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On symmetry and uniform rectifiability arising from some overdetermined elliptic and parabolic boundary conditions

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Abstract. We discuss some recent free boundary - inverse type results of the author and coauthors on elliptic and parabolic pde of p Laplacian type.

1. Elliptic Problems.

Denote points in Euclidean n - space, $\mathbb{R}^n$, by $x = (x_1, ..., x_n)$ and let $\bar{E},\ \partial E$, denote the closure and boundary of $E \subseteq \mathbb{R}^n$, respectively. Let $\langle x, y\rangle$ denote the standard inner product in $\mathbb{R}^n$, $|x| = \langle x, x\rangle^{1/2}$, and set $B(x, r) = \{y \in \mathbb{R}^n : \ |y-x| < r\}$ whenever $x \in \mathbb{R}^n, r > 0$. Define k dimensional Hausdorff measure, $1 \leq k \leq n$, in $\mathbb{R}^n$ as follows: For fixed $\delta > 0$ and $E \subseteq \mathbb{R}^n$, let $L(\delta) = \{B(x_i, r_i)\}$ be such that $E \subseteq \cup B(x_i, r_i)$ and $0 < r_i < \delta,\ i = 1, 2, ...$ Set

$$\phi^k_\delta(E) = \inf_{L(\delta)} \left(\sum \alpha(k) r_i^k \right)$$

where $\alpha(k)$ denotes the volume of the unit ball in $\mathbb{R}^k$. Then

$$H^k(E) = \lim_{\delta \to 0} \phi^k_\delta(E),\ 1 \leq k \leq n.$$

Next let $D \subset \mathbb{R}^n$ be a bounded domain, and N an open neigborhood of ∂D. We suppose for fixed $p, 1 < p < \infty$, that $u \geq 0$ is a weak solution to the p Laplacian partial differential equation $(\nabla \cdot [|\nabla u|^{p-2} \nabla u] = 0)$ in $D \cap N$. with

$$u(x) \to 0 \text{ as } x \to \partial D. \tag{1.1}$$

Put $u \equiv 0$ in $\mathbb{R}^n \setminus D$ and note that u has distributional derivatives which are p th power integrable on $[\mathbb{R}^n \setminus \bar{D}] \cup N$. Moreover there exists a finite positive Borel measure μ corresponding to u such that

$$\int_{D \cap N} |\nabla u|^{p-2} \langle \nabla u, \nabla \phi \rangle \, dx \ = \ - \int \phi \, d\mu \tag{1.2}$$

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whenever $\phi \in C_0^\infty([\mathbb{R}^n \setminus \bar{D}] \cup N)$ (see [HKM]) for results of this type. We remark that if ∂D is sufficiently smooth, then $|\nabla u|$ extends continuously to ∂D and $d\mu = |\nabla u|^{p-1}\, dH^{n-1}$. First consider the boundary condition,

$$\alpha\, r^{n-1} \,\leq\, \mu[B(x,r)] \,\leq\, \beta\, r^{n-1} \tag{1.3}$$

for $0 \leq \alpha \leq \beta < \infty$, $0 < r \leq 2$ diam D, and all $x \in \partial D$.

Question 1. What do assumptions (1.1)-(1.3) imply about ∂D?

One can show for example from basic measure theory arguments that $H^{n-1} \approx \mu$ on ∂D where $\approx$ means the two quantities are constant multiples of each other. Thus ∂D is Ahlfors regular in the sense that for some $1 \leq M < \infty$,

$$M^{-1}\, r^{n-1} \,\leq H^{n-1}[\partial D \cap B(x,r)] \,\leq\, M\, r^{n-1} \tag{1.4}$$

whenever $x \in \partial D$ and $0 < r \leq$ diam D. Can anything more be said about ∂D? Before answering this question, we discuss a related free boundary problem.

Given $F \subset \mathbb{R}^n$ a compact convex set with nonempty interior, $a > 0$, and $1 < p < \infty$, find a function v and a domain $\Omega = \Omega(a,p) \supset F$ with

$$(*) \qquad \nabla \cdot (|\nabla v|^{p-2} \nabla v) = 0 \text{ weakly in } \Omega \setminus F.$$

$$(**) \qquad v(x) \to 1 \text{ whenever } x \to y \in F \text{ and } v(x) \to 0 \text{ as } x \to y \in \partial\Omega. \tag{1.5}$$

$$(***) \qquad |\nabla v(x)| \to a \text{ whenever } x \to y \in \partial\Omega.$$

This problem was solved in [HS] (see also [HS1], [HS2] for related problems). They proved

Theorem A. *Given $F, 1 < p < \infty$, there exists a unique $\hat{v}, \hat{\Omega}$ satisfying (1.5). Moreover $\hat{\Omega}$ is convex with a smooth (C^∞) boundary.*

Note from Theorem A and our earlier remark that if $\hat{\nu}$ denotes the measure associated with $\hat{v}$ on $\partial\hat{\Omega}$, then $\hat{\nu} = a^{p-1} H^{n-1}$ on $\partial\Omega$.

Question 2. Let ν be the positive Borel measure associated with v in (1.5) $(*), (**)$ (defined as in (1.2), with D, u, μ replaced by Ω, v, ν) . Replace $(***)$ in (1.5) by

$$\nu = a^{p-1} H^{n-1} \text{ on } \partial\Omega. \tag{1.6}$$

Is $\hat{v}$ still the unique solution to the resulting free boundary problem?

If $p = 2$ the answer to Question 2 is that $\hat{v}$ need not be unique. To explain why, suppose $F = B(0,1)$. Then $\hat{\Omega}(a,p) = B(0,R)$ for some $R > 1$, as can be seen from Theorem A and the fact that radial solutions exist. On the other hand using the argument in [LV2], one can construct Ω, v, ν satisfying $(*), (**)$, and (1.6) with $\Omega \neq$ ball. In fact $\Omega = f(B(0,1))$ where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a K quasiconformal mapping and $K > 1$ can be arbitrarily near 1 (see [IM] for definitions). Thus further assumptions on ν are needed in order to insure uniqueness in Question 2. For each v, constructed in [LV2], it was true that $|\nabla v| \to \infty$ through a sequence $\to \partial D$. Moreover,

$$|\nabla v| \leq A < \infty \text{ in } D \cap N \tag{1.7}$$

is equivalent to the righthand inequality in (1.3) (with μ replaced by ν). Hence,

Question 3. For fixed $F, p, 1 < p < \infty$, is $\hat{v}$ the unique solution to the free boundary problem obtained when $(***)$ is replaced by (1.6) and (1.3) (with μ replaced by ν) ?

In [LV] we were able to show that the answer to Question 3 is yes, for $p = 2$, and domains Ω for which H^{n-1} almost every point in $\partial\Omega$ has measure theoretic tangent planes in the sense of geometric measure theory. This assumption, for example, allowed us to rule out Cantor sets with positive H^{n-1} measure. Our argument used results of [D] concerning absolute continuity of harmonic measure and surface area in a Lipschitz domain. We had no idea (see [LV, section 5, problems 2 and 5]) how to remove this tangent plane assumption and in general how to handle the case $p \neq 2$, $1 < p < \infty$.

In [LV1] we revisited Question 3 when $p = 2$ and showed that the tangent plane assumption on $\partial\Omega$ could be removed. The catalyst for this paper was the following observation. From (1.7), the Riesz representation formula for subharmonic functions, and the $T1$ theorem, it can be shown that Riesz transforms are bounded operators on the usual Lebesgue space of functions square integrable with respect to H^1 measure on $\partial\Omega$. Using [MMV] it then follows that $\partial\Omega$ is uniformly rectifiable and finally from [J] that $\partial\Omega$ is contained in a chord arc curve or equivalently in later terminology ([DS], [DS1]) that $\partial\Omega$ is locally uniformly rectifiable. Now it is well known that chord arc curves have tangents (in the measure theoretic sense) H^1 almost everywhere. Hence from [LV], $\hat{v}$ is unique when $n = 2, p = 2$. In view of this observation our game plan in [LV1] for the case $n > 2, p = 2$, of Question 3, was to first show that $\partial\Omega$ is locally uniformly rectifiable in the sense of [DS], [DS1]. We could then use results of these authors to conclude first the existence of the necessary tangent planes and then once again invoke [LV]. The proof in [LV1] actually turned out somewhat simpler. That is, for a similar problem (involving the Green's function and harmonic measure), we were able to rework the argument in [LV] and use a square function estimate, as in (1.8) below, to conclude uniqueness or symmetry, since in this case Ω is a ball. Moreover, our argument only assumed the righthand inequality in (1.3) for harmonic measure, which is optimal to prove uniqueness (as noted below (1.7)). To return to Question 1 (and u), we prove the following theorem in [LV3].

Theorem 1. *Fix $p, 1 < p < \infty$, and suppose that u, D, μ satisfy (1.1) - (1.3). If $z \in \partial D$ and $0 < r \leq$ diam D, then*

$$\int_{D \cap B(z,r)} u\, |\nabla u|^p \sum_{i,j=1}^{n} u^2_{x_i x_j}\, dx \leq c_1\, r^{n-1} \tag{1.8}$$

where c_1 depends only on n, p, α, β.

Using Theorem 1 and results from [DS1] we then prove,

Theorem 2. *Under the assumptions in Theorem 1, ∂D is locally uniformly rectifiable.*

For numerous equivalent definitions of uniform rectifiability, see [DS1]. For intuition, we mention that one can think of a uniformly rectifiable domain as being a countable union of Lipschitz domains, together with some 'bad balls,' connecting the domains.
Question 4. Can Theorems 1 and 2 be used to show uniqueness of $\hat{v}$ in Question 3 when $p \neq 2$?

To discuss, our efforts, we first show in [LV3] that uniqueness in Question 3 follows from

$$\limsup_{x \longrightarrow \partial\Omega} |\nabla v(x)| \leq a \tag{1.9}$$

whenever v satisfies $(*), (**)$, and ν satisfies (1.6), (1.3). Indeed, from Theorem A and a continuity argument we first find $b > 0$ with $\hat{\Omega} = \hat{\Omega}(b, p) \subset \Omega$ and $z \in \partial\Omega \cap \partial\hat{\Omega}$. If $\hat{v} = \hat{v}(\cdot, b, p)$, then by the maximum principle for solutions to the p Laplacian we see that, $v \geq \hat{v}$ in $\hat{\Omega}$. From smoothness of $\partial\hat{\Omega}$ and Schauder type estimates we also deduce that $\lim_{x \longrightarrow z} |\nabla \hat{v}(x)| = b$. Using these facts, (1,9), smoothness of $\partial\hat{\Omega}$, and the mean value theorem from differential calculus we deduce that

$$b = \lim_{x \longrightarrow z} |\nabla \hat{v}|(x) \leq \limsup_{x \longrightarrow z} |\nabla v|(x) \leq a. \tag{1.10}$$

Next we use the well known inequality,

$$I(v) = \int_\Omega |\nabla v|^p \, dx \leq I(w) \tag{1.11}$$

whenever w in (1.11) satisfies boundary condition (1.5)$(**)$ and has a distributional gradient that is p th power integrable, with equality for continuous w only if $w \equiv v$. Using (1.11) with $w = \hat{v}$ and essentially integration by parts we get

$$a^{p-1} H^{n-1}(\partial\Omega) = I(v) \leq I(\hat{v}) = b^{p-1} H^{n-1}(\partial\hat{\Omega}) \tag{1.12}$$

where strict inequality holds in (1.12) unless $v \equiv \hat{v}$. Finally we note that

$$H^{n-1}(\partial O) \geq H^{n-1}(\partial\hat{\Omega})$$

whenever O is an open set with $\hat{\Omega} \subset O$, as follows from convexity of $\hat{\Omega}$ and the fact that H^{n-1} measure does not increase under a projection. From this inequality with $O = \Omega$ and (1.10) we conclude that equality must hold in (1.12) so $b = a$ and $\hat{v} = v$. Thus it suffices to prove (1.9) in order to show uniqueness in Question 3.

As for the proof of (1.9), one can use (1.7), blowup type arguments and the fact that $\partial\Omega$ has measure theoretic tangent planes H^{n-1} a.e on $\partial\Omega$ (a consequence of Theorem 2 and results in [DS]) to get that (1.9) holds ' nontangentially ' outside a set of H^{n-1} measure 0. However, a complete proof of (1.9) appears to be linked with delicate questions involving absolute continuity of certain elliptic measures in a uniformly rectifiable domain. In retrospect our proof in [LV3] for $p = 2$ and the Green's function g of Laplace's equation (with pole at 0), relied heavily on the fact that $g, |\nabla g|^2$ are a solution, subsolution, respectively to Laplace's equation (away from 0) and self adjointness of this equation. Our first attempt at proving (1.9) for $1 < p < \infty, p \neq 2$, by modifying the proof in [LV1], was to investigate the partial differential equation,

$$\nabla \cdot (|\nabla v|^{p-2} \nabla w) = 0 \text{ in } \Omega.$$

Clearly v is a weak solution to this equation in Ω. Unfortunately, though, we found that $w = |\nabla v|^k$ is not a weak subsolution to this equation in Ω for any positive integer k, but instead satisfies weakly a partial differential equation of the form, $\nabla \cdot (|\nabla v|^{p-2} w) + \nabla \cdot (bw) = 0$ where $b = (b_1, \dots, b_n)$ involves second derivatives of v. Our second attempt was to write the partial differential equation for the p Laplacian in nondivergence form. In this case a smooth solution v at a point where $\nabla v \neq 0$, satisfies locally a partial differential equation of the form,

$$\sum_{i,j=1}^{n} \hat{b}_{ij}\, v_{x_i x_j} = 0 \tag{1.13}$$

where

$$\hat{b}_{ij} = \, (p-2)\, |\nabla v|^{-2}\, v_{x_i}\, v_{x_j} \; + \delta_{ij} \;\text{ for }\; 1 \leq i,\, j \,\leq\, n \tag{1.14}$$

and δ_{ij} is the Kronecker δ. Moreover for $k \geq p$, $|\nabla v|^k$ is also a subsolution to this equation. However the partial differential equation defined in (1.13), (1.14) is not self adjoint. To outline our third more successful attempt, suppose

$$\limsup_{x \longrightarrow \partial\Omega} |\nabla v(x)| = \beta. \tag{1.15}$$

Given τ, a small positive number, put

$$w(x) = \max[|\nabla v|^p(x) - (\beta - \tau)^p, 0], \text{ for } x \in \Omega.$$

If $\tau < \beta/100$, $\phi \in C_0^\infty(\beta/4, 4\beta)$ with $\phi \equiv 1$ on $[\beta/2, 2\beta]$ and $0 \leq \phi \leq 1$, we put

$$b_{ij}(x) = \delta_{ij} + \phi(|\nabla u(x)|)\,(\hat{b}_{ij}(x) - \delta_{ij}), \;\; 1 \leq i, j \leq n. \tag{1.16}$$

Then w is a weak subsolution to

$$Lf = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(b_{ij} f_{x_j}) = 0 \tag{1.17}$$

in Ω. Moreover thanks to Theorem 1 and interior estimates for solutions to the p Laplacian, we have

$$\int_{B(z,r)\cap\Omega} d(x) \max_{y \in B(x, d(x)/2)} \left[\sum_{i,j=1}^{n} |\nabla b_{ij}|^2(y) \right] dx \;\leq\; c\, r^{n-1} \tag{1.18}$$

for some $c \geq 1$, whenever $z \in \partial\Omega$ and $0 < r \leq$ diam $\partial\Omega$. In (1.18), $d(x)$ denotes the distance of x to $\partial\Omega$. Let ω denote elliptic measure (in the sense of Perron - Wiener - Brelot) defined with respect to the partial differential equation in (1.17). At this point we note that if Ω were a Lipschitz domain (i.e $\partial\Omega$ is locally a Lipschitz graph and Ω lies above the graph), then we could deduce from (1.18) and results of [KP] that $\omega(\cdot, x)$ is absolutely continuous with respect to H^{n-1} measure on $\partial\Omega$ at any $x \in \Omega$. Furthermore, if $h = d\omega(\cdot, x)/dH^{n-1}$, then for some $\lambda > 0, L < \infty$,

$$\int_{B(y,\rho)\cap\partial\Omega} h^{1+\lambda} dH^{n-1} \;\leq\; L\, \rho^{(1-n)\lambda} \left(\int_{B(y,\rho)\cap\partial\Omega} h\, dH^{n-1} \right)^{1+\lambda} \tag{1.19}$$

whenever $y \in \partial\Omega$ and $x \in \Omega \setminus B(y, 2\rho)$. Finally we remark that the reverse Hölder inequality, (1.19), is what we need to apply the method in [LV] and thus conclude first (1.9) and second, uniqueness in Question 3. Since Lipschitz domains are locally uniformly rectifiable and we have Theorem 2, a natural question is,

Question 4. Is $\omega(\cdot, x)$, defined with respect to the uniformly elliptic partial differential in (1.17) (whose coefficients satisfy (1.18)), absolutely continuous with respect to H^{n-1} on $\partial\Omega$?

We do not know the answer to Question 4, but mention that an example in [BJ, section 8] shows that harmonic measure on a compact uniformly rectifiable set, K, can be singular with respect to H^{n-1} measure on K. Thus one needs more information about $\partial\Omega$ in order to answer yes to Question 4. However, it turns out for the method in [LV] to work, one only needs to answer yes to Question 4 for a locally uniform rectifiable domain $\Omega_1 \subset \Omega$ with $w = 0$ on $\Omega \cap \partial\Omega_1$. Thus in [LV3] we prove

Proposition 1. *There exists $\Omega_1 \subset \Omega$ such that $\partial\Omega_1$ is locally uniformly rectifiable with*
(1.20)
(α) $\Omega \setminus \Omega_1 = \cup \bar{B}(z_j, d(z_j)/\bar{c})$ *for some large $\bar{c}$ depending only on the data and for $i \neq j$, $B(z_j, 10d(z_j)/\bar{c}) \cap B(z_i, 10d(z_i)/\bar{c}) = \emptyset$,*

(β) $w \equiv 0$ *on* $\Omega \cap \partial\Omega_1$.
.

In (1.20), $d(x)$ still denotes the distance from x to $\partial\Omega$. We also prove,

Proposition 2. *Let Ω_1 be as in Proposition 1 and let $\omega_1(\cdot, x)$ be the corresponding elliptic measure defined with respect to L in (1.17) with coefficients satisfying (1.18). Then Ω_1 can also be chosen so that the following statement is true. There exists $1 \leq L < \infty, \lambda > 0$, such that $d\omega_1(\cdot, x)/dH^{n-1} = h_1$ and*

$$\int_{B(y,\rho)\cap\partial D} h_1^{1+\lambda} dH^{n-1} \leq L\, \rho^{(1-n)\lambda} \left(\int_{B(y,2\rho)\cap\partial D} h_1 dH^{n-1} \right)^{1+\lambda} \tag{1.21}$$

whenever $y \in \partial\Omega$ and $x \in \Omega \setminus B(y, 2\rho)$.

We note that $B(y, 2\rho)$ occurs on the righthand side of (1.21) (instead of $B(y, \rho)$ as in (1.19)) which is why (1.21) is called a weak reverse Hölder inequality. Armed with Propositions 1 and 2 we can once again use the argument in [LV] to prove (1.9) and hence finally get

Theorem 3. *For fixed $p, 1 < p < \infty$, and F, the free boundary problem in Question 3 has a unique solution given by, $\hat{v}$, $\hat{\Omega}$.*

To outline the proofs of Propositions 1 and 2, we note that the balls $\{B(z_j, d(z_j)/\bar{c})\}$ are chosen so that w vanishes on each ball and also in such a way that a given z_j lies near a point, ζ_j, with $d(\zeta_j) \approx d(z_j)$, and

$$c \int_{\partial\Omega\cap B(\zeta_j,(1-\epsilon)d(\zeta_j))} v\, |\nabla v|^p \sum_{i,j=1}^{n} v_{x_i x_j}^2 \, dx \geq d(\zeta_j)^{n-1} \tag{1.22}$$

for some fixed small $\epsilon > 0$ and $c = c(\epsilon) \geq 1$. We then use (1.22), Theorem 1, and (1.4) to get Ahlfors regularity of $\partial\Omega_1$. Uniform rectifiability of $\partial\Omega_1$ follows from the fact that $\partial\Omega_1$ contains ' big pieces of uniformly rectifiable domains ' and a theorem in [DS1]. Next for Ω_1 as in Proposition 1, let $d_1(x)$ be the distance from x to $\partial\Omega_1$. To prove Proposition 2 we show for given $x \in \Omega_1$ that there exists small positive numbers ξ_1, ξ_2 and $T, 2 \leq T < \infty$, such that whenever $E \subset \partial\Omega_1 \cap B(x, T d_1(x))$ is Borel with

$$\frac{H^{n-1}(E)}{H^{n-1}[B(x, T\, d_1(x)) \cap \partial\Omega_1]} \geq 1 - \xi_1 \tag{1.23}$$

then

$$\omega_1(E, x) \geq \xi_2. \tag{1.24}$$

Given the implication (1.23) implies (1.24), it is not difficult to show the existence of ξ_3, $0 < \xi_3 < 1$, such that if $z \in \partial D$, $0 < r \leq$ diam D, F Borel $\subset B(z, 2r) \cap \partial\Omega$, and

$$\frac{H^{n-1}(F)}{H^{n-1}[B(z, 2r) \cap \partial\Omega_1]} \geq 1 - \xi_3,$$

then for each $\epsilon, 0 < \epsilon < 1$, there exists $\tilde{c} = \tilde{c}(\epsilon)$ for which

$$\omega_1(B(z, r) \cap \partial\Omega_1, x) \leq \epsilon\, \omega_1(B(z, 2r) \cap \partial\Omega_1, x) + \tilde{c}(\epsilon)\, \omega_1(F, x) \tag{1.25}$$

whenever $x \in \Omega_1 \setminus B(z, 2r)$. (1.21) follows from (1.25) and a measure theoretic type argument (see [BL]).

The proof that (1.23) implies (1.24) is somehat involved (see [LV3, sections 6 and 7]). A key ingredient in the proof is to develop an algorithm which produces a Lipschitz domain. We then start a process in which we either stop or apply the algorithm once again. In all cases we show, using Theorem 1 and a corona decomposition type argument as in [S], [DS], [DS1], that after at most N times (where N depends only on the data), that this process must stop with an end result that produces at the N th step a family of Lipschitz domains, $\{D_j\}$, for which the members of a subsequence, say $\{D'_k\}$, contain ' big pieces of $\partial\Omega_1$.' Using the theorem of [KP] mentioned above, it then follows that $\omega_1(F, y_k) \geq \xi_4 > 0$ for some $y_k \in D'_k$. Next we consider the Lipschitz domains produced in the $N-1$ st step and estimate in a quantitative way the number of domains whose boundaries contain big pieces on which $\omega_1(F, \cdot) \geq \xi_4/c$, ($c =$ an absolute constant). For these domains we can apply the above mentioned theorem of [KP] and get that $\omega_1(F, \cdot) \geq \xi_5 > 0$ at a certain distinguished point in each domain. Because the process is finite and we are essentially at liberty to choose ξ_3, our argument will ultimately arrive at the first step and x, yielding (1.24) for sufficiently small ξ_2.

2. Parabolic Problems.

In the parabolic case much work remains to be done, but we have good reason to believe that eventually we will find complete analogues of Theorems 1 - 3. To discuss this case we need some new notation. For notational purposes we assume $n \geq 2$. We write, $(x, t) = (x', x_n, t), x \in \mathbb{R}^n, x' \in \mathbb{R}^{n-1}$, $t, x_n \in \mathbb{R}$, for points in $\mathbb{R}^{n+1}$ and for given $r > 0$ set $C_r(x, t) = \{(y, s) : |y - x| < r, |t - s| < r^2\}$. Given a Borel set $F \subset \mathbb{R}^{n+1}$ put $\sigma(F) = \int_F d\sigma_t\, dt$ where $d\sigma_t$ is $n-1$ dimensional Hausdorff measure on the time slice $F \cap (\mathbb{R}^n \times \{t\})$. We note that σ replaces Hausdorff measure as the natural measure in our parabolic free boundary problems. We also remark that analogues of the results in [D] were obtained in [LM] where we showed that caloric measure (defined in essentially the same way as harmonic measure, but with respect to the heat equation), is absolutely continuous with respect to σ in certain time varying graph domains,

$$\Omega = \{(x,t) : x_n > A(x',t), (x',t) \in \mathbb{R}^n\}.$$

In fact we showed that if $d\omega(\cdot, x, t)/d\sigma = h$, then (1.19) holds for some $\lambda > 0$ with $B(y,\rho)$ replaced by $C_r(y,s)$, provided $|y-x| \leq \Lambda(t-s)$, $(t-s) \geq 4r^2$, and $0 < \Lambda < \infty$ is fixed. Our assumptions on A were that

(2.1)

(a) $\quad |A(x',t) - A(y',t)| \leq c_1 |x'-y'|$ for all $x', y' \in \mathbb{R}^{n-1}$, $t \in \mathbb{R}$,

(b) $\quad$ For H^{n-1} almost every $x' \in \mathbb{R}^{n-1}$, $A(x', \cdot)$, has a time derivative of order $1/2$ which is of bounded mean oscillation on caloric cylinders in $\mathbb{R}^n$.

By caloric cylinders in $\mathbb{R}^n$, we mean cylinders as in the definition of $C_r(x,t)$ except that x, y is replaced by x', y'. Also the space of functions of bounded mean oscillation is defined in the usual way but with $C_r(x',t)$, replacing balls. We note that (2.1)(a), (b), imply

(2.2) $\quad |A(x',t) - A(y',s)| \leq c_2 \, [|x'-y'| + |t-s|^{1/2}]$ whenever $(x',t), (y',s) \in \mathbb{R}^{n+1}$

but (2.2) does not imply (2.1). In fact examples in [KW], [LS], show that caloric measure need not be absolutely continuous with respect to σ. The nonequivalence of (2.1) and (2.2) often causes difficulties in trying to generalize elliptic results to the parabolic setting. If $n = 1$, delete the x' variable in (2.1), (2.2) and the above definitions.

We say that $\partial\Omega \subset \mathbb{R}^{n+1}$ is Ahlfors regular in the caloric or parabolic setting, provided for some $M < \infty$ and all $(x,t) \in \partial\Omega$, $r > 0$,

$$M^{-1} r^{n+1} \leq \sigma(\partial\Omega \cap C_r(x,t)) \leq M r^{n+1}. \tag{2.3}$$

Let

$$\delta(F_1, F_2) = \inf\{|x-y| + |s-t|^{1/2} : (x,t) \in F_1, \, (y,s) \in F_2\}$$

denote the parabolic distance between the sets F_1, F_2 and for Ω as in (2.3), $(z,\tau) \in \partial\Omega$, and $r > 0$ set

$$\gamma(z,\tau,r) = \inf_P \left[r^{-n-3} \int_{\partial\Omega \cap C_r(z,\tau)} \delta(\{(y,s)\}, P)^2 d\sigma(y,s) \right] \tag{2.4}$$

where the infimum is taken over all n dimensional planes P containing a line parallel to the t axis. Let

$$d\nu(z,\tau,r) = \gamma(z,\tau,r) \, d\sigma(z,\tau) \, r^{-1} dr.$$

We say that $\partial\Omega$ is uniformly rectifiable in the parabolic or caloric sense if and only if ν is a Carleson measure on $\partial\Omega \times (0,\infty)$. That is, there exists $M_1 < \infty$ such that whenever $(\hat{x}, \hat{t}) \in \partial\Omega, R > 0$, we have

$$\nu([C_R(\hat{x},\hat{t}) \cap \partial\Omega] \times (0,R)) \leq M_1 R^{n+1}. \tag{2.5}$$

The least such M_1 in (2.5) is called the Carleson norm of ν, written $\|\nu\|$. We note that if Ω is a graph domain, with graph function satisfying (2.1), then $\partial\Omega$ is uniformly rectifiable in the caloric sense.

Let $u > 0$ be a solution to the heat equation ($u_t = \Delta u$) in the domain, Ω (possibly unbounded). Let $t_1 < t_2, t_1, t_2 \in \mathbb{R}$, $S = \mathbb{R}^n \times (t_1, t_2)$, and suppose that

$$u(x,t) \to 0 \text{ as } (x,t) \to \partial\Omega \cap S. \tag{2.6}$$

Put $u \equiv 0$ in $S \setminus \Omega$. Then there is a positive Borel measure μ on $S \cap \partial\Omega$ satisfying

$$\int [\nabla u \cdot \nabla \phi \,-\, u\,\phi_t\,]\, dx\, dt \;=\; -\int \phi\, d\mu \tag{2.7}$$

whenever $\phi \in C_0^\infty(S)$. Here ∇u denotes the gradient of u in the space variable x, only. Boundary conditions (1.3), (1.6) in the parabolic setting are

$$\partial\Omega \text{ is Ahlfors regular and } \mu = a\sigma \text{ on } \partial\Omega. \tag{2.8}$$

We remark that Petrosyan [P] has obtained a partial analogue of the results in [HS] for the heat equation. One can ask whether uniqueness in his theorem still holds under assumption (2.8). More optimistically, we would like to prove an analogue of [HS] for the parabolic p Laplacian ($p > 2$), and then prove uniqueness under assumption (2.8). Indeed, operationally (i.e neglecting boundary terms), we have shown the square function estimate in (1.8) is valid for the parabolic p Laplacian with $B(z,r)$ replaced by $C_r(z,\tau)$ and $n-1$ by $n+1$. We observe that in [LV3] we obtained Theorem 2 from Theorem 1 by first showing that ∂D satisfies a weak exterior convexity condition (WEC). Second we applied a theorem of [DS1] which states that the WEC condition for an Ahlfors regular set (∂D) is equivalent to uniform rectifiability. In the parabolic case, caloric uniform rectifiability implies WEC but not vice-versa. In fact examples from [LS] which show the nonequivalence also show that (2.2) does not imply (2.1). However, we can show that Theorem 2 for a caloric u (under boundary assumption (2.8)) and a graph domain satisfying (2.2) implies caloric uniform rectifiability, which is an important prototype case for the validity of Theorem 2. We suspect in fact that the technique, used in proving Proposition 2, can also be used in the parabolic setting to get parabolic analogues of these propositions, as well as Theorem 2.

Finally we discuss a problem left open in [HLN1] that illustrates the techniques used in proving Theorems 1 and 2 for graph domains. For simplicity we work in $\mathbb{R}^2$. Let $\Omega = \{(x,t) \in \mathbb{R}^2 : x > A(t)\}$ and suppose that (2.2) is valid for A with $(x',t), (y',s)$ replaced by t, s. If as above, $\omega(\cdot, x, t)$ denotes caloric measure for Ω, then we say that $\omega(\cdot, x, t)$ is weakly asymptotically optimal doubling if and only if whenever $t_1 < t_2 < t$, we have

$$\lim_{r \to 0} \sup_{s \in [t_1,t_2]} \frac{\omega[C_{\tau r}(A(s),s),x,t]}{\omega[C_r(A(s),s),x,t]} = \lim_{r \to 0} \inf_{s \in [t_1,t_2]} \frac{\omega[C_{\tau r}(A(s),s),x,t])}{\omega[C_r(A(s),s),x,t]} = \tau^{n+1}. \tag{2.9}$$

In [HLN1, section 5] we conjectured that $\omega(\cdot, x, t)$ is asymptotically optimal doubling in a graph domain satisfying (2.2) with $c_2 \leq 1$, if and only if whenever $t_1, t_2 \in \mathbb{R}$ and $t_1 < t_2$,

$$\theta(\delta) = \sup\{|t-s|^{-1/2}|A(t) - A(s)| \,:\, s,t \in [t_1,t_2], 0 < |s-t| \leq \delta\} \to 0 \text{ as } \delta \to 0. \tag{2.10}$$

In fact we proved this conjecture under the stronger assumption, that A satisfies (2.1). To outline the proof in this case, the implication (2.10) $\to$ (2.9) follows easily from compactness arguments, the maximum principle for the heat equation, and estimates for caloric measure in a half space. The proof of (2.9) $\to$ (2.10) was by contradiction. Assuming the contrary (see the argument following (5.14) in [HLN1]), we obtained a positive solution u to the adjoint heat equation ($u_t = -u_{xx}$) in $\Omega^* = \{(x,t) : x > A^*(t), t \in \mathbb{R}\}$, with u vanishing continously on $\partial\Omega^*$. Moreover,

$$\frac{|A^*(s) - A^*(t)|}{|t-s|^{1/2}} \leq 1 \text{ whenever } s,t \in \mathbb{R} \text{ and } A^* \not\equiv A^*(0). \tag{2.11}$$

Let μ be the positive locally finite Borel measure on $\partial\Omega^*$ associated with u as in (2.7) but with $-\phi_t$ replaced by ϕ_t. Using only (2.2) we showed in [HLN1] that

$$\mu[C_r(A^*(s), s) \cap \partial\Omega^*] \,=\, 2r^2, \text{ whenever } s \in \mathbb{R}, r > 0. \tag{2.12}$$

We write $\delta(x,t)$ for $\delta(\{(x,t)\}, \partial\Omega^*)$. Then from (2.2) we also found for some $1 \leq c_3 < \infty$ and all $(x,t) \in \Omega^*$ that

$$\begin{aligned} &(\alpha) \quad \delta(x,t)\,|u_{xx}(x,t)| \,\leq\, c_3 \\ &(\beta) \quad c_3^{-1}\,\delta(x,t) \,\leq\, u(x,t) \,\leq\, c_3\,\delta(x,t) \\ &(\gamma) \quad c_3^{-1} \,\leq\, u_x(x,t) \,\leq\, c_3. \end{aligned} \tag{2.13}$$

In view of (2.12), (2.13), we could then use (2.1) and results of [LM] to get first that

$$\lim_{\lambda \to 0^+} u_x(A^*(t) + \lambda, t) = 1/2 \text{ for } H^1 \text{ almost every } t \in \mathbb{R}$$

and thereupon that $u_x \equiv 1/2$ in Ω^*. Thus, $u(x,t) = x - A^*(t)$ and since $u_{xx} = -u_t$ in Ω^* we have $A^* \equiv A^*(0)$ which contradicts (2.11). Thus, (2.9) implies (2.10) under assumption (2.1). To remove (2.1) we first prove a square function estimate and then use this estimate to prove caloric uniform rectifiability. For this purpose, let $\phi \geq 0 \in C_0^\infty[-1,1]$ be an even function with $\int_{\mathbb{R}} \phi = 1$ and put $A^*_\epsilon = 4\,\epsilon^{1/2} + A^* * \phi_\epsilon$ for $\epsilon > 0$ where $*$ denotes convolution on $\mathbb{R}$, and $\phi_\epsilon(\cdot) = \epsilon^{-1}\phi(\cdot/\epsilon)$. Observe from (2.11) that

$$\left| \frac{\partial A^*_\epsilon(t)}{\partial t} \right| \leq c_4\,\epsilon^{-1/2} \tag{2.14}$$

for some $c_4, 1 \leq c_4 < \infty$. Given $t \in \mathbb{R}, r > 0,\ 0 < \epsilon^{1/2} < r/100$, let

$$G_r(t) = \{(y,s) \in \Omega^* : A^*(s) < y < A^*(t) + 4r,\ t - r^2 < s < t + r^2\}$$

$$G_r(t,\epsilon) = \{(y,s) \in \Omega^* : A^*_\epsilon(s) < y < A^*(t) + 4r,\ t - r^2 < s < t + r^2\}.$$

From (2.11) we see that

$$G_r(t,\epsilon) \subset G_r(t) \subset \Omega^*.$$

We claim for some $c_5, 1 \leq c_5 < \infty$, that

$$\int_{C_r(A^*(t),t) \cap \Omega^*} u\,(u_{yy})^2\,dy\,ds \,\leq\, \int_{G_r(t)} u\,(u_{yy})^2\,dy\,ds \,\leq\, c_5\,r^2. \tag{2.15}$$

Indeed, integrating by parts and using (2.13), we get for some $1 \leq c_6 < \infty$,

$$
\begin{aligned}
&\int_{G_r(t,\epsilon)} u\,(u_{yy})^2\,dy\,ds = \\
&\qquad -\int_{G_r(t,\epsilon)} [u_y^2\,u_{yy} + u_y\,u\,u_{yyy}]dy\,ds + \int_{t-r^2}^{t+r^2} u u_y u_{yy}\Big|_{(A_\epsilon^*(s),s)}^{(A^*(t)+4r,s)} ds \\
&\qquad = \tfrac{1}{2}\int_{G_r(t,\epsilon)} u\,\frac{\partial u_y^2}{\partial t}\,dyds + \int_{t-r^2}^{t+r^2} [u u_y u_{yy} - \tfrac{1}{3}\,u_y^3\,]\Big|_{(A_\epsilon^*(s),s)}^{(A^*(t)+4r,s)} ds \\
&\qquad\qquad \leq \tfrac{1}{2}\int_{G_r(t,\epsilon)} u\,\frac{\partial u_y^2}{\partial t}\,dyds + c_7 r^2 = T + c_7\,r^2.
\end{aligned}
\tag{2.16}
$$

To handle T we integrate by parts in t and use (2.13), (2.14) to find for some $1 \leq c_8 < \infty$ that

$$
\begin{aligned}
T = &-\tfrac{1}{2}\int_{G_r(t,\epsilon)} u_t\,u_y^2\,dyds + \tfrac{1}{2}\int_{t-r^2}^{t+r^2} \tfrac{\partial A_\epsilon^*(s)}{\partial s}\,[u\,u_y^2](A_\epsilon^*(s),s)\,ds \\
&+ \tfrac{1}{2}\int_{A_\epsilon^*(t+r^2)}^{A^*(t)+4r} u\,u_y^2(y,t+r^2)\,dy - \int_{A_\epsilon^*(t-r^2)}^{A^*(t)+4r} u\,u_y^2(y,t-r^2)\,dy \\
&\qquad \leq \tfrac{1}{2}\int_{G_r(t,\epsilon)} u_{yy}\,u_y^2\,dyds + c_8\,r^2 \leq (c_7+c_8)\,r^2
\end{aligned}
\tag{2.17}
$$

since the last integral was estimated in (2.16). Using (2.17) in (2.16) and letting $\epsilon\to 0$, we conclude that (2.15) is true.

Finally we show that (2.15) implies caloric uniform rectifiability. Observe that in $\mathbb{R}^2$, the only candidates for P in (2.4) are lines on which the x coordinate is constant. Let $(A^*(\hat{t}),\hat{t}) = (\hat{x},\hat{t}) \in \partial\Omega^*$ and $R > 0$. If $(z,\tau) = (A^*(\tau),\tau) \in \partial\Omega^* \cap C_R(\hat{x},\hat{t})$, set $x = z + 4r$ and $I = (\tau - r^2, \tau + r^2)$. Note from (2.11) that if $s \in I$, then $\delta(x,s) \approx r$. Moreover, if $s \in I$ and $(y,s) \in \Omega^*$, we see from Taylor's formula or just integration by parts that

$$
u(y,s) = u(x,s) + u_x(x,s)(y-x) + \int_x^y u_{ww}(w,s)\,(y-w)dw. \tag{2.18}
$$

Put

$$
\Theta(x,y,s) = u(x,s) + u_x(x,s)(y-x) \text{ when } s \in I, y \in \mathbb{R}. \tag{2.19}
$$

Letting $(y,s)\to\partial\Omega^*$, in (2.18), using $u = 0$ on $\partial\Omega^*$, (2.13), and Hölder's inequality we get for $s \in I$ and $(y,s) = (A^*(s),s) \in \partial\Omega^* \cap C_r(z,\tau)$ and some $c, 1 \leq c < \infty$, that

$$
|\Theta(x,y,s)|^2 \leq cr \int_x^y \delta(w,s)^2\,(u_{ww})^2(w,s)dw \tag{2.20}
$$

Next we observe from the mean value theorem in calculus, the fact that $|u_{wt}| = |u_{www}|$, and well known third derivative interior estimates for solutions to the adjoint heat equation in terms of second derivatives of this equation that for $(y,s) \in \partial\Omega^* \cap C_r(z,\tau)$ (so $s \in I$) and some $c, c' < \infty$,

(2.21)
$$|\Theta(x,y,s) - \Theta(x,y,\tau)|^2 \leq c \max_{(w,\hat\tau)\in \bar C_r(x,\tau)} \left(r^4 |u_{\hat\tau}(w,\hat\tau)|^2 + r^6 |u_{w\hat\tau}|^2(w,\hat\tau)\right)$$
$$\leq c' r^{-1} \int_{C_{10r}(z,\tau)\cap\Omega^*} \delta(w,s)^2 \, (u_{ww})^2(w,s) dw ds.$$

Let $\hat\delta(y,s)$ denote the distance from $(y,s) \in C_r(z,\tau) \cap \partial\Omega^*$ to the line

$$P = \{(w,\hat\tau) : \Theta(x,w,\tau) = 0\} = \{(w,\hat\tau) : w = x - \frac{u(x,\tau)}{u_x(x,\tau)}\}$$

which is a Newton approximation to z. From (2.13) we see that

$$\hat\delta(y,s) \approx \Theta(x,y,\tau). \tag{2.22}$$

Let $G_r(\tau)$ be defined as above (2.15) with τ replacing t. Integrating with respect to $d\sigma = ds$ and using (2.20) - (2.22), we find for c^*, c large enough that

(2.23)
$$\gamma(z,\tau,r) = r^{-4} \int_{C_r(z,\tau)\cap\partial\Omega^*} \hat\delta(y,s)^2 ds$$
$$\leq c^* r^{-3} \int_{G(\tau)} \delta(w,s)^2 \, (u_{ww})^2(w,s) dw\, ds$$
$$+ c^* r^{-3} \int_{C_{10r}(z,\tau)\cap\Omega^*} \delta(w,s)^2 \, u_{ww}^2(w,s) dw ds$$
$$\leq c r^{-3} \int_{C_{10r}(z,\tau)\cap\Omega^*} \delta(w,s)^2 \, (u_{ww})^2(w,s) dw ds.$$

Integrating (2.23) over $(z,\tau) \in C_R(\hat x,\hat t) \cap \partial\Omega^*$ and $r \in (0,R)$, we obtain after interchanging the order of integration and using (2.13), (2.15) that for some large $c, \tilde c$,

(2.24)
$$\nu([C_R(\hat x,\hat t) \cap \partial\Omega^*] \times (0,R)) \leq \int_0^R \int_{C_R(\hat x,\hat t)\cap\partial\Omega^*} r^{-1} \gamma(z,\tau,r)\, d\tau\, dr$$
$$\leq c \int_{C_{40R}(\hat x,\hat t)\cap\partial\Omega^*} \delta(w,s)^2 \, (u_{ww})^2(w,s) \, [\int_{10^{-3}\,\delta(w,s)}^{\infty} r^{-2} dr] \, dw ds$$
$$= 1000 c \int_{C_{40R}(\hat x,\hat t)\cap\partial\Omega^*} \delta(w,s) \, u_{ww}^2(w,s) \, dw ds \leq \tilde c R^2.$$

Since $R > 0, (\hat x,\hat t) \in \partial\Omega^*$ are arbitrary, we conclude that $\partial\Omega^*$ is uniformly rectifiable in the caloric sense. From this fact and [HLN] we deduce first that $\partial\Omega^*$ contains big pieces of time varying graphs in the sense of [LM]. Second from [LM], [HLN1, section 3], we conclude that the L^p Dirichlet problem has a solution for some $p < \infty$ which is all that is needed to conclude $u_x \equiv 1/2$ (see the discussion below (2.13)). Thus once again we get a contradiction to (2.11). From this contradiction we obtain that (2.9)→(2.10) under assumption (2.2).

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