned} |f_{\tilde{C}}| &\leq cr^{-(n+1)} \int_{\{z:|z-x|<2\tilde{r}\}} \left(\int_{\mathbb{R}} |D^{\tau}_{1/2} f|^2(z,\cdot)\, d\tau \right) dz \\ &\leq\; cr^{-(n+1)}\; \Phi \;\leq\; c\delta^{2a}, \end{aligned} \tag{7.11}$$

where the last inequality follows easily from (6.15) and (6.12). Using (7.9),(7.11), we can argue as in (6.50) to get first $G \subset \tilde{C}, G$ closed with $f \leq \delta^a$ on G. Moreover, $\tilde{C} \setminus G$ has Lebesgue n measure at most, $e^{-1/(c\delta^a)} r^{n+1}$. Second we obtain as in (6.51)

$$\int_{\tilde{C}} f^2 dzd\tau \leq\; c\delta^a r^{n+1} f_{\tilde{C}} + ce^{-1/(c\delta^a)} r^{n+1} \;\leq\; c\delta^a \Phi + ce^{-1/(c\delta^a)} r^{n+1}.$$

We conclude from this inequality and (7.8) that

$$\|\tilde{\chi} J_3(g\hat{h})\|_2 \;\leq\; c\,(\theta/r)^{n+1}\,[\delta^{5/2} r^{(n+1)/2} + \delta^a\, \Phi^{1/2}]. \tag{7.12}$$

To analyze J_1, J_2, we use the method of rotations. We begin with J_1. Let $\mathcal{S} = \{x : |x| = 1\}$ denote the unit sphere in $\mathbb{R}^{n-1}$. Given $\eta \in \mathcal{S}$ and $\epsilon > 0$, let

$$\begin{aligned} &U_{\epsilon,\eta}(g\hat{h})(z,\tau) \\ &= \int_{|\rho|\geq\epsilon} \int_{\tau+\epsilon^2}^{\infty} (s-\tau)^{-(n/2+1)} \exp\left[-\frac{\rho^2}{4(s-\tau)} \right] \Bigg\{ [1 - \frac{(\tilde{\psi}(z,s) - \tilde{\psi}(z+\rho\eta,s))^2}{2(s-\tau)}] \\ &\cdot(\tilde{\psi}(z,\tau) - \tilde{\psi}(z,s)) \exp\left[-\frac{(\tilde{\psi}(z,s) - \tilde{\psi}(z+\rho\eta,s))^2}{4(s-\tau)} \right] - (\tilde{\psi}(z,\tau) - \tilde{\psi}(z,s)) \Bigg\} \\ &\qquad\qquad \cdot (g\hat{h})(z+\rho\eta,s)|\rho|^{n-2}\, d\rho ds. \end{aligned} \tag{7.13}$$

For almost every $(z,\tau) \in \mathbb{R}^n$ we can integrate by parts in this integral to get

$$U_{\epsilon,\eta}(g\hat{h})(z,\tau) \;=\; V_\epsilon(kg\hat{h})(z,\tau) = \int_{\mathbb{R}^2} \hat{V}_\epsilon(z,\tau,\rho,s)\; (kgh)(z+\rho\eta,s) d\rho ds$$

where $k(z+\eta\rho, s) = \langle \nabla\tilde{\psi}(z+\rho\eta, s)\,,\, \eta\rangle$ when $\rho \in \mathbb{R}$. Here

$$\hat{V}_\epsilon(z,\tau,\rho,s) = H_\epsilon(\rho, s-\tau)\tilde{V}(z,\tau,\rho,s)$$

and for $\lambda \geq \epsilon^2$, $\rho > 0$,

$$H_\epsilon(\rho,\lambda) = \lambda^{-n/2-1} \int_{\max(\rho,\epsilon)}^{\infty} \exp[-u^2/(4\lambda)]\; u^{n-2}\, du$$

while $H(\rho,\lambda) = -H(-\rho,\lambda)$ when $\rho < 0$ and $\lambda \geq \epsilon^2$. Put $H_\epsilon(\rho,\lambda) \equiv 0$ for $\lambda \leq \epsilon^2$. Also,

$$\begin{aligned} &\tilde{V}(z,\tau,\rho,s) \\ &= \left([1 - \frac{(\tilde{\psi}(z,s) - \tilde{\psi}(z+\rho\eta,s)^2}{2(s-\tau)}]\,[\frac{\tilde{\psi}(z,s) - \tilde{\psi}(z+\rho\eta,s)}{2(s-\tau)}](\tilde{\psi}(z,\tau) - \tilde{\psi}(z,s)) \right. \\ &\quad \left. + \frac{(\tilde{\psi}(z,\tau) - \tilde{\psi}(z,s))(\tilde{\psi}(z,s) - \tilde{\psi}(z+\rho\eta,s))}{(s-\tau)} \right) \exp\left[-\frac{(\tilde{\psi}(z,s) - \tilde{\psi}(z+\rho\eta,s))^2}{4(s-\tau)} \right]. \end{aligned}$$

Suppose that $n > 2$ and $z = z^* + \xi\eta$, where z^* is orthogonal to η and $\xi \in \mathbb{R}$. We note that if

$$q(\mu,\tau) = (kg\hat{h})(z^* + \mu\eta, \tau),\ \mu,\tau \in \mathbb{R},$$

then for fixed z^*, the operator $T_\epsilon\, q(\mu,\tau) = U_{\epsilon,\eta}\, g\hat{h}(z^* + \mu\eta, \tau)$ is of the type considered in $[LM]$. In fact expanding the exponential term in the kernel of V as a series in powers of $(\tilde{\psi}(z,s) - \tilde{\psi}(z+\rho\eta,s))$ and using [LM, Lemma 4.2, page 44], we see that the maximal operator $\sup_{\epsilon>0}|T_\epsilon \hat{f}|$ is bounded for $\hat{f} \in L^2(\mathbb{R}^2)$ with norm $\leq c\delta^a$. Let N denote the vector space orthogonal to η and let dz^* be $n-2$ dimensional Lebesgue measure on N. From boundedness of the above maximal operator we see that

$$\begin{aligned} \int_{\mathbb{R}^n} \sup_\epsilon |U_{\epsilon,\eta}(g\hat{h})(z,\tau)|^2\, dz d\tau \;&=\; \int_N \left(\int_{\mathbb{R}^2} \sup_\epsilon |T_\epsilon\, q|^2(\mu,\tau) d\mu d\tau \right) dz^* \\ \leq\; c\delta^{2a} \int_N &\left(\int_{\mathbb{R}^2} |(g\hat{h}|\nabla\tilde{\psi}|)(z^* + \mu\eta, \tau)|^2 d\mu d\tau \right) dz^* \;\leq c\delta^{2a}(\theta/r)^{2(n+1)}\Phi. \end{aligned} \tag{7.14}$$

Next we note that if $J_{1,\epsilon}$ is the operator whose kernel equals $\hat{J}_1$ when $\min\{|y-z|^2, (s-\tau)\} \geq \epsilon^2$ and is 0, otherwise, then

$$J_{1,\epsilon}(g\hat{h})(z,\tau) = c(1 + |\nabla\tilde{\psi}|^2)^{-1/2}(z,\tau) \int_{\mathcal{S}} U_{\epsilon,\eta}(g\hat{h})(z,\tau)\, d\eta$$

where $d\eta$ denotes surface area or H^{n-2} measure on $\mathcal{S}$. Using this equality, Minkowski's inequality for integrals, and (7.14) we conclude for $n > 2$ that

$$\| \sup_{\epsilon>0} |J_{1,\epsilon}(g\hat{h})\|_2 \;\leq\; c\delta^a\, (\theta/r)^{n+1}\, \Phi^{1/2}.$$

Letting $\epsilon \to 0$ and using the results in [LM, ch 1, section 5] it follows that

$$\|J_1(g\hat{h})\|_2 \leq c\delta^a (\theta/r)^{n+1}\Phi^{1/2}. \tag{7.15}$$

If $n = 2$, we get (7.15) by the above argument with only a few notational changes.

A similar argument applies to J_2. That is we can write the truncated operator $J_{2,\epsilon}$ as an integral over $\mathcal{S}$ of certain operators acting on $L^2(\mathbb{R}^2)$. Each of these operators can be integrated by parts, to get an operator acting on $kg\hat{h}$ whose corresponding maximal operator has norm $\leq c\delta^a$ on $L^2(\mathbb{R}^2)$. We obtain that (7.15) holds with J_1 replaced by J_2. From (7.15) for J_1, J_2, (7.12), and (7.3) we see first that

$$\|\tilde{\chi}K_{12}\hat{h}\|_2 \leq c(\theta/r)^{n+1}\,[\delta^{5/2}r^{(n+1)/2} + \delta^a\Phi^{1/2}]$$

and thereupon from (6.56) that

$$\|\tilde{\chi}K_{11}\hat{h}\|_2 \leq c[|(\sqrt{1+|\nabla\tilde{\psi}|^2}K_{11}\hat{h})_{\tilde{C}}|r^{(n+1)/2} + (\theta/r)^{n+1}\,[\delta^{5/2}r^{(n+1)/2} + \delta^a\Phi^{1/2}] \tag{7.16}$$

Thus Lemma 7.2 is valid. □

It remains to consider $K_{11}\hat{h}$. We observe that

$$\begin{aligned}&\sqrt{1+|\nabla\tilde{\psi}|^2}(z,\tau)\ \hat{K}_{11}(z,\tau,y,s)\\ &\qquad = \frac{\tilde{\psi}(z,\tau)-\tilde{\psi}(y,s)}{(s-\tau)^{n/2+1}}\exp\left[-\frac{|y-z|^2}{4(s-\tau)}\right]\end{aligned} \tag{7.17}$$

when $s > \tau$ and is 0, otherwise. Let r' be as defined above (6.38) and let χ' be the characteristic function of $C_{r'}(x,t)$. Using (7.17) and the same argument as in (6.47) we see that

$$|(\sqrt{1+|\nabla\tilde{\psi}|^2}K_{11}\hat{h})_{\tilde{C}}| \leq c|[\sqrt{1+|\nabla\tilde{\psi}|^2}K_{11}(\chi'\hat{h})]_{\tilde{C}}| + c\delta^{5/2}\,(\theta/r)^{n+1}.$$

Moreover using singular integral estimates from either [H] or [LM, ch 1], (6.29), and Hölder's inequality we get as in (6.39) that

$$|[\sqrt{1+|\nabla\tilde{\psi}|^2}K_{11}(\chi'\hat{h})]_{\tilde{C}}| \leq c|[\sqrt{1+|\nabla\tilde{\psi}|^2}K_{11}(\tilde{\chi}\hat{h})]_{\tilde{C}}| + c\delta^{5/2}\,(\theta/r)^{n+1}.$$

A similar argument using (6.28), (6.29) and the definition of $\hat{h}$ also gives,

$$|[\sqrt{1+|\nabla\tilde{\psi}|^2}K_{11}(\tilde{\chi}\hat{h})]_{\tilde{C}}| \leq c(\theta/r)^{n+1}|(\sqrt{1+|\nabla\tilde{\psi}|^2}K_{11}\tilde{\chi})_{\tilde{C}}| + c\delta^{5/2}\,(\theta/r)^{n+1}.$$

Combining the three above inequalities we conclude that

$$|(\sqrt{1+|\nabla\tilde{\psi}|^2}K_{11}\hat{h})_{\tilde{C}}| \leq c(\theta/r)^{n+1}|(\sqrt{1+|\nabla\tilde{\psi}|^2}K_{11}\tilde{\chi})_{\tilde{C}}| + c\delta^{5/2}\,(\theta/r)^{n+1}. \tag{7.18}$$

Also

$$\begin{aligned}\left|\int_{\tilde{C}}\sqrt{1+|\nabla\tilde{\psi}|^2}K_{11}\tilde{\chi}\,dzd\tau\right| &\leq \left|\int_{\tilde{C}}\int_{\tilde{C}}\frac{\tilde{\psi}(z,\tau)-\tilde{\psi}(y,\tau)}{(s-\tau)^{n/2+1}}\exp\left[-\frac{|y-z|^2}{4(s-\tau)}\right]dydsdzd\tau\right|\\ &+\left|\int_{\tilde{C}}\int_{\tilde{C}}\frac{\tilde{\psi}(y,\tau)-\tilde{\psi}(y,s)}{(s-\tau)^{n/2+1}}\exp\left[-\frac{|y-z|^2}{4(s-\tau)}\right]dydsdzd\tau\right| = P_1+P_2.\end{aligned} \tag{7.19}$$

Now $P_1 = 0$ since the integrand in y, z is antisymmetric. As for P_2 we use (6.12) and change variables to get that

$$P_2 \le \int_{|x-z|<\tilde r} \int_{|x-y|<\tilde r} \int_0^{t+\tilde r^2-\tau} \left| \int_{t-\tilde r^2}^{t+\tilde r^2} \frac{\tilde\psi(y,\tau)-\tilde\psi(y,\tau+u)}{u^{n/2+1}} d\tau \right| \exp\left[-\frac{|y-z|^2}{4u}\right] dudydz$$
$$\le cre^{-1/(c\delta)} \int_{|x-z|<\tilde r} \int_{|x-y|<\tilde r} \int_0^{t+\tilde r^2-\tau} u^{-n/2} \exp\left[-\frac{|y-z|^2}{4u}\right] dudydz \le ce^{-1/(c\delta)}\, r^{n+1} \tag{7.20}$$

Using (7.18) - (7.20) we deduce first that

$$|(\sqrt{1+|\nabla\tilde\psi|^2} K_{11}\hat h)_{\tilde C}| \le c(\theta/r)^{n+1}[\delta^{5/2}\, r^{(n+1)/2} + \delta^a\, \Phi^{1/2}]$$

and second from (7.16) that

$$\|\tilde\chi K_{11}\hat h\|_2 \le c(\theta/r)^{n+1}\, [\delta^{5/2} r^{(n+1)/2} + \delta^a \Phi^{1/2}] \tag{7.21}$$

To continue the proof of (6.26) we note from the same argument as in (7.18), (6.38), (6.39) that

$$(\theta/r)^{n+1}\|K_{11}1\|_2 \le c(\theta/r)^{n+1}\|K_{11}\tilde\chi\|_2 + c\delta^{5/2}(\theta^2/r)^{(n+1)/2}$$
$$\le c^2\|\tilde\chi K_{11}\hat h\|_2 + c^2\,\delta^{5/2}\,(\theta^2/r)^{(n+1)/2}. \tag{7.22}$$

Let $\mathcal F$ denote the Fourier transform on $\mathbb R^{n+1}$. Changing variables in the integral defining $K_{11}1$ and taking the Fourier transform under the integral sign, we see that

$$\mathcal F(\sqrt{1+|\nabla\tilde\psi|^2}\, K_{11}1)(z,\tau) = \mathcal F(\tilde\psi)(z,\tau)\, \left[\int_0^\infty \int_{\mathbb R^n} \{1-e^{-i(\langle z,w\rangle + u\tau)}\} u^{-(n/2+1)}\, e^{-|w|^2/u} dwdu\right]$$
$$= \mathcal F(\tilde\psi)(z,\tau)\, I(z,\tau). \tag{7.23}$$

We claim that

$$c\, I(z,\tau) \ge |z| + |\tau|^{1/2}.$$

This inequality follows easily from the fact that the real part of the integrand is always nonnegative and the inequality

$$1-\cos\theta = 2[\sin(\theta/2)]^2 \ge \theta^2/8$$

$(0<\theta<1/2)$ with $\theta = (\langle z,w\rangle + u\tau)$ and (w,u) chosen appropriately. For example if $|z| \ge |\tau|^{1/2}$, then

$$c^2 I(z,\tau)$$
$$\ge c\int_{\{1/100<|w||z|<1/50\}} \int_{\{1/100<u^{1/2}|z|<1/50\}} |z|^{n+2}\,(\langle z,w\rangle + u\tau)^2\, dudw \ge |z|.$$

Using our claim it follows from Plancherel's theorem that

$$c^2\|\sqrt{1+|\nabla\tilde\psi|^2} K_{11}1\|_2^2 = c^2\|\mathcal F(\sqrt{1+|\nabla\tilde\psi|^2}K_{11}1)\|_2^2 \ge \int_{\mathbb R^n} c(|z|^2+|\tau|)|\mathcal F(\tilde\psi)|^2(z,\tau)dzd\tau$$
$$= c\int_{\mathbb R^n} (|D^t_{1/2}\tilde\psi|^2 + |\nabla\tilde\psi|^2)dzd\tau = c\Phi. \tag{7.24}$$

Using (7.22), (7.24), (6.15) in (7.21) we obtain that

$$(\theta/r)^{n+1}\,\Phi^{1/2}\;\leq\;c(\theta/r)^{n+1}\,[\delta^{5/2}r^{(n+1)/2}+\delta^{a}\Phi^{1/2}]. \tag{7.25}$$

Clearly this inequality implies for δ' small enough that $\Phi\leq c\delta^{5}r^{n+1}$, which is (6.26). The proof of (6.26) is now complete.

Next define $\tilde{\gamma},\tilde{\nu}$, as in (1.5), (1.6) relative to $\tilde{\Omega},\tilde{\sigma}$. We assert that

$$\tilde{\nu}(\tilde{\Delta}(X,t,2r)\times(0,2r))\leq c\Phi\leq c\delta^{5}r^{n+1} \tag{7.26}$$

where the last inequality follows from (6.26). To prove this inequality let P_λ be the approximate identity defined above (2.7) and for $\lambda>0$, let

$$\tilde{\kappa}(z,\tau,\lambda)=\lambda^{-(n+3)}\inf_{L}\int_{C_\lambda(z,\tau)}|\tilde{\psi}(y,s)-L|^{2}dyds$$

where the infimimum is taken over linear functions, L, of y only. Then from (2.5) we deduce

$$\begin{aligned}\tilde{\nu}(\tilde{\Delta}(X,t,2r)\times(0,2r))&\leq c\int_{0}^{\infty}\int_{\mathbb{R}^n}\lambda^{-1}\tilde{\kappa}(z,\tau,\lambda)dzd\tau d\lambda\\ &\leq c\int_{0}^{\infty}\int_{\mathbb{R}^n}\lambda^{-(n+4)}\left[\int_{C_\lambda(z,\tau)}|\tilde{\psi}(y,s)-\tilde{\psi}(z,\tau)-\langle\nabla_z P_\lambda\tilde{\psi}(z,\tau)\,,\,y-z\rangle|^{2}dyds\right]dzd\tau d\lambda\\ &\leq c\,\Phi\end{aligned}$$

where the last line follows from interchanging the order of integration, taking Fourier transforms on $\mathbb{R}^n$ in the (z,τ) variable, and using Plancherel's theorem, as well as making estimates similar to those in the proof of (7.24). Thus assertion (7.26) is true.

We use (7.26) to estimate first γ and then $\nu[\Delta(X,t,r/4)\times(0,r/4)]$. To begin, let $\hat{\chi}$ be the characteristic function of $\Delta(X,t,2r)\setminus\tilde{\Omega}$ and for $(Y,s)\in\partial\Omega$, let

$$\bar{M}(\hat{\chi})\,(Y,s)=\sup_{\rho>0}\frac{\sigma[\Delta(Y,s,\rho)\cap\Delta(X,t,2r)\setminus\partial\tilde{\Omega}]}{\sigma[\Delta(Y,s,\rho)]}.$$

From the Hardy-Littlewood maximal theorem and (6.9) we see that there exists $c>0$ and a compact set $E'\subset\tilde{\Delta}(X,t,2r)\cap\Delta(X,t,2r)$ with

$$\sigma(\Delta(X,t,2r)\setminus E')\leq\,e^{-1/(c\delta^{a})}r^{n+1}\text{ and }\bar{M}(\hat{\chi})(Y,s)\leq e^{-1/(c\delta^{a})}\text{ for }(Y,s)\in E'. \tag{7.27}$$

Let $\{Q_i\}$ be a Whitney decompositon of $\mathbb{R}^{n+1}\setminus E'$ by rectangles with sides parallel to the coordinate axes and the following properties:

$$\begin{aligned}&(a)\quad Q_i=Q_{r_i}(Z_i,\tau_i)=\{(Y,s):|y_j-z_j|<r_i,\;0\leq j\leq n-1,\;|s-\tau_i|<r_i^2\},\\ &(b)\quad Q_i\cap Q_j=\emptyset, i\neq j,\\ &(c)\quad 10^{-10n}d(Q_i,E')\;\leq\;r_i\;\leq 10^{-5n}d(Q_i,E'),\\ &(d)\quad \text{For each } i,\ \{j:\bar{Q}_j\cap\bar{Q}_i\neq\emptyset\}\text{ has cardinality at most } c,\\ &(e)\quad \mathbb{R}^{n+1}\setminus E'=\bigcup\bar{Q}_i.\end{aligned} \tag{7.28}$$

Given $(\hat{Z},\hat{\tau}) \in E'$ and $0<\rho\le r/4$, let $\xi=\xi(\hat{Z},\hat{\tau},\rho)=\{i:\bar{Q}_i\cap\Delta(\hat{Z},\hat{\tau},\rho)\setminus E'\neq\emptyset\}$. We claim that

$$\gamma(\hat{Z},\hat{\tau},\rho)\;\le\; c\tilde{\gamma}(\hat{Z},\hat{\tau},\tfrac{5}{4}\rho)\;+\;c\sum_{i\in\xi(\hat{Z},\hat{\tau},\rho)}(r_i/\rho)^{n+1}. \tag{7.29}$$

In fact let P be a plane containing a line parallel to the t axis with

$$\tilde{\gamma}(\hat{Z},\hat{\tau},\tfrac{5}{4}\rho)\ge\; 2^{-(n+1)}\,\rho^{-n-3}\int_{\partial\tilde{\Omega}\cap C_{(5/4)\rho}(\hat{Z},\hat{\tau})} d(\{(Y,s)\},P)^2\,d\tilde{\sigma}(Y,s). \tag{7.30}$$

Then

$$\begin{aligned}\gamma(\hat{Z},\hat{\tau},\rho)\le\;&\rho^{-n-3}\int_{E'\cap\Delta(\hat{Z},\hat{\tau},\rho)} d(\{(Y,s)\},P)^2\,d\sigma(Y,s)\\ &+\rho^{-n-3}\sum_{i\in\xi}\int_{\bar{Q}_i\cap\Delta(\hat{Z},\hat{\tau},\rho)} d(\{(Y,s)\},P)^2\,d\sigma(Y,s)=L_1+L_2.\end{aligned} \tag{7.31}$$

From (7.30) and the fact that $E'\subset\partial\Omega\cap\partial\tilde{\Omega}$, we see that

$$L_1\le c\tilde{\gamma}(\hat{Z},\hat{\tau},\tfrac{5}{4}\rho).$$

As for L_2, let $\xi=\xi_1\cup\xi_2$ where

$$\xi_1=\{i\in\xi(\hat{Z},\hat{\tau},\rho): d(\{(Y,s)\},P)\le r_i \text{ for some } (Y,s)\in\bar{Q}_i\}.$$

Then we have the trivial estimate

$$\rho^{-n-3}\sum_{i\in\xi_1}\int_{\bar{Q}_i\cap\Delta(\hat{Z},\hat{\tau},\rho)} d(\{(Y,s)\},P)^2\,d\sigma(Y,s)\le c\sum_{i\in\xi_1}(r_i/\rho)^{n+3}. \tag{7.32}$$

If $i\in\xi_2$, then from (7.28)(c), (7.27) and Ahlfors regularity, we find for some $(Y^*,s^*)\in\bar{Q}_i\cap[\Delta(\hat{Z},\hat{\tau},9\rho/8)\setminus E']$ that

$$c\sigma[\Delta(Y^*,s^*,r_i/4)\cap\partial\tilde{\Omega}]\ge r_i^{n+1}.$$

Using these observations we see that

$$\begin{aligned}\int_{\bar{Q}_i\cap\Delta(\hat{Z},\hat{\tau},\rho)} d(\{(Y,s)\},P)^2\,d\sigma(Y,s)\;&\le cr_i^{n+1}\,[d(\{(Y^*,s^*)\},P)+r_i]^2\\ &\le\;c\int_{\Delta(Y^*,s^*,r_i/4)\cap\partial\tilde{\Omega}} d(\{(Y,s)\},P)^2\,d\tilde{\sigma}(Y,s).\end{aligned}$$

Summing this inequality and using (7.28) (c), (d), (7.30), we find that

$$L_2\;\le\;c\tilde{\gamma}(\hat{Z},\hat{\tau},\tfrac{5}{4}\rho).$$

From this inequality, (7.32), (7.31) we get (7.29). We now integrate the inequality in (7.29) over $[E'\cap\Delta(X,t,r/4)]\times(0,r/4)$ and interchange the order of integration. If $r_i'(\hat{Z},\hat{\tau})=d(\{(\hat{Z},\hat{\tau})\},Q_i)$, we obtain from (7.29), (7.26),

$$
\begin{aligned}
\nu[E'\cap\Delta(X,t,r/4)\times(0,r/4)] &= \int_0^{r/4}\int_{E'\cap\Delta(X,t,r/4)}\gamma(\hat Z,\hat\tau,\rho)\,d\sigma(\hat Z,\hat\tau)\,\rho^{-1}\,d\rho \\
&\le c\int_{E'\cap\Delta(X,t,r/4)}\left(\sum_{i\in\xi(\hat Z,\hat\tau,r/4)}\int_{r_i'(\hat Z,\hat\tau)}^{r/4}(r_i/\rho)^{n+3}\,\rho^{-1}\,d\rho\right)d\sigma(\hat Z,\hat\tau) \\
&\qquad + c\nu[E'\cap\Delta(X,t,r/2)\times(0,r/2)] \qquad (7.33) \\
&\le c\sum_{i\in\xi(X,t,r/4)}\int_{E'\cap\Delta(X,t,r/4)}\left(\frac{r_i}{r_i'(\hat Z,\hat\tau)}\right)^{n+3}d\sigma(\hat Z,\hat\tau)\,+c\delta^5 r^{n+1} \\
&\le c\sum_{i\in\xi(X,t,r/4)} r_i^{n+1}\,+\,c\delta^5 r^{n+1}\le c\sigma(\Delta(X,t,r)\setminus E')+c\delta^5 r^{n+1}\le c\delta^5 r^{n+1},
\end{aligned}
$$

where the last line was obtained using (7.27), (7.28) and Ahlfors regularity. To estimate $\nu[\Delta(X,t,r/4)\setminus E'\times(0,r/4)]$ we note that by assumption ν is a Carleson measure on $\partial\Omega\times(0,\infty)$ with norm $\|\nu\|_+\le 10^{-10}$. Using this fact and applying the same argument as in (7.9) we deduce

$$\int_{\Delta(X,t,r)}\left(\int_0^r\gamma(\hat Z,\hat\tau,\rho)\rho^{-1}d\rho\right)^2 d\sigma(\hat Z,\hat\tau)\le\, cr^{n+1}.$$

From this inequality, Hölder's inequality, and (7.27) we get

$$\nu[\,\Delta(X,t,r/4)\setminus E'\times(0,r/4)\,]\,\le\, c\delta^5 r^{n+1}. \tag{7.34}$$

We conclude from (7.34), (7.33) that

$$\nu[\,\Delta(X,t,r/4)\times(0,r/4)\,]\le\, c\delta^5\, r^{n+1}. \tag{7.35}$$

Since $(X,t),r$ are arbitrary subject to $(X,t)\in\Delta(Y,s,\delta^\beta r_1)$ for some $(Y,s)\in F$ and $r\le\delta^\beta r_1$, we conclude first that Proposition 6.2 is true for δ'' small enough with $\hat r=\delta^\beta r_1/4$ and thereupon that Theorem 3 is valid. □

Remark: For $n=1$ Theorem 3 remains valid when (a) is replaced by (5.12). In fact the only use we made of (a) was through Proposition 5.1 which states that (a) implies $\partial\Omega$ is Reifenberg flat with vanishing constant. For $n=1$ the weaker assumption (5.12) still implies that $\partial\Omega$ is Reifenberg flat with vanishing constant as we showed in Proposition 5.13.

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