The Boundary Harnack Inequality for Infinity Harmonic Functions in the Plane

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Abstract

We prove the boundary Harnack inequality for positive infinity harmonic functions vanishing on a portion of the boundary of a bounded domain $\Omega \subset \mathbf{R}^2$ under the assumption that $\partial\Omega$ is a quasicircle.

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1 Introduction

In this paper we prove the boundary Harnack inequality for positive infinity harmonic functions vanishing on a portion of the boundary of a bounded domain $\Omega \subset \mathbf{R}^2$ under the assumption that $\partial\Omega$ is a Jordan curve and Ω is a uniform domain. The geometric restrictions imposed are equivalent to the statement that $\partial\Omega$ is a quasicircle.

Before we can state our main result we need to introduce some notation. In particular, we let $\bar{E}, \partial E$ and diam E be the closure, boundary and diameter of the set $E \subset \mathbf{R}^2$ and we define $d(y, E)$ to equal the distance from $y \in \mathbf{R}^2$ to E. $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbf{R}^2$ and $|x| = \langle x, x \rangle^{1/2}$ is the Euclidean norm of x. $B(x, r) = \{y \in \mathbf{R}^n : |x - y| < r\}$ is defined whenever $x \in \mathbf{R}^2$, $r > 0$, and we let dx denote two-dimensional Lebesgue measure on $\mathbf{R}^2$. If $O \subset \mathbf{R}^2$ is open, $1 \leq q \leq \infty$, then by $W^{1,q}(O)$ we denote the space of equivalence classes of functions g with distributional gradient $\nabla g = (g_{x_1}, g_{x_2})$, both of which are q th power integrable on O. We let $\|g\|_{1,q} = \|g\|_q + \| \, |\nabla g| \, \|_q$ be the norm in $W^{1,q}(O)$ where $\| \cdot \|_q$ denotes the usual Lebesgue q norm in O. Note that if $q = \infty$, then $\|g\|_{1,\infty} = \|g\|_\infty + \| |\nabla g| \, \|_\infty$ where $\| \cdot \|_\infty$ denotes the essential supremum on O. $C_0^\infty(O)$ is the set of infinitely differentiable functions with compact support in O and we let $W_0^{1,q}(O)$ be the closure of $C_0^\infty(O)$ in the norm of $W^{1,q}(O)$. Finally, $C(O)$ is the set of continuous functions on O.

The infinity Laplace equation, in a bounded domain $G \subset \mathbf{R}^2$, is the partial differential equation

$$\Delta_\infty u := \sum_{i,j=1}^{2} u_{x_i}(x) u_{x_j}(x) u_{x_i x_j}(x) = 0 \tag{1.1}$$

defined for smooth functions u in G. As the equation in (1.1) is a nonlinear and highly degenerate partial differential equation the mere concept of solutions to (1.1) is a non-trivial one and solutions to (1.1) have to be understood in the viscosity sense. In particular, u is said to be infinity harmonic in G if u is continuous and if u is a solution to (1.1) in the viscosity sense. Moreover, if $f \in W^{1,\infty}(\mathbf{R}^n)$ then there exists a viscosity solution $u \in W^{1,\infty}(G) \cap C(\bar{G})$, see [J, Theorem 1.8], to the Dirichlet problem

$$\Delta_\infty u = 0 \text{ in } G, \quad \lim_{x \in \Omega, \ x \to \xi} u(x) = f(\xi) \text{ for } \xi \in \partial G. \tag{1.2}$$

Using the maximum principle of Jensen, see [J, Theorem 3.11], we also see that u is the unique viscosity solution to (1.2) in G. In addition Jensen proved, see [J, Corollary 3.14], that if $f : \partial G \to \mathbf{R}$ is a continuous function, then the Dirichlet problem in (1.2) has a unique viscosity solution in G and u attains the boundary data at every boundary point $\xi \in \partial G$. Note that viscosity solutions are by definition continuous and that the conclusions on the solvability of the Dirichlet problem in (1.2) are valid for any bounded domain $G \subset \mathbf{R}^2$.

The equation in (1.1) was first derived by Aronsson [A1-A4] and plays an important role as the governing equation for so called absolute minimizers. An absolute minimizer u in G is a continuous function $u : \bar{G} \to \mathbf{R}$, which is Lipschitz continuous in G, and which has the following property: if $\hat{G}$ is a subset of G and if v is a function, continuous on the closure of $\hat{G}$ and Lipschitz continuous in $\hat{G}$, which satisfies $v = u$ on $\partial\hat{G}$, then ess-sup$_{\hat{G}}|\nabla u| \leq$ ess-sup$_{\hat{G}}|\nabla v|$. Moreover, the solution u to (1.2) is the unique absolutely minimizing Lipschitz extension of

f. For more on the infinity Laplacian, its generalizations as well as equivalent definitions of infinity harmonic functions we refer to [BEJ], [CEG] and [ACJ]. Moreover, for applications of the infinity Laplacian to image processing and game theory we refer to [CMS] and [PSSW] respectively.

The purpose of this paper is to prove the boundary Harnack inequality for positive infinity harmonic functions vanishing on a portion of a quasicircle. We also mention that our main result, Theorem 1, is completely new, except for sufficiently smooth domains where barrier type estimates can be used to get Theorem 1 (see [B]).

A Jordan curve J is said to be a k quasicircle, $0 < k < 1$, if $J = h(\partial B(0,1))$ where $h \in W^{1,2}(\mathbf{R}^2)$ is a homeomorphism of $\mathbf{R}^2$ and

$$|h_{\bar{z}}| \leq k|h_z|, \, dx \text{ almost everywhere in } \mathbf{R}^2. \tag{1.3}$$

Here we are using complex notation, $i = \sqrt{-1}$, $z = x_1 + ix_2$, $2h_{\bar{z}} = h_{x_1} + ih_{x_2}$, $2h_z = h_{x_1} - ih_{x_2}$. We say that J is a quasicircle if J is a k quasicircle for some $0 < k < 1$. Let w_1, w_2 be distinct points on the Jordan curve J and let J_1, J_2 be the arcs on J with endpoints w_1, w_2. Then J is said to satisfy the Ahlfors three point condition provided there exists $1 \leq \tilde{M} < \infty$ such that whenever $w_1, w_2 \in J$, we have

$$\min\{\mathrm{diam} J_1, \mathrm{diam} J_2\} \leq \tilde{M}|w_1 - w_2|. \tag{1.4}$$

Ω is said to be a uniform domain provided there exists $\hat{M}$, $1 \leq \hat{M} < \infty$, such that if $w_1, w_2 \in \Omega$, then there is a rectifiable curve $\gamma : [0, 1] \to \Omega$ with $\gamma(0) = w_1$, $\gamma(1) = w_2$, such that if $H^1(\cdot)$ denotes the one-dimensional Hausdorff measure on γ, then

$$\begin{array}{ll} (i) & H^1(\gamma) \leq \hat{M}|w_1 - w_2|, \\ (ii) & \min\{H^1(\gamma([0,t])), H^1(\gamma([t,1]))\} \leq \hat{M} d(\gamma(t), \partial\Omega). \end{array} \tag{1.5}$$

Let $d(E, F)$ denote the Hausdorff distance between the nonempty sets E and F. Recall that if $1 \leq M < \infty$ and if $\Omega \subset \mathbf{R}^2$ is a domain, then a ball $B(w, r)$ is said to be M non tangential provided

$$M^{-1} r < d(B(w, r), \partial\Omega) < Mr.$$

Furthermore, if $w_1, w_2 \in \Omega$, then a Harnack chain from w_1 to w_2 in Ω is a sequence of M non tangential balls such that the first ball contains w_1, the last ball contains w_2, and consecutive balls intersect. A domain Ω is called non tangentially accessible (NTA) if there exist M (as above) such that:

$$\begin{array}{ll} (i) & \text{corkscrew condition: for any } w \in \partial\Omega, 0 < r \leq r_0, \text{ there exists } a_r(w) \in \Omega \text{ such that} \\ & M^{-1} r < |a_r(w) - w| < r, d(a_r(w), \partial\Omega) > M^{-1} r, \\ (ii) & \mathbf{R}^2 \setminus \bar{\Omega} \text{ satisfies the corkscrew condition,} \\ (iii) & \text{Harnack chain condition: given } \epsilon > 0, w_1, w_2 \in \Omega, \ d(w_j, \partial\Omega) > \epsilon, \text{ and} \\ & |w_1 - w_2| < c\epsilon, \text{there is a Harnack chain from } w_1 \text{ to } w_2 \text{ whose length} \\ & \text{depends on } c \text{ but not on } \epsilon. \end{array} \tag{1.6}$$

Note that Ω is a uniform domain if and only if (1.6) (i) and (iii) hold. Moreover, if $\partial\Omega = J$ is a Jordan curve, then the conditions J is a quasicircle, J satisfies the Ahlfors three point condition, Ω is a uniform domain and Ω is non tangentially accessible all imply each other and constants in one definition can be determined from the constants in any of the other definitions. For more on these geometric notions and proofs of the stated statements we refer to [G]. Finally we let $\Delta(w,r) = \partial\Omega \cap B(w,r)$ whenever $w \in \partial\Omega$, $0 < r$.

We can now state the main theorem proved in this paper.

Theorem 1. *Let $\Omega \subset \mathbf{R}^2$, assume $\partial\Omega$ is a Jordan curve and assume that Ω is a uniform domain with constant $\hat{M}$. Let $w \in \partial\Omega$, $0 < r \leq r_0$. Suppose u and v are positive infinity harmonic functions in $\Omega \cap B(w,4r)$, that u and v are continuous in $\bar{\Omega} \cap B(w,4r)$ and that $u = 0 = v$ on $\Delta(w,4r)$. Then there exists a constant $\hat{c}_1 \in [1,\infty)$, which only depends on $\hat{M}$, such that if $\tilde{r} = r/\hat{c}_1$, $u(a_{\tilde{r}}(w)) = 1 = v(a_{\tilde{r}}(w))$ and $x \in \Omega \cap B(w, r/\hat{c}_1)$, then*

$$\hat{c}_1^{-1} \leq \frac{u(x)}{v(x)} \leq \hat{c}_1.$$

Concerning the proof of Theorem 1 we note that problems for the infinity Laplacian can often be understood by considering limits of the corresponding problems posed for the p Laplacian, Δ_p, for finite p, $1 < p < \infty$, and our proof of Theorem 1 is, in fact, based on a uniform in p boundary Harnack inequality, for large values of p, stated as Theorem 2 below.

Recall that given a bounded domain G and $1 < p < \infty$, then u is said to be p harmonic in G provided $u \in W^{1,p}(G)$ and

$$\int |\nabla u|^{p-2} \langle \nabla u, \nabla \psi \rangle \, dx = 0 \tag{1.7}$$

whenever $\psi \in W_0^{1,p}(G)$. Observe that if u is smooth and $\nabla u \neq 0$ in G, then

$$\Delta_p u := \nabla \cdot (|\nabla u|^{p-2} \nabla u) \equiv 0 \text{ in } G \tag{1.8}$$

and u is a classical solution in G to the p Laplace partial differential equation. Here, as in the sequel, $\nabla\cdot$ is the divergence operator. If $f \in W^{1,p}(\mathbf{R}^n)$ we let u_p denote the unique weak solution to the problem

$$\Delta_p u_p = 0 \text{ in } G, \quad \lim_{x\in\Omega,\ x\to\xi} u_p(x) = f(\xi) \text{ for } \xi \in \partial G \tag{1.9}$$

and we note that (1.9) means that u_p solves (1.7) and that $u_p - f \in W_0^{1,p}(G)$.

To describe the relation between the problems in (1.2) and (1.9), we assume $f \in W^{1,\infty}(\mathbf{R}^n)$ and we let $u_\infty \in W^{1,\infty}(G) \cap C(\bar{G})$ be the unique viscosity solution to (1.2) with boundary data defined by the restriction of f to ∂G. We also let u_p, for $p \in (1,\infty)$, denote the unique weak solution to the problem in (1.9) with data defined by f. Arguing as in [J, Theorem 1.22] we see that there exists a subsequence $\{p_j\}$, $p_j \to \infty$ as $j \to \infty$, such that $u_{p_j} \to u_\infty$, uniformly in $C(\bar{G})$ as $j \to \infty$. In particular, the unique solution $u = u_\infty$ to the problem in (1.2) is the uniform limit of the corresponding problems in (1.9) for finite p.

We base our proof of Theorem 1 on the following uniform in p boundary Harnack inequality.

Theorem 2. *Let $\Omega \subset \mathbf{R}^2$, assume $\partial\Omega$ is a Jordan curve and assume that Ω is a uniform domain with constant $\hat{M}$. Then there exists $\hat{p} > 2$ such that the following is true. Let p be given, $\hat{p} \leq p < \infty$, $w \in \partial\Omega$, $0 < r \leq r_0$ and suppose that u and v are positive p harmonic functions in $\Omega \cap B(w, 4r)$. Assume also that u and v are continuous in $\bar{\Omega} \cap B(w, 4r)$ and that $u = 0 = v$ on $\Delta(w, 4r)$. Then there exists a constant $\hat{c}_2 \in [1, \infty)$, which depends on $\hat{M}$ and $\hat{p}$ but is independent of p, such that if $\tilde{r} = r/\hat{c}_2$, $u(a_{\tilde{r}}(w)) = 1 = v(a_{\tilde{r}}(w))$ and $x \in \Omega \cap B(w, r/\hat{c}_2)$, then*

$$\hat{c}_2^{-1} \leq \frac{u(x)}{v(x)} \leq \hat{c}_2.$$

Concerning the proof of Theorem 2 we note that Theorem 2 was proved in [BL, Lemma 2.16] for a fixed and given p, $1 < p < \infty$, with a constant $\hat{c}_2 = \hat{c}_2(p, \hat{M})$. Hence in Theorem 2 we refine the result in [BL, Lemma 2.16] by proving that there exists $\hat{p} > 2$ such that $\hat{c}_2$ can be chosen to depend only on $\hat{p}$ whenever $\hat{p} \leq p < \infty$. We also note that in [LN] the boundary Harnack inequality for p harmonic functions, $1 < p < \infty$, vanishing on a portion of the boundary of a Lipschitz domain $\Omega \subset \mathbf{R}^n$, $n \geq 2$, was proved. The techniques used in [LN] to establish the boundary Harnack inequality differ significantly from those used in [BL]. In particular, currently we are not able to extend the approach in [LN] to the case $p = \infty$.

The rest of the paper is organized in the following way. In section 2 we prove a number of uniform in p estimates for p harmonic functions and in section 3 we prove Theorem 1 and Theorem 2.

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2 Uniform (in p) estimates for p harmonic functions

In the following we consider a bounded domain $\Omega \subset \mathbf{R}^2$ and at certain instances we will assume that $\partial\Omega$ is a Jordan curve and that Ω is a uniform domain with constant $\hat{M}$. We intend to state and prove certain interior and boundary estimates, with uniform in p constants, for u assuming that u is a positive weak solution to the p Laplacian in $\Omega \cap B(w, 2r)$, u is continuous in $\bar{\Omega} \cap \bar{B}(w, 2r)$ with $u = 0$, in the Sobolev sense, on $\Delta(w, 2r)$ when this set is nonempty. Recall that $\Delta(w, r) = \partial\Omega \cap B(w, r)$ whenever $w \in \partial\Omega$, $0 < r$. More specifically we assume that $u \in W^{1,p}(\Omega \cap B(w, 2r))$ and that (1.7) holds whenever $\psi \in W_0^{1,p}(\Omega \cap B(w, 2r))$. Also $\zeta u \in W_0^{1,p}(\Omega \cap B(w, 2r))$ whenever $\zeta \in C_0^\infty(B(w, 2r))$. Often we extend u to $B(w, 2r)$ by defining $u \equiv 0$ on $B(w, 2r) \setminus \Omega$. Moreover, we let $\max\limits_{B(z,s)} u$, $\min\limits_{B(z,s)} u$ be the essential supremum and infimum of u on $B(z, s)$ whenever $B(z, s) \subset \mathbf{R}^2$ and whenever u is defined on $B(z, s)$. Throughout the paper c will denote, unless otherwise stated, a positive constant ≥ 1, not necessarily the same at each occurrence, which is independent of p but may depend on $\hat{M}$ and $\hat{p}$. In general, $c(a_1, \ldots, a_n)$ denotes a positive constant ≥ 1, not necessarily the same at each occurrence, which is independent of p but depends on $a_1, \dots, a_n$.

The following lemmas, Lemma 2.1-2.6, can be found in [BL] with constants depending on p. Here we show that the constants can be chosen independently of p.

Lemma 2.1. *(Energy estimate) Let $\Omega \subset \mathbf{R}^2$ be a bounded domain. There exists $\hat{p} > 2$ such that the following is true. Assume $w \in \bar{\Omega}$, $0 < r \leq \operatorname{diam} \Omega$ and assume that either $B(w, 2r) \subset \Omega$ or $w \in \partial\Omega$. Let p be given, $\hat{p} \leq p < \infty$, and suppose that u is a non negative weak solution to (1.7) in $\Omega \cap B(w, 2r)$ and that $u = 0$ on $\Delta(w, 2r)$ whenever this set is non empty. Then there exists a constant c_1, $1 \leq c_1 < \infty$, which depends on $\hat{p}$ but is independent of p, such that*

$$r^{p-n} \int\limits_{\Omega \cap B(w,r)} |\nabla u|^p \, dx \;\leq\; c_1^p \,(\max_{\Omega \cap B(w,2r)} u)^p.$$

Lemma 2.2. *(Interior Harnack inequality) There exists $\hat{p} > 2$ such that the following is true. Let p be given, $\hat{p} \leq p < \infty$, $w \in \mathbf{R}^2$, $0 < r$, and let u be a positive weak solution to (1.7) in $B(w, 2r)$. Then there exists a constant c_2, $1 \leq c_2 < \infty$, which depends on $\hat{p}$ but is independent of p, such that*

$$\max_{B(w,r)} u \;\leq c_2 \min_{B(w,r)} u.$$

Lemma 2.3. *(Hölder estimate) Let $\Omega \subset \mathbf{R}^2$ be a bounded domain. There exists $\hat{p} > 2$ such that the following is true. Assume $w \in \bar{\Omega}$, $0 < r \leq \operatorname{diam} \Omega$ and assume that either $B(w, 2r) \subset \Omega$ or $w \in \partial\Omega$. Let p be given, $\hat{p} \leq p < \infty$, and suppose that u is a non negative weak solution to (1.7) in $\Omega \cap B(w, 2r)$ and that $u = 0$, in the Sobolev sense, on $\Delta(w, 2r)$ whenever this set is non empty. Then there exist $\alpha_3 \in (0, 1)$ and a constant c_3, $1 \leq c_3 < \infty$, both which depend on $\hat{p}$ but are independent of p, such that if $x, y \in \Omega \cap B(w, r/2)$ then*

$$|u(x) - u(y)| \leq c_3 \left(\frac{|x-y|}{r} \right)^{\alpha_3} \max_{\Omega \cap B(w,2r)} u.$$

Lemma 2.4. *(Carleson type estimate) Let $\Omega \subset \mathbf{R}^2$ be a bounded domain, assume $\partial\Omega$ is a Jordan curve and assume that Ω is a uniform domain with constant $\hat{M}$. There exists $\hat{p} > 2$ such that the following is true. Let p be given, $\hat{p} \leq p < \infty$, $w \in \partial\Omega$, $0 < r \leq r_0$, and suppose that u is a positive continuous p harmonic function in $\Omega \cap B(w, 2r)$ and that $u = 0$, in the Sobolev sense, on $\Delta(w, 2r)$. Then there exists a constant c_4, $1 \leq c_4 < \infty$, which depends on $\hat{M}$ and $\hat{p}$ but is independent of p, such that if $\tilde{r} = r/c_4$, then*

$$\max_{\Omega \cap B(w,\tilde{r})} u \;\leq\; c_4 \, u(a_{\tilde{r}}(w)).$$

Lemma 2.5. *Let $\Omega \subset \mathbf{R}^2$ be a bounded domain, assume $\partial\Omega$ is a Jordan curve and assume that Ω is a uniform domain. Given $p, 1 < p < \infty, w \in \partial\Omega, 0 < r \leq r_0$, suppose that u is a positive p harmonic function in $\Omega \cap B(w, 2r)$, continuous on $\bar{\Omega} \cap \bar{B}(w, 2r)$ with $u = 0$ on $\Delta(w, 2r)$. Extend*

u to $B(w,2r)$ by defining $u \equiv 0$ on $B(w,2r)\setminus\Omega$. Then there exists a unique finite positive Borel measure μ on $\mathbf{R}^2$ with support in $\Delta(w,2r)$, such that whenever $\psi \in C_0^\infty(B(w,2r))$ then

$$\int\limits_{\mathbf{R}^2} |\nabla u|^{p-2}\,\langle \nabla u,\, \nabla \psi\rangle\, dx \;=\; -\int\limits_{\mathbf{R}^2} \psi\, d\mu.$$

Lemma 2.6. *Let Ω, u, w, r, r_0 and μ be as in Lemma 2.5. Then there exists $\hat p > 2$ such that the following is true for p given, $\hat p \leq p < \infty$. There exists a constants c_6, $1 \leq c_6 < \infty$, which depends on $\hat M$ and $\hat p$ but is independent of p, such that if $\tilde r = r/c_6$, then*

$$c_6^{-p} r^{p-n} \mu(\Delta(w, \tilde r/2)) \leq (u(a_{\tilde r}(w)))^{p-1} \leq c_6^p r^{p-n}\, \mu(\Delta(w, 2\tilde r)).$$

In the following we prove Lemmas 2.1-2.6. In the proof of Lemmas 2.1-2.3 we will assume, as we may by a simple translation, scaling and normalization argument, that

$$w = 0,\ r = 1 \text{ and } \max_{\Omega\cap B(w,2r)} u = 1. \tag{2.7}$$

Assumption (2.7) is permissible since the p Laplace equation is invariant under translations and scalings and since the same is true for the parameter $\hat M$ in the definition of a uniform domain. We will frequently also make use the following test function,

$$\theta \in C_0^\infty(B(0,2)),\ \theta \equiv 1 \text{ on } B(0,3/2),\ \theta \geq 0 \text{ and } |\nabla\theta| \leq c. \tag{2.8}$$

Proof of Lemma 2.1. If $w \in \partial\Omega$ then we extend the restriction of u to the set $\Omega \cap B(0,2)$, to $B(0,2)$, by defining $u \equiv 0$ on $B(0,2)\setminus\Omega$. We note that u is continuous on $B(0,2)$ by Morrey's lemma whenever $p > 2$. Then u is a non negative subsolution to the p Laplace equation in $B(0,2)$. In particular, if $\psi \in C_0^\infty(B(0,2))$, $\psi \geq 0$, then

$$\int\limits_{\mathbf{R}^2} |\nabla u|^{p-2}\,\langle \nabla u, \nabla\psi\rangle dx \leq 0. \tag{2.9}$$

To actually prove that u satisfies (2.9) one can show that if ψ is as above and if we let $\tilde\psi = [(\eta + \max[u-\epsilon, 0])^\epsilon - \eta^\epsilon]\psi$, with $\eta > 0$ small, then $\tilde\psi$ is an admissible test function for (1.7). Moreover, using (1.7) we see that

$$\int\limits_{\{u\geq\epsilon\}} [(\eta + \max[u-\epsilon,0])^\epsilon - \eta^\epsilon]|\nabla u|^{p-2}\,\langle\nabla u, \nabla\psi\rangle dx \leq 0.$$

Using dominated convergence, first letting $\eta \to 0$ and then $\epsilon \to 0$ we see that u satisfies (2.9).

Hence, to prove Lemma 2.1 we can assume that u satisfies (2.9) whenever $\psi \in C_0^\infty(B(0,2))$ and $\psi \geq 0$. Moreover, assume (2.7), let θ be as in (2.8) and define $\psi = \theta^p(1+u)^p$. Then we first see that

$$\nabla\psi = p\theta^{p-1}(1+u)^p\nabla\theta + p\theta^p(1+u)^{p-1}\nabla u \tag{2.10}$$

and using (2.10) in (2.9) we can conclude that

$$p\int_{\mathbf{R}^2} |\nabla u|^{p-2}\langle \nabla u, \nabla\theta\rangle \theta^{p-1}(1+u)^p dx + p\int_{\mathbf{R}^2} |\nabla u|^p\theta^p(1+u)^{p-1}dx \leq 0.$$

Therefore,

$$\int_{\mathbf{R}^2} |\nabla u|^p\theta^p(1+u)^{p-1}dx \leq c\int_{\mathbf{R}^2} |\nabla u|^{p-1}\theta^{p-1}(1+u)^p dx,$$

and, using the normalization of u in (2.7),

$$\int_{\mathbf{R}^2} |\nabla u|^p\psi dx \leq c\int_{\mathbf{R}^2} |\nabla u|^{p-1}\psi^{(p-1)/p}dx. \tag{2.11}$$

Based on (2.11) we can complete the proof of Lemma 2.1 by using Hölder's inequality.

Proof of Lemma 2.2. The Harnack inequality, Lemma 2.2, for non negative p harmonic functions can be proved by the iteration methods of DeGiorgi and Moser, see [S] and [DBT]. Unfortunately, in these methods the constant in the Harnack inequality blows up as $p \to \infty$. Another approach to the Harnack inequality, valid only when $p > 2$, follows from energy bounds on $\nabla(\log u)$. In fact, using this approach the Harnack inequality can also be established for the infinity Laplacian, see [LM].

Assume (2.7) and let θ be as in (2.8). Let $\epsilon > 0$ and define $u_\epsilon = (\epsilon + u)$. Using the test function $\theta^p u_\epsilon^{1-p}$ in (1.7) we see that

$$0 = (1-p)\int_{\mathbf{R}^2} |\nabla u|^{p-2}\langle \nabla u, \nabla u_\epsilon\rangle \theta^p u_\epsilon^{-p}\, dx + p\int_{\mathbf{R}^2} |\nabla u|^{p-2}\langle \nabla u, \nabla\theta\rangle \theta^{p-1}u_\epsilon^{1-p}\, dx$$

and hence

$$\int_{\mathbf{R}^2} |\nabla u|^p\theta^p u_\epsilon^{-p}\, dx \leq c\left(\frac{p}{p-1}\right)\int_{\mathbf{R}^2} |\nabla u|^{p-1}\theta^{p-1}u_\epsilon^{1-p}\, dx.$$

Therefore, using the Hölder inequality and letting $\epsilon \to 0$ we see that

$$\int_{\mathbf{R}^2} |\nabla u|^p\theta^p u^{-p}\, dx \leq c^p\left(\frac{p}{p-1}\right)^p. \tag{2.12}$$

Assuming (2.7) we note that (2.12) in fact states that

$$\int_{\mathbf{R}^2} |\theta\nabla(\log u)|^p\, dx \leq c^p\left(\frac{p}{p-1}\right)^p. \tag{2.13}$$

Let $\hat p > 2$. Using the Sobolev embedding theorem and the Hölder inequality we see that if $x \in B(0,1)$ and $p \geq \hat p$ then

$$(|\log u(x) - \log u(0)|)^{\hat p} \leq c^{\hat p} \int\limits_{B(0,1)} |\nabla(\log u)|^{\hat p} dx \leq c^{\hat p+1-\hat p/p}\Bigg(\int\limits_{B(0,1)} |\nabla(\log u)|^{p} dx\Bigg)^{\hat p/p}. \quad (2.14)$$

Based on (2.14) we then see, using (2.13), that

$$|\log u(x) - \log u(0)| \leq c^{2+1/\hat p-1/p} \leq c^{2+1/\hat p}$$

whenever $x \in B(0,1)$ and for a constant c which is independent of p. Hence, $c_2^{-1}u(0) \leq u(x) \leq c_2 u(0)$ whenever $x \in B(0,1)$ for some c_2 which is independent of p.

Proof of Lemma 2.3. Assume (2.7) and let u denote the restriction of u to the set $\Omega \cap B(0,2)$. Assume first that $w \in \partial\Omega$. Extend u to $B(0,2)$ by defining $u \equiv 0$ on $B(0,2) \setminus \Omega$. As $u = 0$ on $\Delta(0,2)$ it follows from Lemma 2.1 that $u \in W^{1,p}(B(0,1))$. Hence to prove Lemma 2.3 we can, in both cases, simply use the Sobolev embedding theorem in the following way. Let $\hat p > 2$. Then using the Sobolev embedding theorem, the Hölder inequality and Lemma 2.1 we see that

$$\sup_{x,y\in B(0,1/2)} \frac{|u(x)-u(y)|}{|x-y|^{1-2/\hat p}} \leq c^{2+1/\hat p-1/p}\Bigg(\int\limits_{\Omega\cap B(0,1)} |\nabla u|^p dx\Bigg)^{1/p} \leq c^{2+1/\hat p}$$

for a constant c which is independent of p. Hence Lemma 2.3 is valid with $\alpha_3 = 1 - 2/\hat p$.

Proof of Lemma 2.4. This follows from a general argument using Lemma 2.2 and Lemma 2.3 often attributed to Carleson (see [CFMS]). Hence the constant c_4 is a function of the constants c_2, c_3 and α_3.

Proof of Lemma 2.5. Existence of μ in Lemma 2.5 follows from (2.9), basic Caccioppoli estimates as in Lemma 2.1, and the same argument as in the proof of the Riesz representation theorem for positive linear functionals on the space of continuous functions.

Proof of Lemma 2.6. Note that using Lemma 2.2 and Lemma 2.4 we see that there exists $\tilde c$, $1 \leq \tilde c < \infty$, which depend on $\hat M$ and $\hat p$ but is independent of p, such that if $\tilde r = r/\tilde c$, then

$$\max_{\Omega\cap B(w,2\tilde r)} u \approx u(a_{\tilde r}(w)).$$

Moreover, in the following we modify (2.7) and we assume, as we may, that

$$w = 0,\ \tilde r = 1 \text{ and } u(a_{\tilde r}(w)) = 1. \quad (2.7')$$

Let θ be as in (2.8) and define $\hat\theta(x) = \theta(2x)$. Using Lemma 2.5, Lemma 2.1, the normalization in (2.7') and the inequality in the display above (2.7') we see that

$$\mu(\Delta(0,1/2)) \leq \int\limits_{\mathbf{R}^2} \hat\theta\, d\mu \leq \int\limits_{\mathbf{R}^2} |\nabla u|^{p-1}|\nabla\hat\theta| dx \leq c\Bigg(\int\limits_{\Omega\cap B(0,1)} |\nabla u|^p dx\Bigg)^{(p-1)/p} \leq c c_1^{p-1} \leq c^{p-1}.$$

Hence, there exist $\hat{p} > 2$ and a constant c, independent of p, such that if $2 < \hat{p} \leq p < \infty$, then $\mu(\Delta(0, 1/2)) \leq c^{p-1}(u(a_1(0))^{p-1}$. Hence the left hand side inequality in Lemma 2.6 is proved.

Next we prove the right hand side inequality in Lemma 2.6 and our proof is based on [KZ], see also [EL]. We define

$$M(\rho) = \sup_{B(0,\rho)} u(x) \text{ whenever } \rho \in [0, 2]$$

and we let h be p harmonic in $B(0, 2)$ with boundary values equal to u on $\partial B(0, 2)$. Note that by assumption u is continuous on $\bar{\Omega} \cap \bar{B}(0, 2)$ and hence u is well defined on $\partial B(0, 2)$. By the weak maximum principle for p harmonic functions we see that $0 \leq u \leq h$ on $B(0, 2)$. Moreover, considering $p \geq \hat{p}$ and applying the Harnack inequality in Lemma 2.2 to the function h we see that

$$\inf_{B(0,1)} h \geq c_2^{-1} \sup_{B(0,1)} h \geq c_2^{-1} \sup_{B(0,1)} u = c_2^{-1} M(1). \tag{2.15}$$

c_2 is the constant appearing in Lemma 2.2 and this constant is independent of p. Using Lemma 2.3 we see that

$$u(x) \leq c_3 t^{\alpha_3} M(1) \text{ whenever } x \in B(0, t),\ t < 1/2. \tag{2.16}$$

Let $\beta = c_2^{-1}/2$ and restrict t to the interval $[0, (\beta/c_3)^{1/\alpha_3})$. Using (2.16) it then follows that $M(t) \leq \beta M(1)$. Under the same conditions we also see, using (2.15), that, whenever $x \in B(0, t)$,

$$h(x) - u(x) \geq \inf_{B(0,1)} h - \sup_{B(0,t)} u \geq 2\beta M(1) - \beta M(1) = \beta M(1). \tag{2.17}$$

Next we note that the function

$$\psi = \min_{B(0,2)} \{h - u, \beta M(1)\} \tag{2.18}$$

is non negative in $B(0, 2)$ and belongs to the space $W_0^{1,p}(B(0, 2))$. Using (2.17) we also see that $\psi = \beta M(1)$ on $B(0, t)$. Let Γ denote the points where $\nabla\psi$ exists and is non zero. Then

$$\int\limits_{B(0,2)} |\nabla\psi|^p dx \ \leq \ \int\limits_{B(0,2)\cap\Gamma} \Big(|\nabla h| + |\nabla u|\Big)^{p-2} |\nabla h - \nabla u|^2 dx. \tag{2.19}$$

Next we claim that if $p \geq 2$ then

$$\Big(|\xi| + |\zeta|\Big)^{p-2} |\xi - \zeta|^2 \leq c 2^p \langle |\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta, \xi - \zeta\rangle \tag{2.20}$$

whenever $\xi, \zeta \in \mathbf{R}^2$. In fact, to prove this we first note that

$$\langle |\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta, \xi - \zeta\rangle = \frac{1}{2}(|\xi|^{p-2} + |\zeta|^{p-2})|\xi - \zeta|^2 + \frac{1}{2}(|\xi|^{p-2} - |\zeta|^{p-2})(|\xi|^2 - |\zeta|^2) \tag{2.21}$$

and hence

$$\langle|\xi|^{p-2}\xi-|\zeta|^{p-2}\zeta,\xi-\zeta\rangle \geq \frac{1}{2}(|\xi|^{p-2}+|\zeta|^{p-2})|\xi-\zeta|^2 \geq c^{-1}2^{-p}\biggl(|\xi|+|\zeta|\biggr)^{p-2}|\xi-\zeta|^2. \quad (2.22)$$

Combining (2.19) and (2.22) we get

$$\begin{aligned}\int\limits_{B(0,2)}|\nabla\psi|^p dx &\leq c2^p\int\limits_{B(0,2)}\langle(|\nabla h|^{p-2}\nabla h-|\nabla u|^{p-2}\nabla u),\nabla\psi\rangle dx\\ &= -c2^p\int\limits_{B(0,2)}\langle|\nabla u|^{p-2}\nabla u,\nabla\psi\rangle dx = c2^p\int\limits_{B(0,2)}\psi d\mu\end{aligned}$$

based on which we can conclude that

$$\int\limits_{B(0,2)}|\nabla\psi|^p dx \leq c2^p\beta M(1)\mu(B(0,2)). \quad (2.23)$$

But on the other hand using Hölder's inequality as well as a Sobolev type inequality we see that

$$(\beta M(1))^p\pi t^2 \leq \int\limits_{B(0,2)}|\psi|^p dx \leq c^p\int\limits_{B(0,2)}|\nabla\psi|^p dx. \quad (2.24)$$

That the c in (2.24) can be chosen independent of p for large values of p is intuitively clear and an explicit constant, independent of p, for which the right hand inequality in (2.24) is true can be found in [GT, p.164]. Combining (2.23) and (2.24) we can therefore conclude that

$$(\beta M(1))^p\pi t^2 \leq c^p\beta M(1)\mu(B(0,2)) \quad (2.25)$$

for a constant c independent of p. Hence, if we choose $t=(\beta/c_3)^{1/\alpha_3}/2$, then

$$c^p\mu(B(0,2)) \geq (\beta M(1))^{p-1}\pi t^2 \geq \hat{c}^p(u(a_1(0))^{p-1}. \quad (2.26)$$

In the last deductions we have used Lemma 2.4, Lemma 2.2, the normalization in (2.7') and the inequality in the display above (2.7'). In (2.26) both constants, c and $\hat{c}$, are independent of p. This completes the proof of the right hand side inequality and hence the proof of Lemma 2.6.

3 Proof of Theorem 1 and Theorem 2

In this section we prove Theorem 1 and Theorem 2.

Proof of Theorem 1. We here prove Theorem 1 assuming Theorem 2. Let u and v be as in the statement of Theorem 1 and note that this implies, in particular, that $u,v\in W^{1,\infty}(\Omega\cap B(w,3r))\cap C(\bar{\Omega}\cap\bar{B}(w,3r))$. We let, for $p\in(2,\infty)$, u_p and v_p be the unique weak solutions to the problem in (1.9) in $\Omega \cap B(w, 2r)$ with continuous boundary values, on $\partial(\Omega \cap B(w, 2r))$, equal to u and v respectively. Using the uniqueness result of Jensen [J, Theorem 3.11] it follows that for every $\epsilon > 0$ and for every compact set $K \subset \Omega \cap B(w, 2r)$ there exists $\tilde{p} = \tilde{p}(\epsilon, K) > 2$ such that if $p \geq \tilde{p}$ then

$$|u_p(x) - u(x)| < \epsilon \text{ and } |v_p(x) - v(x)| < \epsilon \tag{3.1}$$

whenever $x \in K$. In particular u_p and v_p converge to u and v, uniformly on compact subsets of $\Omega \cap B(w, 2r)$, as $p \to \infty$. Let $\hat{p}$ and $\hat{c}_2$ be as in Theorem 2. Then

$$\hat{c}_2^{-1} \frac{u_p(a_{\tilde{r}}(w))}{v_p(a_{\tilde{r}}(w))} \leq \frac{u_p(x)}{v_p(x)} \leq \hat{c}_2 \frac{u_p(a_{\tilde{r}}(w))}{v_p(a_{\tilde{r}}(w))} \tag{3.2}$$

whenever $x \in \Omega \cap B(w, r/(2\hat{c}_2))$ if $p \geq \hat{p}$. Let $z \in \Omega \cap B(w, r/(2\hat{c}_2))$ and note that using (3.1) and (3.2) we can conclude that $u(x) \leq 2\hat{c}_2 v(x)$ whenever $x \in B(z, d(z, \partial\Omega)/2)$. As z is arbitrary Theorem 1 follows.

Proof of Theorem 2. The proof of Theorem 2 is based on the proof of Lemma 2.16 in [BL]. In fact to prove Theorem 2 we repeat the argument in [BL, Lemma 2.16] making sure that the constants of our estimates can be chosen independently of p whenever $p \geq \hat{p} > 2$. In the following we let $\hat{p}$ be large enough to ensure the validity of the statements in Lemmas 2.1-2.4 and in Lemma 2.6. Moreover, we let $\tilde{r} = r/(4c_6)$ with c_6 as in Lemma 2.6.

Let $\gamma : (1, 1) \to \mathbf{R}^2$ be a parametrization of $\partial\Omega$ such that $\gamma(0) = w$. Let $r_1 = \tilde{r}/c_1$ where $c_1 = c_1(\hat{M}) \geq 1$ will be chosen later. Let $t_1 = \sup\{t < 0 : |\gamma(t) - w| = r_1\}$, $t_2 = \inf\{t < 0 : |\gamma(t) - w| = r_1\}$, $z_1 = \gamma(t_1)$ and $z_2 = \gamma(t_2)$. Then $|w - z_1| = |w - z_2| = r_1$ and the part of $\partial\Omega$ between z_1 and z_2 is contained in $B(w, r_1)$. If $r_2 = r_1/c_1$, then from (1.4) we see, for c_1 large enough, that $B(z_1, r_2) \cap B(z_2, r_2) = \emptyset$. For any two points $\zeta_1 \in \Delta(z_1, r_2)$ and $\zeta_2 \in \Delta(z_2, r_2)$ we can use (1.5) to construct a curve with endpoints ζ_1, ζ_2 in the following way. Take ρ such that $B(\zeta_i, \rho) \subset B(z_i, r_2)$ for $i = 1, 2$. Draw the curve from $a_\rho(\zeta_1)$ to $a_\rho(\zeta_2)$ guaranteed by (1.5). Similarly, connect $a_\rho(\zeta_1)$ to $a_{\rho/2}(\zeta_1)$ and then $a_{\rho/2}(\zeta_1)$ to $a_{\rho/4}(\zeta_1)$ and so on. Since $a_{\rho/2^n}(\zeta_1) \to \zeta_1$, as $n \to \infty$, this curve ends up at ζ_1. We can advance from $a_\rho(\zeta_2)$ to ζ_2 in the same way. The total curve from ζ_1 to ζ_2 is denoted by Γ. From our construction and (1.5) we note that, for c_1 large enough,

$$\begin{aligned} &(i) && \Gamma \setminus \{\zeta_1, \zeta_2\} \subset \Omega \cap B(w, \tilde{r}), \\ &(ii) && H^1(\Gamma) \leq c_1 \tilde{r}, \\ &(iii) && \min\{H^1(\Gamma([0, t])), H^1(\Gamma([t, 1]))\} \leq c_1 d(\Gamma(t), \partial\Omega). \end{aligned} \tag{3.3}$$

Recall that $H^1(\cdot)$ denotes the one-dimensional Hausdorff measure on Γ. In the following we let c_1 be fixed and satisfying the above requirements.

Next we consider the functions u, v in Theorem 2 and we extend both of these to $B(w, 2r)$ in the standard way. We let μ and ν be the measures, in the sense of Lemma 2.5, corresponding to u and v respectively. Let $M_+ = M_+(\hat{M})$ be a constant to be chosen and assume that M_+ is so large that $\Gamma \cap B(w, r_1/M_+) = \emptyset$ independently of $\zeta_1 \in \Delta(z_1, r_2)$ and $\zeta_2 \in \Delta(z_2, r_2)$. Existence of M_+ follows from (3.3). Suppose that $u/v > \lambda$ at some point in $\Omega \cap B(w, \tilde{r}/M_+)$. Our intention if to prove that λ can not be too large. Using the continuity of u and v in $\bar{\Omega} \cap B(w, r)$ and the weak maximum principle for solutions to the p Laplacian, we see that $u/v > \lambda$ at some point $\xi \in \Gamma$. Making use of the point ξ and using (3.3), Lemma 2.6 and Lemma 2.2 we deduce for some $s, 0 < s < \tilde{r}/2$ and $i \in \{1,2\}$, that

$$\frac{\mu(\Delta(\zeta_i, s))}{\nu(\Delta(\zeta_i, s))} \geq \frac{c_6^{-p}(u(a_s(\xi)))^{p-1}}{c_6^p(v(a_s(\xi)))^{p-1}} > \tilde{c}_6^{-p}\lambda^{p-1}, \tag{3.4}$$

where $\tilde{c}_6$ is a constant independent of p. Allowing ζ_i in this construction to vary in $\Delta(z_i, r_2)$, $i = 1, 2$, we get a covering of either $\Delta(z_1, r_2)$ or $\Delta(z_2, r_2)$ by balls of the form $\Delta_\zeta = \Delta(\zeta, s)$. Assume for example that $\Delta(z_1, r_2)$ is covered by balls of this type. Then using a standard covering argument we get a subcovering, $\{\Delta_{\zeta_n}\}$, of $\Delta(z_1, r_2)$ such that the balls, with one-fifth the diameter of the original balls but the same centers, denoted $\{\Delta^*_{\zeta_n}\}$, are disjoint. From (3.3), (3.4) and Lemmas 2.2 and 2.6 we then deduce

$$\tilde{c}_6^{-p}\lambda^{p-1}\nu(\Delta(z_1, r_2)) \leq \tilde{c}_6^{-p}\lambda^{p-1}\nu(\cup_n \Delta_{\zeta_n}) < \sum \mu(\Delta_{\zeta_n}) \leq \hat{c}_6^p \sum \mu(\Delta^*_{\zeta_n})$$

for yet another constant $\hat{c}_6$ which is independent of p. Hence

$$\tilde{c}_6^{-p}\lambda^{p-1}\nu(\Delta(z_1, r_2)) < \hat{c}_6^p \mu(\Delta(w, 2\tilde{r})). \tag{3.5}$$

Moreover, using Lemma 2.6 and Lemma 2.2 once more we also see that

$$\nu(\Delta(w, 2\tilde{r}) \leq c^p \nu(\Delta(z_1, r_2)) \tag{3.6}$$

for some c independent of p. Combining (3.5) and (3.6) we first see that

$$c^{-p}\tilde{c}_6^{-p}\hat{c}_6^{-p}\lambda^{p-1} < \frac{\mu(\Delta(w, 2\tilde{r}))}{\nu(\Delta(w, 2\tilde{r}))} \tag{3.7}$$

and then, using Lemma 2.6, Lemma 2.2 and the normalization $u(a_{\tilde{r}}(w)) = 1 = v(a_{\tilde{r}}(w))$,

$$c^{-p}\tilde{c}_6^{-p}\hat{c}_6^{-p}\lambda^{p-1} < \frac{\mu(\Delta(w, 2\tilde{r}))}{\nu(\Delta(w, 2\tilde{r}))} \leq \hat{c}^p \tag{3.8}$$

for yet another constant $\hat{c}$ which is independent of p. (3.8) implies that $\lambda < c$ with c independent of p. Hence $u/v \leq c$ in $\Omega \cap B(w, \tilde{r}/M_+)$ and the proof of Theorem 2 is complete.

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