Boundary Behaviour and the Martin Boundary Problem for p Harmonic Functions in Lipschitz Domains

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Abstract

In [LN] we proved a boundary Harnack inequality for the ratio of two positive p harmonic functions, vanishing on a portion of the boundary of a Lipschitz domain. In the current paper we continue our study by showing that this ratio is Hölder continuous up to the boundary. We also consider the Martin boundary of certain domains and the corresponding question of when a minimal positive p harmonic function (with respect to a given boundary point w) is unique up to constant multiples. In particular we show that the Martin boundary can be identified with the topological boundary in domains that are convex or C^1. Minimal positive p harmonic functions relative to a boundary point w in a Lipschitz domain are shown to be unique, up to constant multiples, provided the boundary is sufficiently flat at w.

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1 Introduction

In this paper, which is the second in a series, we continue our study in [LN] concerning the boundary behaviour of positive p harmonic functions, $p \neq 2$ and $1 < p < \infty$, in bounded Lipschitz domains $\Omega \subset \mathbf{R}^n$. More specifically in [LN] (see Theorem 1 below) we established the boundary Harnack inequality for positive p harmonic functions, $1 < p < \infty$, vanishing on a portion of the boundary of a Lipschitz domain $\Omega \subset \mathbf{R}^n$ and we carried out an in depth analysis of p capacitary functions in starlike Lipschitz ring domains. As a result of this analysis we were able to prove Hölder continuity up to the boundary for quotients of p capacitary functions, $p \neq 2$ and $1 < p < \infty$, in starlike Lipschitz ring domains (see Theorem 2 in [LN]). Still at that point in time we were unable to extend our Hölder continuity results to the quotients of all p harmonic function considered in our boundary Harnack inequality. The first part of this paper is devoted to this extension (see Theorem 2) with constants which depend only on p, n and the Lipschitz constant for Ω.

In the second part of this paper we study and resolve the p Martin boundary problem in certain subsets of $\Omega \subset \mathbf{R}^n$ by proving that the quotient of two minimal positive p harmonic functions in Ω, minimal relative to $w \in \partial\Omega$, equals a constant. This implies, in particular, that the Martin boundary can be identified with the topological boundary at $w \in \partial\Omega$. Our argument consists of first establishing a sufficient criteria for the above statement to hold (see Theorem 3). We then show that this criteria is fulfilled in a number of interesting cases including the cases when (a) Ω is convex and (b) $\partial\Omega$ is C^1.

To put the results of [LN] and this paper into perspective, we mention that if $p = 2$, i.e., in the case of harmonic functions, the term boundary Harnack inequality was first coined in [K] and later proved independently by [A], [D], [W]. This boundary Harnack inequality for positive harmonic functions vanishing on a portion of the boundary of a Lipschitz domain was later extended in [JK] to non-tangentially accessible (NTA) domains, where it is also shown that the corresponding ratio is Hölder continuous. For $p = 2$, it follows easily from the boundary Harnack inequality for harmonic functions that the Martin boundary of an NTA domain agrees with its topological boundary.

Extension of these results to Lipschitz domains when $p \neq 2$, $1 < p < \infty$, have eluded the experts until now primarily because the p Laplace operator is non-linear when $p \neq 2$. In fact we believe that the results and techniques of [LN] and this paper provide a starting point for far reaching developments concerning the p Laplace operator in Lipschitz domains and beyond.

To properly state our results we need to introduce some notation. Points in Euclidean n-space $\mathbf{R}^n$ are denoted by $x = (x_1, \dots, x_n)$ or (x', x_n) where $x' = (x_1, \dots, x_{n-1}) \in \mathbf{R}^{n-1}$. We let $\bar{E}, \partial E$, diam E, be the closure, boundary, diameter, of the set $E \subset \mathbf{R}^n$ and we define $d(y, E)$ to equal the distance from $y \in \mathbf{R}^n$ to E. $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbf{R}^n$ and we let $|x| = \langle x, x \rangle^{1/2}$ be the Euclidean norm of x. $B(x, r) = \{y \in \mathbf{R}^n : |x - y| < r\}$ is defined whenever $x \in \mathbf{R}^n$, $r > 0$, and dx denotes Lebesgue n-measure on $\mathbf{R}^n$. If $O \subset \mathbf{R}^n$ is open and $1 \leq q \leq \infty$, then by $W^{1,q}(O)$, we denote the space of equivalence classes of functions f with distributional gradient $\nabla f = (f_{x_1}, \dots, f_{x_n})$, both of which are q th power integrable on O. Let $\|f\|_{1,q} = \|f\|_q + \| \, |\nabla f| \, \|_q$ be the norm in $W^{1,q}(O)$ where $\| \cdot \|_q$ denotes the usual Lebesgue q norm in O. Next let $C_0^\infty(O)$ be set of infinitely differentiable functions with compact support in O and let $W_0^{1,q}(O)$ be the closure of $C_0^\infty(O)$ in the norm of $W^{1,q}(O)$.

Given G a bounded domain (i.e, a connected open set) and $1 < p < \infty$, we say that u is p

harmonic in G provided $u \in W^{1,p}(G)$ and

$$\int |\nabla u|^{p-2} \langle \nabla u, \nabla \theta \rangle \, dx = 0 \tag{1.1}$$

whenever $\theta \in W_0^{1,p}(G)$. Observe that if u is smooth and $\nabla u \neq 0$ in G, then

$$\nabla \cdot (|\nabla u|^{p-2} \, \nabla u) \equiv 0 \text{ in } G \tag{1.2}$$

so u is a classical solution in G to the p Laplace partial differential equation. Here, as in the sequel, $\nabla\cdot$ is the divergence operator. We note that $\phi : E \to \mathbf{R}$ is said to be Lipschitz on E provided there exists $b, 0 < b < \infty$, such that

$$|\phi(z) - \phi(w)| \, \leq \, b \, |z - w|, \;\; \text{whenever } z, w \in E. \tag{1.3}$$

The infimum of all b such that (1.3) holds is called the Lipschitz norm of ϕ on E, denoted $\|\phi\hat{\|}_E$. It is well known that if $E = \mathbf{R}^{n-1}$, then ϕ is differentiable almost everywhere on $\mathbf{R}^{n-1}$ and $\|\phi\hat{\|}_{\mathbf{R}^{n-1}} = \| \, |\nabla \phi| \, \|_\infty$.

In the following we let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain, i.e., we assume that there exists a finite set of balls $\{B(x_i, r_i)\}$, with $x_i \in \partial\Omega$ and $r_i > 0$, such that $\{B(x_i, r_i)\}$ constitutes a covering of an open neighbourhood of $\partial\Omega$ and such that, for each i,

$$\begin{aligned} \Omega \cap B(x_i, 4r_i) \;\; &= \;\; \{y = (y', y_n) \in \mathbf{R}^n : y_n > \phi_i(y')\} \cap B(x_i, 4r_i), \\ \partial\Omega \cap B(x_i, 4r_i) \;\; &= \;\; \{y = (y', y_n) \in \mathbf{R}^n : y_n = \phi_i(y')\} \cap B(x_i, 4r_i), \end{aligned} \tag{1.4}$$

in an appropriate coordinate system and for a Lipschitz function ϕ_i. The Lipschitz constant of Ω is defined to be $M = \max_i \| |\nabla \phi_i| \|_\infty$. If Ω is Lipschitz and $r_0 = \min r_i$, then for each $w \in \partial\Omega$, $0 < r < r_0$, we can find points $a_r(w) \in \Omega \cap \partial B(w, r)$ with $d(a_r(w), \partial\Omega) \geq c^{-1} r$ for a constant $c = c(M)$. In the following we let $a_r(w)$ denote one such point. Furthermore, if $w \in \partial\Omega$, $0 < r < r_0$, then we let $\Delta(w, r) = \partial\Omega \cap B(w, r)$ be the naturally defined surface ball. Finally let $e_i, 1 \leq i \leq n$, denote the point in $\mathbf{R}^n$ with one in the i th coordinate position and zeroes elsewhere.

In [LN] we proved,

Theorem 1. *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain with constant M. Given $p, 1 < p < \infty, w \in \partial\Omega$, $0 < r < r_0$, suppose that u and v are positive p harmonic functions in $\Omega \cap B(w, 2r)$. Assume also that u and v are continuous in $\bar{\Omega} \cap \bar{B}(w, 2r)$ and $u = 0 = v$ on $\Delta(w, 2r)$. Under these assumptions there exists $c_1, 1 \leq c_1 < \infty$, depending only on p, n, and M such that if $\tilde{r} = r/c_1, u(a_{\tilde{r}}(w)) = v(a_{\tilde{r}}(w)) = 1$, and $y \in \Omega \cap B(w, \tilde{r})$, then*

$$\left| \log \frac{u(y)}{v(y)} \right| \, \leq \, c_1.$$

In the first part of this paper we refine Theorem 1 and prove,

Theorem 2. *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain with constant M. Given $p, 1 < p < \infty, w \in \partial\Omega$, $0 < r < r_0$, suppose that u and v are positive p harmonic functions in $\Omega \cap B(w, 2r)$.*

Assume also that u and v are continuous in $\bar\Omega \cap \bar B(w, 2r)$ and $u = 0 = v$ on $\Delta(w, 2r)$. Under these assumptions there exist $c_2, 1 \leq c_2 < \infty$, and α, $\alpha \in (0, 1)$, both depending only on p, n, and M, such that if $y_1, y_2 \in \Omega \cap B(w, r/c_2)$ then

$$\left|\log \frac{u(y_1)}{v(y_1)} - \log \frac{u(y_2)}{v(y_2)}\right| \leq c_2 \left(\frac{|y_1 - y_2|}{r}\right)^{\alpha}.$$

In [LN], as discussed above, Theorem 1 and Theorem 2 were proved in the setting of starlike Lipschitz ring domains and p capacitary functions. Furthermore, in [LN] we were able, by a comparison argument, to establish Theorem 1. While the proof of Theorem 2 in this paper is our genuinely new contribution (as compared to [LN]), we shall also for completeness and to make the paper self contained, outline a somewhat simpler proof of Theorem 1.

To describe the key features of the proof of Theorem 1 and Theorem 2 suppose $x \in \Omega$ and let $N \subset \Omega$ be a neighbourhood of x. Moreover, let $\{u(\cdot, \tau), \tau \in [0, 1]\}$ be a sequence of positive p harmonic functions with $\nabla u(\cdot, \cdot) \neq 0$ and suppose that $u(\cdot, \cdot)$ is sufficiently smooth in $N \times [0, 1]$. If $\zeta = \langle \nabla u, \xi\rangle$, for some $\xi \in \mathbf{R}^n$, or $\zeta = u_\tau(\cdot, \tau)$, then ζ satisfies, at x, the partial differential equation

$$L\zeta = \nabla \cdot [(p-2)|\nabla u|^{p-4}\langle \nabla u, \nabla \zeta\rangle \nabla u + |\nabla u|^{p-2}\nabla \zeta] = 0. \tag{1.5}$$

This follows from differentiating (1.2) for u with respect to $\{x_k\}_{k=1}^n$ or τ. In (1.5) we have written ∇u for $\nabla u(\cdot, \tau)$. (1.5) can be written in the form

$$L\zeta = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} [\, b_{ij}(x)\zeta_{x_j}(x)\,] = 0, \tag{1.6}$$

where, at $x \in N \cap \Omega$,

$$b_{ij}(x) = |\nabla u|^{p-4}[(p-2)u_{x_i}u_{x_j} + \delta_{ij}|\nabla u|^2](x),\ 1 \leq i, j \leq n, \tag{1.7}$$

and δ_{ij} is the Kronecker δ. Clearly we also have

$$L\, u(x, \cdot) = (p-1)\, \nabla \cdot \left[\, |\nabla u|^{p-2}\, \nabla u(x, \cdot)\,\right] = 0. \tag{1.8}$$

A first key observation is therefore that $u(\cdot, \tau)$, $\langle \nabla u(\cdot, \tau), \xi\rangle$ for $\xi \in \mathbf{R}^n$ as well as $u_\tau(\cdot, \tau)$ all satisfy the divergence form partial differential equation (1.6). Based on this insight the proof of Theorem 1 and Theorem 2 can be seen as decomposed into the following well defined steps.

Step 1. Fix $\tau \in [0, 1]$ and let $u' = u(\cdot, \tau), L$, be as in (1.6)-(1.7). Suppose also that $u' > 0$ is p harmonic in $\Omega' \cap B(w', r)$ and vanishes continuously on $\partial\Omega' \cap B(w', r)$, where Ω' is Lipschitz with constant M' and $w' \in \partial\Omega'$. We first study positive solutions to L vanishing on a portion of $\partial\Omega'$. In particular, for this linear operator we prove a boundary Harnack inequality and Hölder continuity for the corresponding quotient under the assumption that $|\nabla u'|$ satisfies a uniform non-degeneracy condition in $\Omega' \cap B(w', 2r)$. By definition this means that we assume that there exists a constant $\delta > 1$ such that, for all $x \in \Omega' \cap B(w', 2r)$,

$$\delta^{-1} \frac{u'(x)}{d(x, \partial\Omega')} \leq |\nabla u'(x)| \leq \delta \frac{u'(x)}{d(x, \partial\Omega')}. \tag{1.9}$$

In particular, if this condition is fulfilled then L is a locally uniformly elliptic operator in $\Omega' \cap B(w', r)$ with ellipticity constant, at $x \in \Omega' \cap B(w', r)$, bounded above and below by $|\nabla u'(x)|^{p-2} \approx (u'(x)/d(x, \partial\Omega'))^{p-2}$. The proportionality constants depend only on p, n and δ and in general, in our applications, $\delta = \delta(p, n, M')$. We also assume that if $x \in \Omega' \cap B(w', 2r)$, then

$$0 < \, |\nabla u'|(x) \leq \delta' \, \langle \nabla u'(x) \, , \, \xi \rangle \tag{1.10}$$

for some $\xi \in \partial B(0, 1)$ and $\delta' > 1$. Again, usually $\delta' = \delta'(p, n, M')$. Given (1.9), (1.10), we can use some Rellich inequalities and Carleson measure type estimates from [LN, section 2] in order to deduce that a theorem in [KP] can be applied to elliptic measure defined relative to L. From this theorem and arguments in [LN, section 3] one eventually obtains the desired boundary Harnack inequality as well as Hölder continuity for the ratio of solutions to L.

Step 2. The proofs of Theorems 1 and 2 are based on certain deformations of p harmonic functions, using which we are able to make use of the results established in Step 1. In this step we describe the deformation technique used in [LN] to establish Theorem 1 and Theorem 2 for certain starlike Lipschitz ring domains. To sketch the argument of [LN] we note that a bounded domain $\Omega \subset \mathbf{R}^n$ is said to be starlike Lipschitz with center $\hat{x} \in \Omega$ provided

$$\partial\Omega = \{\hat{x} + R(\omega)\omega : \omega \in \partial B(0, 1)\} \text{ where } \log R : \partial B(0, 1) \to \mathbf{R} \text{ is Lipschitz on } \partial B(0, 1). \tag{1.11}$$

We denote the Lipschitz constant for $\log R$ by M. Let $\Omega_i, i = 1, 2$, be two starlike Lipschitz domains with $\Omega_1 \subset \Omega_2$ and let $\rho > 0$ satisfy

$$d(\hat{x}, \partial\Omega_1) \approx \rho \leq d(\hat{x}, \partial\Omega_1)/4.$$

We say that $D_i = \Omega_i \setminus \bar{B}(\hat{x}, \rho), i = 1, 2$, are starlike Lipschitz ring domains with center $\hat{x}$. Let $R_i, i = 1, 2$, be the graph functions defining $\Omega_i, i = 1, 2$, and let, for fixed $p, 1 < p < \infty$, u_1, u_2, be the p capacitary functions for D_1, D_2, respectively. Furthermore, assume that $R_1, R_2 \in C_0^\infty(\partial B(0, 1))$ and also that $w \in \partial\Omega_1 \cap \partial\Omega_2$ and $r > 0$ are such that

$$B(w, 2r) \cap D_1 = B(w, 2r) \cap D_2 \tag{1.12}$$

while $\bar{B}(w, 8r) \cap \partial B(\hat{x}, 2\rho) = \emptyset$. Define, for $0 \leq \tau \leq 1$ and $\omega \in \partial B(0, 1)$,

$$R(\tau, \omega) = [R_2(\omega)]^\tau \, [R_1(\omega)]^{1-\tau}. \tag{1.13}$$

We then let $\Omega(\tau)$ be the starlike Lipschitz domain with center $\hat{x}$ and graph function $R(\tau, \cdot)$. We also define $D(\tau) = \Omega(\tau) \setminus \bar{B}(\hat{x}, \rho)$ as the corresponding ring domain and, for fixed $p, 1 < p < \infty$, we let $u(\cdot, \tau), \tau \in [0, 1]$, be the p capacitary function for $D(\tau)$. Then $u(\cdot, \tau) = 1$ on $\partial B(\hat{x}, \rho)$ and $u(\cdot, \tau) = 0$ on $\partial\Omega(\tau)$ in the $W_0^{1,p}$ Sobolev sense for $\tau \in [0, 1]$. Under these assumptions one can prove (see [LN, Lemma 4.5]) that $u_\tau(\cdot, \tau)$ is well defined and that

$$\log\left(\frac{u_2(x)}{u_1(x)}\right) = \int_0^1 \frac{u_\tau(x, \tau)}{u(x, \tau)} \, d\tau. \tag{1.14}$$

It follows from (1.12) that

$$D(\tau) \cap B(w, 2r) = D_1 \cap B(w, 2r) \text{ for all } \tau \in [0, 1], \tag{1.15}$$

and from a boundary maximum principle, as well as smoothness of $\partial\Omega(\tau)$, we see that $u_\tau > 0$ in $D(\tau)$ while $u_\tau = 0$ continuously on $\partial D_1 \cap B(w, 2r)$. Furthermore, using starlikness of the ring domains it follows (see [LN, Lemma 2.5]) that $u(\cdot,\tau)$ satisfies (1.9), (1.10) with constants $\delta = \delta(p,n,M)$, $\delta' = \delta'(p,n,M)$ independent of $\tau \in [0,1]$. From these facts and (1.14) we see that the proof of Theorem 1 and Theorem 2 for the starlike Lipschitz ring domains defined above, is reduced to proving a boundary Harnack inequality and Hölder continuity for $u_\tau(\cdot,\tau)/u(\cdot,\tau)$ with constants depending only on p, n, and the Lipschitz constants for $\Omega_i, i = 1,2$. Thus we can use step 1 to conclude Theorems 1 and 2 in the above setting. Finally we get Theorem 1 for a general bounded Lipschitz domain from the above special case, a comparison type argument, and the maximum principle for p harmonic functions. More details on the proof of Theorem 1 are given in sections 2 and 3.

Step 3. In this step another deformation technique is introduced in order to prove Theorem 2 by an argument based on induction. To briefly discuss the argument one first observes that, due to the arbitrariness of v in the statement of Theorem 2, it suffices, by the triangle inequality, to prove Theorem 2 with u replaced by u' and with r replaced by r/c_1, for c_1 large enough, where u' is the p capacitary function for a starlike Lipschitz ring domain D' such that $\partial\Omega \cap B(w, r/c_1) = \partial D' \cap B(w, r/c_1)$. Writing (hence redefining) u for u' we let $r' = r/(4c_1^2)$ and define $u(\cdot,\tau), 0 \le \tau \le 1$, to be the p harmonic function in $\Omega \cap B(w, 2r')$ which has boundary values on $\partial[\Omega \cap B(w, 2r')]$, equal to $\tau v(\cdot) + (1-\tau)u(\cdot)$ for $0 \le \tau \le 1$. Note that $u(\cdot,0) = u(\cdot)$, $u(\cdot,1) = v(\cdot)$ and that we are in fact introducing a new set of deformations adapted to our problem. We first show (see Lemma 4.3) that there exists $\epsilon_0 > 0$ small such that if for some positive $\hat{L}$, $\tau \in [0,1]$ and some s, $0 < s \le r'$, we have

$$(1-\epsilon_0)\hat{L} \le \frac{u(\cdot,\tau)}{u} \le (1+\epsilon_0)\hat{L} \text{ in } \Omega \cap B(w,s), \tag{1.16}$$

then (1.9), (1.10) holds for $u(\cdot,\tau)$ with $\delta = \delta(p,n,M)$. Next we use $\xi_1 = 0 < \xi_2 < ... < \xi_m = 1$ to divide [0,1] into $\{[\xi_k, \xi_{k+1}]\}, 1 \le k \le m-1$, such that all of these intervals have a length of $\epsilon_0'/2$ with the possible exception of the interval containing $\xi_m = 1$ which is of length $\le \epsilon_0'/2$. From an initial application of Theorem 1, one deduces that $\epsilon_0' = \epsilon_0'(p,n,M)$ can be chosen so small (see (4.1)) that

$$1 - \epsilon_0/2 \le \frac{u(\cdot,\xi_{k+1})}{u(\cdot,\xi_k)} \le 1 + \epsilon_0/2 \text{ in } \Omega \cap B(w, 2r') \tag{1.17}$$

for every k, $1 \le k \le m-1$, see (4.23). From (1.16), (1.17), and the fact that $u(\cdot,\xi_1) = u(\cdot)$, it follows easily that $u(\cdot,\tau)$ satisfies (1.9), (1.10) on $\Omega \cap B(w, r'/4)$ whenever $\tau \in [0,\xi_2]$ and with constants independent of τ. Then, by an argument similar to the one described in step 2, we are able to make use of the results established in step 1 to conclude that Theorem 2 is in fact true in the special case of the functions $u(\cdot,\xi_1) = u(\cdot)$ and $u(\cdot,\xi_2)$. We then iterate along these lines and finally prove, by an inductive argument, that Theorem 2 is true for the functions $u(\cdot,\xi_1) = u(\cdot)$ and $u(\cdot,\xi_m) = v(\cdot)$. We stress that our argument depends heavily on the fact that u is a p capacitary function for which we have good control of ∇u.

In the second part of this paper we consider the Martin boundary problem for p harmonic functions in the setting of bounded Lipschitz domains. More specifically recall that $\tilde{u}$ is said to be a minimal positive p harmonic function in a bounded Lipschitz domain Ω and relative to $w \in \partial\Omega$, provided $\tilde{u} > 0$ is p harmonic in Ω and $\tilde{u}$ has continuous boundary value 0 on $\partial\Omega \setminus \{w\}$. $\tilde{u}$ is said to be unique up to constant multiples if $\tilde{v} = \lambda\tilde{u}$, for some constant λ, whenever $\tilde{v}$ is a minimal positive p harmonic function relative to $w \in \partial\Omega$. Finally we say that the p Martin boundary of Ω can be identified with $\partial\Omega$ provided each $w \in \partial\Omega$ corresponds to a unique (up to constant multiples) minimal positive p harmonic function. We note that for $p = 2$ one can easily use Theorem 1 to get that the Martin boundary of a bounded Lipschitz domain Ω agrees with its topological boundary. Indeed, if $w \in \partial\Omega$ and if u, v are minimal harmonic functions corresponding to w, one first uses Theorem 1 for harmonic functions to show that $\gamma = \inf_\Omega u/v > 0$. Next one applies this result to $u - \gamma v, v$ in order to conclude that $u = \gamma v$. Note however that this argument depends heavily on linearity of the Laplacian and thus the argument fails for the p Laplacian when $p \neq 2$. In fact, at this time we cannot prove that the p Martin boundary of a bounded Lipschitz domain always agrees with its topological boundary when $p \neq 2$.

To state our results we need another definition. Let Ω be a bounded Lipschitz domain. We call $\tilde{\Omega} \subset \Omega$ a non-tangential approach region at $w \in \partial\Omega$ if the intersection of the closure of $\tilde{\Omega}$ and the closure of Ω equals w and if, for some $\tilde{\eta} > 0$, $d(x, \partial\Omega) \geq \tilde{\eta}|x - w|$ for all $x \in \tilde{\Omega}$. To indicate w and $\tilde{\eta}$ we write $\tilde{\Omega}(w, \tilde{\eta})$. Using Theorem 2 and its proof it follows (see Lemma 4.28) that if u is a minimal positive p harmonic function in Ω relative to $w \in \partial\Omega$, then there exists $\tilde{\eta}, \tilde{r}, c$ depending only on p, n, M such that

$$
\begin{aligned}
&(a) \quad u \text{ satisfies (1.9) in } [\Omega \setminus \tilde{\Omega}(w, \tilde{\eta})] \cap B(w, \tilde{r}) \text{ with a constant } \delta = \delta(p, n, M),\\
&(b) \quad \text{given } y \in [\Omega \setminus \tilde{\Omega}(w, \tilde{\eta})] \cap B(w, \tilde{r}) \text{ there exists } z \in \partial\Omega \text{ with } y \in B(z, |z - w|/c) \text{ and}\\
&\qquad \xi = \xi(z), \delta' = \delta'(p, n, M), \text{ for which (1.10) holds for } x \in \Omega \cap B(z, 2|z - w|/c). \qquad (1.18)
\end{aligned}
$$

Our first result on the Martin boundary problem is Theorem 3 which gives a sufficient criteria for a minimal positive p harmonic function in a Lipschitz domain Ω to be unique (relative to $w \in \partial\Omega$).

Theorem 3. *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain with constant M and let $u, \tilde{\Omega}(w, \tilde{\eta})$, be as in (1.18). Suppose that there exist a sequence of positive numbers $\{\rho_l\}$ with $\lim_{l \to \infty} \rho_l = 0$, a fixed number $\tilde{b} > 1$ and $\delta > 0$ such that (1.9) holds for all*

$$x \in \hat{\Omega} = \bigcup_l \tilde{\Omega}(w, \tilde{\eta}/2) \cap [B(w, \tilde{b}\rho_l) \setminus B(w, \rho_l/\tilde{b})].$$

If $\tilde{b} = \tilde{b}(p, n, M, \delta)$ is large enough, then u is unique up to constant multiples.

From Theorem 3 we see that the determination of the p Martin boundary at a boundary point w is reduced, for Lipschitz domains, to proving the existence of a certain sequence of positive numbers tending to zero and a corresponding minimal positive p harmonic function satisfying the non-degeneracy condition (1.9) in $\hat{\Omega}$. We are able to verify this sufficiency condition for a number of interesting cases and in particular we prove the following theorem.

Theorem 4. *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain. Then the p Martin boundary of Ω can be identified with the topological boundary of Ω in the following cases.*

1. Ω is convex.

2. $\partial\Omega$ is C^1.

Also if $\partial\Omega$ has a tangent plane at w, then a minimal positive p harmonic function relative to w is unique up to constant multiples.

We also consider the exterior Martin boundary problem for a Lipschitz domain Ω. In this case, given p, $1 < p < \infty$, we say that $u > 0$ is a minimal positive p harmonic function relative to $w \in \partial\Omega$, provided u is p harmonic in $\mathbf{R}^n \setminus \bar{\Omega}$ with continuous boundary values 0 on $\partial\Omega \setminus \{w\}$ and $\lim_{x\longrightarrow\infty} u(x) = 0$. In this paper we do not define minimal positive p harmonic functions relative to ∞. We prove,

Theorem 5. *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain. Then the p Martin boundary of $\mathbf{R}^n \setminus \bar{\Omega}$ can be identified with the topological boundary of Ω in the following cases.*

1. Ω is convex.

2. $\partial\Omega$ is C^1.

Also if $\partial\Omega$ has a tangent plane at w, then a minimal positive p harmonic function in $\mathbf{R}^n \setminus \bar{\Omega}$ relative to w, is unique up to constant multiples.

Concerning proofs the proof of Theorem 3 is based on a variation of the deformation technique briefly described in step 3 above. In this case though, if u and v are two minimal positive p harmonic function in Ω relative to $w \in \partial\Omega$, we let $u(\cdot, \tau), 0 \leq \tau \leq 1$, be the p harmonic function in $\Omega \setminus B(w, r')$ with continuous boundary values $\tau v(\cdot) + (1 - \tau)u(\cdot)$ for $0 \leq \tau \leq 1$. Note that $u(\cdot, 0) = u(\cdot)$, $u(\cdot, 1) = v(\cdot)$. Using the hypotheses of Theorem 3, (1.18), Theorem 2, and an argument similar to the one described in step 3 we are able by way of an iterative induction argument to prove that $|\nabla u(\cdot, \tau)|$ satisfies a uniform non-degeneracy condition in $\hat{\Omega}$ with constants independent of τ. Versions of the results described in step 1 are then used to establish certain decay estimates for the oscillation of u/v in $\Omega \setminus B(w, r')$. These estimates are easily seen to imply Theorem 3. We refer to section 5 for details. Theorem 4 is proved by constructing a minimal positive p harmonic function in Ω, relative to $w \in \partial\Omega$, which satisfies the criteria of Theorem 3. A similar game plan is used to prove Theorem 5.

The rest of the paper is organized as follows. In section 2 we first state some basic estimates for p harmonic functions in Lipschitz domains. We then point out that (1.9) and (1.10) imply that the arguments of sections 2 and 3 in [LN] can be reused to deduce a boundary Harnack inequality, and the corresponding Hölder continuity for the quotients of positive solutions to the partial differential equation in (1.6), (1.7). In this section we focus in particular on estimates of the decay for the oscillation of quotients of positive solutions as this is the kind of estimates we need in the proof of Theorem 3. In section 3 we prove Theorem 1 while in section 4 we develop the deformation technique briefly described in step 3 above and in particular we prove Theorem 2. By similar arguments, in section 5, we get Theorem 3. Section 6 is devoted to the proof of Theorem 4. At the end of section 6 we also state, Corollary 6.22, a local flatness criteria at $w \in \partial\Omega$, (more general than a tangent plane), which implies uniqueness of the corresponding minimal positive p harmonic function (up to constant multiples). In section 7 we prove Theorem

5 and also point out (see Closing Remarks) that in $\mathbf{R}^2$ it is always true that the boundary of a Lipschitz domain (or its complement) can be identified with the p Martin boundary of the domain.

2 Estimates for p harmonic functions in Lipschitz domains

In the following we start by stating and proving a number of estimates for p harmonic functions in a bounded Lipschitz domain $\Omega \subset \mathbf{R}^n$ having Lipschitz constant M. Recall that $\Delta(w,r) = \partial\Omega \cap B(w,r)$ whenever $w \in \partial\Omega$, $0 < r$. Throughout the paper c will denote, unless otherwise stated, a positive constant ≥ 1, not necessarily the same at each occurrence, which only depends on p, n and M. In general, $c(a_1, \dots, a_n)$ denotes a positive constant ≥ 1, not necessarily the same at each occurrence, which depends on p, n, M and $a_1, \dots, a_n$. If $A \approx B$ then A/B is bounded from above and below by constants which, unless otherwise stated, only depend on p, n and M. Moreover, we let $\max\limits_{B(z,s)} u$, $\min\limits_{B(z,s)} u$ be the essential supremum and infimum of u on $B(z,s)$ whenever $B(z,s) \subset \mathbf{R}^n$ and whenever u is defined on $B(z,s)$.

2.1 Basic estimates

For proofs and references to proofs of Lemmas 2.1-2.6 stated below we refer to [LN].

Lemma 2.1. *Given $p, 1 < p < \infty$, let u be a positive p harmonic function in $B(w,2r)$. Then*

$$(i) \qquad r^{p-n} \int\limits_{B(w,r/2)} |\nabla u|^p \, dx \leq c\,(\max_{B(w,r)} u)^p,$$

$$(ii) \qquad \max_{B(w,r)} u \leq c \min_{B(w,r)} u.$$

Furthermore, there exists $\alpha = \alpha(p,n) \in (0,1)$ such that if $x, y \in B(w,r)$ then

$$(iii) \qquad |u(x) - u(y)| \leq c \left(\tfrac{|x-y|}{r}\right)^{\alpha} \max_{B(w,2r)} u.$$

Lemma 2.2. *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain and suppose that p is given, $1 < p < \infty$. Let $w \in \partial\Omega$, $0 < r < r_0$ and suppose that u is a positive p harmonic function in $\Omega \cap B(w,2r)$ and that u has continuous boundary value 0 on $\Delta(w,2r)$. Then*

$$(i) \qquad r^{p-n} \int\limits_{\Omega \cap B(w,r/2)} |\nabla u|^p \, dx \leq c\,(\max_{\Omega \cap B(w,r)} u)^p.$$

Furthermore, there exists $\alpha = \alpha(p,n,M) \in (0,1)$ such that if $x, y \in \Omega \cap B(w,r)$ then

$$(ii) \qquad |u(x) - u(y)| \leq c \left(\tfrac{|x-y|}{r}\right)^{\alpha} \max_{\Omega \cap B(w,2r)} u.$$

Lemma 2.3. *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain and suppose that p is given, $1 < p < \infty$. Let $w \in \partial\Omega$, $0 < r < r_0$ and suppose that u is a positive p harmonic function in $\Omega \cap B(w, 2r)$ and that u has continuous boundary value 0 on $\Delta(w, 2r)$. There exists $c = c(p, n, M)$ such that if $\tilde{r} = r/c$, then*

$$\max_{\Omega \cap B(w,\tilde{r})} u \leq c\, u(a_{\tilde{r}}(w)).$$

Lemma 2.4. *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain and suppose that p is given, $1 < p < \infty$. Let $w \in \partial\Omega$, $0 < r < r_0$ and suppose that u is a positive p harmonic function in $\Omega \cap B(w, 4r)$ and that $u = 0$ on $\Delta(w, 4r)$. Extend u to $B(w, 4r)$ by defining $u \equiv 0$ on $B(w, 4r) \setminus \Omega$. Then u has a representative in $W^{1,p}(B(w, 4r))$ with Hölder continuous partial derivatives in $\Omega \cap B(w, 4r)$. In particular there exists $\sigma \in (0, 1]$, depending only on p, n, such that if $x, y \in B(\tilde{w}, \tilde{r}/2)$, $B(\tilde{w}, 4\tilde{r}) \subset \Omega \cap B(w, 4r)$, then*

$$c^{-1}\, |\nabla u(x) - \nabla u(y)| \leq (|x - y|/\tilde{r})^{\sigma} \max_{B(\tilde{w},\tilde{r})} |\nabla u| \leq c\, \tilde{r}^{-1}\, (|x - y|/\tilde{r})^{\sigma}\, u(\tilde{w}).$$

Also if $c|\nabla u(x)| \geq u(x)/d(x, \partial\Omega)$, for all $x \in B(\tilde{w}, 2\tilde{r})$, then

$$\max_{B(\tilde{w},\tilde{r}/2)} \sum_{i,j=1}^{n} |u_{x_i x_j}| \leq c \left((\tilde{r})^{-n} \int_{B(\tilde{w},\tilde{r})} \sum_{i,j=1}^{n} |u_{x_i x_j}|^2\, dx \right)^{1/2} \leq \frac{c^2 u(\tilde{w})}{d(\tilde{w}, \partial\Omega)^2}.$$

Lemma 2.5. *Let $1 < p < \infty$. If u is a positive p harmonic function in $\Omega \subset \mathbf{R}^n$, then u is infinitely differentiable in $B(w, r) \setminus \{x : \nabla u(x) = 0\}$ for any $w \in \Omega$, $r > 0$ such that $B(w, r) \subset \Omega$. Furthermore, assume that $w \in \partial\Omega$, $r > 0$ and that u has continuous boundary value 0 on $\Delta(w, 2r)$. Assume also that, in an appropriate coordinate system,*

$$\begin{aligned} \Omega \cap B(w, 2r) &= \{y = (y', y_n) \in \mathbf{R}^n : y_n > \phi(y')\} \cap B(w, 2r), \\ \Delta(w, 2r) &= \{y = (y', y_n) \in \mathbf{R}^n : y_n = \phi(y')\} \cap B(w, 2r), \end{aligned}$$

for an infinitely differentiable function ϕ, i.e., $\phi \in C^\infty(\mathbf{R}^{n-1})$. Then there exists an open neighborhood N of $\Delta(w, 3r/2)$ and $\hat{\epsilon}$, both depending only on the C^3 norm of ϕ, p, n, such that

(*a*) *u has a C^∞ extension to the closure of $N \cap \Omega \cap B(w, 3r/2)$,*

(*b*) *$\hat{\epsilon}\, d(y, \partial\Omega)^{-1}\, u(y) \leq |\nabla u(x)| \leq \hat{\epsilon}^{-1}\, d(y, \partial\Omega)^{-1}\, u(y)$ whenever $B(y, 2d(y, \partial\Omega)) \subset N, y \in \Omega \cap B(w, r)$, and $x \in \Omega \cap B(y, 2d(y, \partial\Omega))$.*

Next suppose for fixed $p, 1 < p < \infty$, that Ω^* is a starlike Lipschitz ring domain with center $\hat{x}$ and constant M^*. Let u^* be the p capacitary function for the starlike Lipschitz ring domain $D^* = \Omega^* \setminus B(\hat{x}, \rho)$ where $c^{-1} d(\hat{x}, \partial D^*) \leq \rho \leq \frac{1}{4} d(\hat{x}, \partial D^*)$. Then $u^* = 1$ on $\partial B(\hat{x}, \rho)$, $u^* = 0$ on $\partial\Omega^*$ in the $W_0^{1,p}(\Omega^*)$ Sobolev sense, and u^* is p harmonic in D^*.

Lemma 2.6. *Let u^*, D^*, p, be as above. Then there exists c^*, depending only on p, n, M^*, such that*

(*i*) $0 < |\nabla u^*(x)| \leq c^* \langle \frac{\hat{x} - x}{|\hat{x} - x|}, \nabla u^*(x) \rangle$ *whenever $x \in D^*$,*

(*ii*) $(c^*)^{-1} u^*(x)/d(x, \partial D^*) \leq |\nabla u^*(x)| \leq c^* u^*(x)/d(x, \partial D^*)$ *whenever $x \in D^*$.*

Remark. We note from (ii) that u^* satisfies the uniform non-degeneracy condition in (1.9) and a condition akin to (1.10). From basic geometry one in fact sees easily that (i) of Lemma 2.6 implies (1.10) for some ξ and u^* on $\Omega^* \cap B(w, r)$, whenever $w \in \partial\Omega^*$ and $r \leq d(\hat{x}, \partial\Omega^*)/c$. Moreover, suppose as in (1.10) that Ω' is Lipschitz, $w' \in \partial\Omega'$, and that u' is a positive p harmonic function in $\Omega' \cap B(w', 2r)$ with continuous boundary value 0 on $\partial\Omega' \cap B(w', 2r)$. If (1.9), (1.10) hold for u' with $\xi = e_n$, then from the mean value theorem of elementary calculus we see that u' is increasing on $\Omega' \cap B(w', r)$ in the direction of ω whenever $\omega \in B(e_n, c^{-1}) \cap \partial B(0, 1)$ and c is large enough (depending only on δ'). Using this fact and once again basic geometry, we deduce that there exits $w'' \in \Omega' \cap B(w', r)$ for which

$$0 < |\nabla u'(x)| \leq c \, \langle \tfrac{w''-x}{|\hat{w}''-x|} \, , \, \nabla u'(x) \rangle$$

on rays connecting w'' to points in $\partial\Omega' \cap B(w', r/c)$. Also, $d(w'', \partial\Omega') \approx |w'' - w'| \approx r$ and all constants depend only on δ, δ'. Finally we remark that Lemmas 2.7, 2.8 stated below were originally proved in [LN] when u was the capacitary function for a starlike ring domain. However the proof used only Lemmas 2.1-2.6 and did not depend on the fact that u was a p capacitary function. From the above discussion it is easily seen that (1.9), (1.10) can be used for u in place of Lemma 2.6. Now Lemmas 2.1-2.5 are standard for positive p harmonic functions vanishing on the boundary of a Lipschitz domain. Thus (1.9), (1.10) will be the only conditions in our proofs which we need to check in order to apply Lemmas 2.7, 2.8.

2.2 Boundary behaviour of positive solutions to the operator L

Repeating the arguments of proof of section 3 in [LN] the following two lemmas can be proven.

Lemma 2.7. *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain, $1 < p < \infty, w \in \partial\Omega, 0 < r < r_0$, and assume that u is a positive p harmonic function in $\Omega \cap B(w, 2r)$, continuous in $\bar{\Omega} \cap \bar{B}(w, 2r)$ with $u = 0$ on $\Delta(w, 2r)$. Assume that $|\nabla u|$ satisfies (1.9), (1.10) in $\Omega \cap B(w, 2r)$ with constants δ, δ' and that, $\phi, \Delta(w, 2r)$, are as in Lemma 2.5. Let h_1, h_2 be positive solutions, in $\Omega \cap B(w, r)$, to the elliptic operator L defined in (1.6), (1.7) and assume that $h_i = 0, i = 1, 2$, continuously on $\Delta(w, r)$. Then there exists $\hat{c}$, depending only on p, n, M, δ and δ', such that if $\hat{r} = r/\hat{c}$, then*

$$\hat{c}^{-1} \frac{h_1(a_{\hat{r}}(w))}{h_2(a_{\hat{r}}(w))} \leq \frac{h_1(y)}{h_2(y)} \leq \hat{c} \, \frac{h_1(a_{\hat{r}}(w))}{h_2(a_{\hat{r}}(w))}$$

whenever $y \in \Omega \cap B(w, \hat{r})$.

Lemma 2.8. *Let Ω, ϕ, p, u, w, r, $\hat{r}$, δ, δ' and h_1, h_2 be as in Lemma 2.7. There exist α, $0 < \alpha < 1$, and c^*, both depending only on p, n, M, δ and δ', such that if $y, y' \in \Omega \cap B(w, \hat{r}/4)$ then,*

$$\left| \frac{h_1(y)}{h_2(y)} - \frac{h_1(y')}{h_2(y')} \right| \leq c^* \left(\frac{|y - y'|}{r} \right)^{\alpha} \frac{h_1(a_{\hat{r}}(w))}{h_2(a_{\hat{r}}(w))}.$$

To briefly outline the proof of Lemma 2.7 one sees, from the remark after Lemma 2.6, that the argument in [LN, Lemma 2.39] can be used, essentially verbatim, to show the existence of

$q > p$, depending only on p, n, δ, δ' and the Lipschitz constant for Ω, such that the following reverse Hölder inequality holds whenever $z \in \partial\Omega$ and $B(z, 4s) \subset B(w, 2r)$,

$$\int_{\Delta(z,s)} |\nabla u|^q \, dH^{n-1} \leq c\, s^{(n-1)(\frac{p-1-q}{p-1})} \left(\int_{\Delta(z,s)} |\nabla u|^{p-1} \, dH^{n-1} \right)^{q/(p-1)} .$$

In the last display H^{n-1} denotes Hausdorff $(n-1)$-measure on $\partial\Omega$. Using this inequality and arguing as in [LN, Lemma 2.45] we see that there exists a starlike Lipschitz domain $\tilde{\Omega} \subset \Omega \cap B(z, s)$ with center $\tilde{z}$, $d(\tilde{z}, \partial\Omega) \geq c^{-1}s$, satisfying

$$\begin{array}{ll} (a) & c\, H^{n-1}[\partial\tilde{\Omega} \cap \Delta(z, s)] \geq s^{n-1}, \\ (b) & c^{-1}s^{-1}\, u(\tilde{z}) \, \leq \, |\nabla u(x)| \, \leq \, cs^{-1}\, u(\tilde{z}) \text{ whenever } x \in \tilde{\Omega}. \end{array}$$

Here c depends only on p, n, and the Lipschitz constant for Ω. Next in [LN, Lemma 2.54] we define

$$d\tilde{\sigma}(x)/dx = d(x, \partial\tilde{\Omega}) \max_{B(x, \frac{1}{2}d(x,\partial\tilde{\Omega}))} \{|\nabla u|^{2p-6} \sum_{i,j=1}^{n} u^2_{y_i y_j}\}$$

whenever $x \in \tilde{\Omega}$, and we use $(a), (b)$ in the above display, as well as Lemma 2.4, to show that $\tilde{\sigma}$ is a Carleson measure on $\tilde{\Omega}$ in the sense that if $z \in \partial\tilde{\Omega}$ and $0 < t < s/4$, then

$$\tilde{\sigma}(\tilde{\Omega} \cap B(z, t)) \, \leq \, c\, t^{n-1}\, (u(\tilde{z})/s)^{2p-4}.$$

We then use this fact and a theorem in [KP] to deduce that if $\tilde{\omega}$ is elliptic measure defined with respect to $L, u, (b_{ij})$ in $\tilde{\Omega}$, then $\tilde{\omega}$ is an A^∞ weight with respect to H^{n-1} measure on $\partial\tilde{\Omega}$ (see [LN, Theorem 3.11]). Finally, we use this result for $\tilde{\omega}$ as well as some arguments on elliptic measure (see [LN, Lemmas 3.13]) to get Lemma 2.7. $\Box$

Below we will give the proof of Lemma 2.8. In fact Lemma 2.8 is an easy consequence of the following estimate for the decay of the oscillation of h_1/h_2.

Lemma 2.9. *Let Ω, p, u, w, r, δ, δ', $\hat{c}$, and h_1, h_2 be as in Lemma 2.7. Define, for $0 < s < \hat{r}$,*

$$M(s) \;=\; M(s, w) = \sup_{\Omega \cap B(w,s)} \frac{h_1(y)}{h_2(y)}, \quad m(s) = m(s, w) = \inf_{\Omega \cap B(w,s)} \frac{h_1(y)}{h_2(y)}.$$

There exists $\theta = \theta(p, n, M, \delta, \delta') \in (0, 1)$ such that if $0 < s < r$, then

$$M(s/\hat{c}) - m(s/\hat{c}) \leq \theta(M(s) - m(s)).$$

Proof. We note that by construction $h_1 - m(s)h_2 \geq 0$ and $M(s)h_2 - h_1 \geq 0$ in $\Omega \cap B(w, s)$ and we observe from Harnack's inequality for positive solutions to L that each of these functions is either positive or identically zero in $\Omega \cap B(w, s)$.

Using Lemma 2.7 applied to the functions $h_1 - m(s)h_2 \geq 0$, $M(s)h_2 - h_1 \geq 0$ and h_2 we find that if $0 < s < r$ then

$$\begin{aligned} M(s/\hat{c}) - m(s) &\leq \hat{c}^2(m(s/\hat{c}) - m(s)) \\ M(s) - m(s/\hat{c}) &\leq \hat{c}^2(M(s) - M(s/\hat{c})). \end{aligned} \tag{2.10}$$

If we define $\psi(t) = M(t) - m(t)$, we then get, adding the inequalities in (2.10),

$$\psi(s) + \psi(s/\hat{c}) \;\leq\; \hat{c}^2(\psi(s) - \psi(s/\hat{c}))$$

or

$$\psi(s/\hat{c}) \;\leq\; \tfrac{\hat{c}^2-1}{\hat{c}^2+1}\,\psi(s). \tag{2.11}$$

Clearly (2.11) implies the conclusion of Lemma 2.9 with $\theta = (\hat{c}^2 - 1)/(\hat{c}^2 + 1)$. $\Box$

Proof of Lemma 2.8. To start with we note that if $y, y' \in \Omega \cap B(w, \hat{r}/4)$ and $|y - y'| > \frac{\hat{r}}{1000}$, then from Lemma 2.7 we see that Lemma 2.8 holds. Furthermore, if $|y - y'| \leq \min[\frac{\hat{r}}{1000}, d(y, \partial\Omega)/2]$ we can use Harnack's inequality and interior Hölder continuity estimates for h_1, h_2 to get Lemma 2.8 in this case. If none of these two cases occur choose $\tilde{y} \in \partial\Omega \cap B(w, \hat{r}/2)$ with $|y - \tilde{y}| = d(y, \partial\Omega)$. Using Lemma 2.9, with $M(s)$ and $m(s)$ defined with respect to $\tilde{y}$, i.e., $M(s) = M(s, \tilde{y})$, $m(s) = m(s, \tilde{y})$, we see for given s, $0 < s < \hat{r}/2$, that

$$M(s/\hat{c}, \tilde{y}) - m(s/\hat{c}, \tilde{y}) \leq \theta(M(s, \tilde{y}) - m(s, \tilde{y}))$$

Iterating this relation, starting from $s = \hat{r}/4$ and finishing with $s \approx 4|y - y'|$, we deduce for some $\alpha, 0 < \alpha < 1$, and c, both depending only on p, n, M, δ, δ',

$$\begin{aligned} \left|\frac{h_1(y)}{h_2(y)} - \frac{h_1(y')}{h_2(y')}\right| &\leq M(2|y - y'|, \tilde{y}) - m(2|y - y'|, \tilde{y}) \\ &\leq c\left(\frac{|y - y'|}{r}\right)^{\alpha} \frac{h_1(a_{\hat{r}}(w))}{h_2(a_{\hat{r}}(w))}. \end{aligned}$$

This completes the proof of Lemma 2.8. $\Box$

We end this section by stating Lemma 2.12, a variation of Lemma 2.9. This lemma will be used in the proof of Theorem 3.

Lemma 2.12. *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain, $1 < \hat{b}, w \in \partial\Omega$, and $0 < \hat{b}^2 t < r < r_0/4$. Assume, for p given, $1 < p < \infty$, that u is a positive p harmonic function in $\Omega \cap [B(w, \hat{b}^2 t) \setminus B(w, t/\hat{b}^2)]$, continuous in the closure of $\Omega \cap [B(w, \hat{b}^2 t) \setminus B(w, t/\hat{b}^2)]$ with $u = 0$ on $\partial\Omega \cap [B(w, \hat{b}^2 t) \setminus B(w, t/\hat{b}^2)]$. Assume also that $\Delta(w, 2r), \phi$ are as in Lemma 2.5 and that the uniform non-degeneracy condition (1.9) holds with u' replaced by u in $\Omega \cap [B(w, \hat{b}^2 t) \setminus B(w, t/\hat{b}^2)]$. Finally assume for each $z \in \partial\Omega \cap [B(w, \hat{b}^2 t) \setminus B(w, t/\hat{b}^2)]$ that there exist $\xi = \xi(z)$ and $c' = c'(p, n, M) \geq 2$ such that (1.10) is true with u' replaced by u in $\Omega \cap B(z, |z - w|/c')$. Let h_1, h_2 be positive solutions, in $\Omega \cap [B(w, \hat{b}^2 t) \setminus B(w, t/\hat{b}^2)]$, to the elliptic operator L defined in (1.6), (1.7), with respect to u. Moreover suppose $h_i, i = 1, 2$, extends continuously to the closure of $\Omega \cap [B(w, \hat{b}^2 t) \setminus B(w, t/\hat{b}^2)]$ with $h_i \equiv 0$ on $\partial\Omega \cap [B(w, \hat{b}^2 t) \setminus B(w, t/\hat{b}^2)]$. Define, for $\rho \in (1/\hat{b}^2, 1)$,*

$$\begin{aligned} \hat{M}(\rho) &= \hat{M}(\rho, w) = \sup_{\Omega \cap [B(w, \hat{b}^2 \rho t) \setminus B(w, t/(\hat{b}^2 \rho))]} \frac{h_1(y)}{h_2(y)}, \\ \hat{m}(\rho) &= \hat{m}(\rho, w) = \inf_{\Omega \cap [B(w, \hat{b}^2 \rho t) \setminus B(w, t/(\hat{b}^2 \rho))]} \frac{h_1(y)}{h_2(y)}. \end{aligned}$$

If $\hat b = \hat b(p,n,M,\delta,\delta')$ is large enough, then there exists $\theta = \theta(p,n,M,\delta,\delta') \in (0,1)$ such that

$$\hat M(1/\hat b) - \hat m(1/\hat b) \leq \theta(\hat M(1) - \hat m(1)).$$

Proof of Lemma 2.12. To begin the proof, we note that the non-degeneracy assumption (1.9) on u allows us to apply the interior Harnack inequality for positive solutions to L as in Lemma 2.9, while (1.9),(1.10) together give the boundary Harnack inequality in Lemma 2.7 with w, r replaced by $z \in \partial\Omega \cap [B(w, \hat b^2 t) \setminus B(w, t/\hat b^2)]$ and $0 < s \leq |z - w|/c'$. We also note that $h_1 - \hat m(1) h_2$ and $h_2 \hat M(1) - h_1$ are positive solutions to the operator L in $\Omega \cap [B(w, \hat b^2 t) \setminus B(w, t/\hat b^2)]$ with continuous boundary values equal to 0 on $\partial\Omega \cap [B(w, \hat b^2 t) \setminus B(w, t/\hat b^2)]$. Given $0 < \gamma << 1$ to be chosen, we define

$$\begin{aligned}
\tilde M(1/\hat b) &= \sup_{\Omega\cap[B(w,\hat b t)\setminus B(w,t/\hat b)]\cap\{x: d(x,\partial\Omega)\geq\gamma t\}} \frac{h_1(y)}{h_2(y)},\\
\tilde m(1/\hat b) &= \inf_{\Omega\cap[B(w,\hat b t)\setminus B(w,t/\hat b)]\cap\{x: d(x,\partial\Omega)\geq\gamma t\}} \frac{h_1(y)}{h_2(y)}.
\end{aligned}$$

Using Lemma 2.7 as in the proof of Lemma 2.9 we get for $\gamma = \gamma(p,n,M,\delta,\delta') > 0$ sufficiently small, that

$$\begin{aligned}
\hat M(1/\hat b) - \hat m(1) &\leq c^*(\tilde M(1/\hat b) - \hat m(1)),\\
\hat M(1) - \hat m(1/\hat b) &\leq c^*(\hat M(1) - \tilde m(1/\hat b)). \qquad (2.13)
\end{aligned}$$

Next if $\hat b$ is large enough we can use the the interior Harnack inequality in $\Omega \cap [B(w, \hat b^2 t) \setminus B(w, t/\hat b^2)]$ to deduce

$$\begin{aligned}
\tilde M(1/\hat b) - \hat m(1) &\leq c_*\,(\tilde m(1/\hat b) - \hat m(1)),\\
\hat M(1) - \tilde m(1/\hat b) &\leq c_*\,(\hat M(1) - \tilde M(1/\hat b)). \qquad (2.14)
\end{aligned}$$

Combining (2.13), (2.14), and using Lemma 2.7 again we deduce that there exists a constant $\tilde c = \tilde c(p,n,\delta,\delta')$ such that

$$\begin{aligned}
\hat M(1/\hat b) - \hat m(1) &\leq \tilde c(\hat m(1/\hat b) - \hat m(1))\\
\hat M(1) - \hat m(1/\hat b) &\leq \tilde c(\hat M(1) - \hat M(1/\hat b). \qquad (2.15)
\end{aligned}$$

Using (2.15) it follows, as in (2.11), that there exists, for $\hat b = \hat b(p,n,M,\delta,\delta')$ large enough, $0 < \theta = \theta(p,n,M,\delta,\delta') < 1$, such that

$$\hat M(1/\hat b) - \hat m(1/\hat b) \leq \theta(\hat M(1) - \hat m(1)).$$

This completes the proof of the lemma. □

3 Deformation of starlike Lipschitz ring domains and the proof of Theorem 1

Recall from section 1 that a bounded domain $\Omega \subset \mathbf{R}^n$ is said to be starlike Lipschitz with center $\hat x \in \Omega$ provided

$$\partial\Omega = \{\hat x + R(\omega)\omega : \omega \in \partial B(0,1)\} \text{ where } \log R : \partial B(0,1) \to \mathbf{R} \text{ is Lipschitz on } \partial B(0,1).$$

If $\bar{B}(\hat{x},\rho)\subset\Omega$, then we say that $D=\Omega\setminus\bar{B}(\hat{x},\rho)$ is a starlike Lipschitz ring domain with center $\hat{x}$. If p is fixed, $1<p<\infty$, then we let $\hat{u}=\hat{u}(\cdot,p)$ be the p capacitary function for D. That is $\hat{u}\equiv 1$ on $B(\hat{x},\rho)$, $\hat{u}\equiv 0$ on $\partial\Omega$ in the sense of $W_0^{1,p}(\Omega)$ and $\hat{u}$ is p harmonic in D. It is well known that $\hat{u}$ is unique and

$$\int_D |\nabla\hat{u}|^p\,dx=\inf\{\int_D |\nabla\theta|^p\,dx\}$$

where the infimum is taken over all $\theta\in C_0^\infty(\Omega)$ with $\theta\equiv 1$ on $\bar{B}(\hat{x},\rho)$.

We first prove Theorem 1 with u,v replaced by c^+u_1,u_2, where $u_i, i=1,2$ are the p capacitary functions of starlike ring domains $D_i=\Omega_i\setminus\bar{B}(\hat{x},\rho), i=1,2$, with $D_1\subset D_2$. The constant c^+ is defined below. We assume that the Lipschitz constant for $D_i, i=1,2$, is bounded above by constants depending only on the Lipschitz constant for Ω and that

$$\max\{16r,\tfrac{d(\hat{x},\partial\Omega_1)}{c}\}\leq\rho\leq\min\{\tfrac{d(\hat{x},\partial\Omega_1)}{4},cr\},\tag{3.1}$$

where $c=c(M,n,p)\geq 4$. If $R_i, i=1,2$, denotes the graph function for Ω_i corresponding to $\hat{x}$, then we also assume, temporarily, that

$$R_i, i=1,2, \text{ is infinitely differentiable on } \partial B(0,1).\tag{3.2}$$

Next we define, for $\tau\in[0,1]$ and $\omega\in\partial B(0,1)$,

$$R(\tau,\omega)=[R_2(\omega)]^\tau\ [R_1(\omega)]^{1-\tau}$$

and we let $\Omega(\tau)$ be starlike Lipschitz domains with center $\hat{x}$ and graph function $R(\tau,\cdot)$. We also define $D(\tau)=\Omega(\tau)\setminus\bar{B}(\hat{x},\rho)$ as the corresponding ring domain. For fixed $p, 1<p<\infty$, let $u(\cdot,\tau),\tau\in[0,1]$, be the p capacitary function for $D(\tau)$. From the hypotheses of Theorem 1 and our assumption that $D_1\subset D_2$, we have

$$\begin{aligned}&D(\tau_1)\subset D(\tau_2)\text{ whenever } 0\leq\tau_1<\tau_2\leq 1 \text{ and}\\ &D(\tau)\cap B(w,2r)=D_1\cap B(w,2r)\text{ whenever }\tau\in[0,1].\end{aligned}\tag{3.3}$$

Extend $u(\cdot,\tau)$ to $\mathbf{R}^n$ by putting $u(\cdot,\tau)\equiv 0$ on $\mathbf{R}^n\setminus\Omega(\tau)$ and $u(\cdot,\tau)\equiv 1$ on $\bar{B}(\hat{x},\rho)$. From Lemma 2.2 *(ii)* we see that each function in the set $\{u(\cdot,\tau),0\leq\tau\leq 1\}$ is Hölder continuous on $\mathbf{R}^n$ with exponent independent of $\tau\in[0,1]$. Next in view of (3.2) we deduce that Lemma 2.5 holds for $u(\cdot,\tau),\tau\in[0,1]$, where $N,\hat{\epsilon}$ can be chosen independent of $\tau\in[0,1]$. Finally observe that Lemma 2.6 holds with u^* replaced by $u(\cdot,\tau)$ and constants independent of $\tau\in[0,1]$. Using these facts we in [LN, Lemma 4.5] prove the following.

Lemma 3.4. *Let $u(\cdot,\cdot)$ be as above. Then $u_\tau(x,\tau)=\frac{\partial}{\partial\tau}u(x,\tau)$ exists continuously whenever $(x,\tau)\in\bigcup_{t\in[0,1]}D(t)\times\{t\}$. Moreover, $u_\tau(\cdot,\tau)$ extends continuously to $\bar{D}(\tau)$ for $\tau\in[0,1]$ and*

- *(i)* $u_\tau(\cdot,\tau)$ *is a solution to (1.6) in* $D(\tau)$ *with* b_{ij} *defined relative to* $u(\cdot,\tau)$,
- *(ii)* $u_\tau(\hat{x}+R(\tau,\omega)\omega,\tau)=-R(\tau,\omega)\langle\omega,\nabla u(\hat{x}+R(\tau,\omega)\omega,\tau)\rangle\log(R_2/R_1)(\omega)$ *when* $\hat{x}+R(\tau,\omega)\omega\in\partial\Omega(\tau)$ *and* $u_\tau\equiv 0$ *on* $\partial B(\hat{x},\rho)$,
- *(iii)* $\log\left(\frac{u_2(x)}{u_1(x)}\right)=\int_0^1\frac{u_\tau(x,\tau)}{u(x,\tau)}\,d\tau$ *whenever* $x\in D_1\cap B(w,2r)$.

To continue the proof of Theorem 1 for the functions u_1 and u_2 we note from (3.3) and the maximum principle for p harmonic functions that $u(\cdot,\tau_1) \leq u(\cdot,\tau_2)$ in $D(\tau_1)$. Using this fact, (3.3), and Lemma 3.4 (ii) we see that

$$u_\tau \geq 0 \text{ in } D(\tau) \text{ and } u_\tau = 0 \text{ on } \partial D(\tau) \cap B(w, 2r). \tag{3.5}$$

From (3.3) and Lipschitzness of $\log R_i, i = 1, 2$ we also see that $\log(R_2/R_1) \leq c$. Using this fact, the fact that both $u_\tau(x,\tau)$ and $\langle \hat{x} - x, \nabla u(x,\tau)\rangle$, both satisfy (1.6) in $D(\tau)$, Lemma 2.6 (ii) for $u(\cdot,\tau)$ and the maximum principle for solutions to (1.6), we get

$$0 \leq u_\tau(x,\tau) \leq c\langle \hat{x} - x, \nabla u(x,\tau)\rangle \text{ in } D_1 \cap B(w, 2r). \tag{3.6}$$

From (3.6), (3.1), and Lemma 2.6 (i) we conclude that

$$0 \leq u_\tau(a_r(w),\tau) \leq cr|\nabla u(a_r(w),\tau)| \leq c^2 u(a_r(w),\tau) \tag{3.7}$$

where, $a_r(w)$ is defined below (1.4) and once again, $c = c(p, n, M)$ is independent of τ. From the remark after Lemma 2.6 we see that Lemma 2.7 can be applied with $h_1 = u_\tau(\cdot,\tau), h_2 = u(\cdot,\tau)$ whenever $\tau \in [0, 1]$. From this lemma, (3.7), Lemma 3.4 (iii), and Harnack's inequality we conclude that

$$\log(u_2/u_1) \leq c \text{ in } B(w, \hat{r})$$

when (3.2) holds. In the last display, $\hat{r} = r/\hat{c}$ as stated in Lemma 2.7. The smoothness assumption in (3.2) is removed by a standard approximation argument, using the fact that the constants in the above argument depend only on p, n, and the Lipschitz constants for Ω_1, Ω_2. If $\tilde{r} = \hat{r}$, we conclude that Theorem 1 is valid when $u = c^+u_1, v = u_2$, where c^+ is chosen so that $c^+u_1(a_{\tilde{r}}(w)) = u_2(a_{\tilde{r}}(w))$.

To continue the proof of Theorem 1 let Ω, M, w, r, u and v be as in Theorem 1 and put $w' = w + \frac{r}{4}e_n$. We observe that if c' is large enough (depending on p, n and M) and if we define $r' = r/c'$, then the domain $\Omega_1 \subset \Omega$, obtained from drawing all open line segments from points in $\Delta(w, r')$ to points in $B(w', r')$, is starlike Lipschitz with center w' and Lipschitz constant bounded by $c(n, M)$. Let, for $y \in \Delta(w, r')$, $R(\omega) = |y - w'|$ if $\omega = (y - w')/|y - w'|$. We also define, for $i = 0, 1, 2$, the sets $K_i = \{(y - w')/|y - w'| : y \in \Delta(w, 2^{-i}r')\}$ and introduce L as the supremum of R over the set $K_0 = \Delta(w, r')$. Choosing c' large enough, we may assume

$$100r' \leq r/4 \leq L \leq r. \tag{3.8}$$

From our construction we observe that, for some c (depending on p, n, and M),

$$\min\{d(K_2, \partial B(0, 1) \setminus K_1), d(K_1, \partial B(0, 1) \setminus K_0)\} \geq c^{-1}. \tag{3.9}$$

Let $0 \leq \alpha \leq 1$, $\alpha \in C_0^\infty(\mathbf{R}^n)$, with $\alpha \equiv 1$ on K_2 and $\alpha \equiv 0$ on $\partial B(0, 1) \setminus K_1$. Using (3.9) we see that we can choose α so that

$$|\nabla\alpha| \leq c^{-1}. \tag{3.10}$$

Let

$$\log R_2(\omega) = \begin{cases} \alpha \log R + (1 - \alpha) \log(2L) \text{ when } \omega \in K_0 \\ \log(2L) \text{ when } \omega \in \partial B(0, 1) \setminus K_0 \end{cases}$$

Using (3.10) it is easily shown that

$$\|\log R_2\hat{\|}_{\partial B(0,1)} \leq c\,(\|\log R\hat{\|}_{\partial K_0}+1). \tag{3.11}$$

Let Ω_2 be the starlike Lipschitz domain with center at w' and graph function R_2. Let $D_i = \Omega_i \setminus B(w', r'/4), i = 1, 2$, and let u_i be the corresponding p capacitary function. Note from (3.8) and the definition of α that $D_1 \subset D_2$ and $D_1 \cap B(w, r'/4) = D_2 \cap B(w, r'/4)$. Also (3.1) is valid with $\rho = r'/4$ and $\hat{x} = w'$. Finally observe that the Lipschitz constants for $D_i, i = 1, 2$, can be estimated above by the Lipschitz constant for Ω. From our construction, the fact that $L \geq 100r'$, Lemma 2.3, and Harnack's inequality we deduce first that

$$u_1 \leq c\min(u, v) \leq c\max(u, v) \leq c^2 u_2 \tag{3.12}$$

on $\partial[\Omega_1 \cap B(w, 3r'/4)]$ and second from the weak maximum principle that this inequality also holds in $\Omega_1 \cap B(w, 3r'/4)$. From our earlier work we now conclude Theorem 1. $\Box$

4 Deformation of p harmonic functions and proof of Theorem 2

In this section we develop a new iterative deformation technique for p harmonic functions in Lipschitz domains which we then use to prove Theorem 2 by induction. Throughout this section we assume that $\Omega \subset \mathbf{R}^n$ is a bounded Lipschitz domain with constant M, $w \in \partial\Omega$ and that p, $1 < p < \infty$, is given. In the following we will assume that u, v are positive p harmonic functions in $\Omega \cap B(w, 2\hat{r})$, for some $0 < \hat{r} < r_0$, which are continuous in $\bar{\Omega} \cap \bar{B}(w, 2\hat{r})$ and vanish on $\Delta(w, 2\hat{r})$. We will also assume, for technical reasons, that

$$u \leq v/2 \leq \hat{c}_1 u \text{ in } \Omega \cap \bar{B}(w, 2\hat{r}) \tag{4.1}$$

for a constant $\hat{c}_1$. Note that if (4.1) is not fulfilled then we can multiply v by a positive constant to get this inequality, using Theorem 1, with $\hat{r}$ replaced by $\hat{r}/c_1$. To define the deformations we let $0 < r' \leq \hat{r}$. We define $u(\cdot, \tau), 0 \leq \tau \leq 1$, to be p harmonic in $\Omega \cap B(w, 2r')$ and such that, for $0 \leq \tau \leq 1$,

$$u(x, \tau) = \tau v(x) + (1 - \tau)u(x), \text{ for } x \in \partial[\Omega \cap B(w, 2r')]. \tag{4.2}$$

4.1 Deformation of positive p harmonic functions vanishing on the boundary

In this subsection we prove the following two lemmas.

Lemma 4.3. *Assume that $\Omega \subset \mathbf{R}^n$ is a bounded Lipschitz domain, $w \in \partial\Omega$ and that p, $1 < p < \infty$, is given. Let u and v be positive p harmonic functions in $\Omega \cap B(w, 2\hat{r})$, continuous in $\bar{\Omega} \cap \bar{B}(w, 2\hat{r})$ with $u = 0 = v$ on $\Delta(w, 2\hat{r})$. Let $0 < r' \leq \hat{r}$ and assume that $u(\cdot, \tau), 0 \leq \tau \leq 1$, is the p harmonic function in $\Omega \cap B(w, 2r')$ with boundary values as in (4.2). Also suppose that u satisfies (1.9), (1.10) in $\Omega \cap B(w, 2\hat{r})$ for some δ, δ', ξ. Then there exists $\epsilon_0 \in (0, 1/4)$ such that if*

$$(1 - \epsilon_0)\hat{L} \leq \frac{u(\cdot, \hat{\tau})}{u(\cdot)} \leq (1 + \epsilon_0)\hat{L}$$

in $\Omega \cap B(w,s)$ *for some* $\hat\tau \in [0,1]$, $\hat L, 0 < \hat L < \infty$, *and* s, $0 < s \leq r'$, *then (1.9) and (1.10) also hold for* $u(\cdot,\hat\tau)$ *in* $\Omega \cap B(w,s/4)$. *In particular,*

$$\hat\delta^{-1}\frac{u(x,\hat\tau)}{d(x,\partial\Omega)} \;\leq\; \langle \nabla u(x,\hat\tau), \xi\rangle \leq |\nabla u(x,\hat\tau)| \leq \hat\delta \frac{u(x,\hat\tau)}{d(x,\partial\Omega)}$$

whenever $x \in \Omega \cap B(w,s/4)$ *and* $\hat\delta > 1$ *depends only on* p,n,M,δ,δ'.

Lemma 4.4. *Let* $\Omega \subset \mathbf{R}^n, p, w, \hat r, r', u, v,$ *and* $u(\cdot,\tau)$ *be as in the statement of Lemma 4.3. Assume also that* u, v *satisfy (4.1) and that* Ω, ϕ, *are as in Lemma 2.5 with* r *replaced by* $\hat r$. *Let* $\xi', \hat\xi \in [0,1]$, $\xi' < \hat\xi$ *and let* s *be such that* $0 < s \leq r'$. *Finally assume that the last display in Lemma 4.3 holds whenever* $x \in \Omega \cap B(w,s/4)$ *and* $\hat\tau \in [\xi',\hat\xi]$. *Then there exist constants* $\tilde c_2, \hat c_2, \alpha$, *depending only on* p,n, M *and the* $\hat\delta$ *in Lemma 4.3, such that if* $x,y \in \Omega \cap B(w,s/\tilde c_2)$ *then*

$$\left|\log\left(\frac{u(x,\hat\xi)}{u(x,\xi')}\right) - \log\left(\frac{u(y,\hat\xi)}{u(y,\xi')}\right)\right| \;\leq\; \hat c_2 \left(\frac{|x-y|}{s}\right)^{\alpha}.$$

Proof of Lemma 4.3. Let $x \in \Omega \cap B(w,s/4)$. Using Lemma 2.4 and the Harnack inequality we see that,

$$|\nabla u(z_1,\hat\tau) - \nabla u(z_2,\hat\tau)| \leq ct^\sigma \max_{B(x,td(x,\partial\Omega))} |\nabla u(\cdot,\hat\tau)| \;\leq\; c^2 t^\sigma\, u(x,\hat\tau)/d(x,\partial\Omega), \tag{4.5}$$

whenever $z_1, z_2 \in \bar B(x, td(x,\partial\Omega))$ and $0 \leq t \leq 1/2$. Here c depends only on p,n. Clearly (4.5) implies the upper bound in (1.9) with u' replaced by $u(\cdot,\hat\tau)$. Thus we only have to prove the bound from below for $\langle u(\cdot,\hat\tau),\xi\rangle$ in order to get (1.9). To prove this bound we argue by contradiction and we start by supposing that there exists a point $x \in \Omega \cap B(w,s/4)$ such that, for $\eta > 0$ to be chosen,

$$\langle \nabla u(x,\hat\tau),\xi\rangle \leq \eta\, u(x,\hat\tau)/d(x,\partial\Omega) \tag{4.6}$$

where ξ is the unit vector in (1.10) corresponding to u. From (4.5) with $z = z_1, x = z_2$ and (4.6) we deduce

$$\langle \nabla u(z,\hat\tau),\xi\rangle \;\leq\; [\eta + c^2 t^\sigma]\; u(x,\hat\tau)/d(x,\partial\Omega) \tag{4.7}$$

whenever $z \in B(x,td(x,\partial\Omega))$. Integrating, it follows that if $y = x + td(x,\partial\Omega)\xi$ and $t = \eta^{1/\sigma}$, then

$$u(y,\hat\tau) - u(x,\hat\tau) \leq c'\eta^{1+1/\sigma}\, u(x,\hat\tau). \tag{4.8}$$

In (4.8) c' depends only on p,n. We also deduce, using (1.9), (1.10) for u and the mean value theorem, that for z as above and c^* (depending only on p,n,M,δ), that the following is true with $y = x + td(x,\partial\Omega)\xi$

$$c^*(u(y) - u(x)) \geq \eta^{1/\sigma} u(x). \tag{4.9}$$

Using the point y we note that if $\epsilon_0, \hat L$ are as in Lemma 4.3, then from (4.8), (4.9) we find that

$$\begin{aligned}(1-\epsilon_0)\hat L \;&\leq\; \frac{u(y,\hat\tau)}{u(y)} \;\leq\; \left(\frac{1+c'\eta^{1+1/\sigma}}{1+\eta^{1/\sigma}/c^*}\right)\frac{u(x,\hat\tau)}{u(x)} \\ &\leq\; (1+\epsilon_0)\left(\frac{1+c'\eta^{1+1/\sigma}}{1+\eta^{1/\sigma}/c^*}\right)\hat L \;<\; (1-\epsilon_0)\hat L \end{aligned}\tag{4.10}$$

provided $1/\tilde{c} \geq \eta^{1/\sigma} \geq \tilde{c}\,\epsilon_0$ for some large $\tilde{c}$ depending only on δ, δ', p, n, M. With $\tilde{c}$ now fixed we put $\epsilon_0 = 1/\tilde{c}^2$ and we assume that the hypotheses of Lemma 4.3 hold for this ϵ_0. Then, in order to avoid the contradiction in (4.10), at y, it must be true that,

$$\langle \nabla u(x, \hat{\tau}), \xi \rangle \geq \frac{u(x, \hat{\tau})}{\tilde{c}^{\sigma}\, d(x, \partial\Omega)}$$

whenever $x \in \Omega \cap B(w, s/4)$. This completes the proof of Lemma 4.3. □

Proof of Lemma 4.4. Using Lemma 2.5 and (1.9) applied to $u(\cdot, \tau)$ we see that

$$u(\cdot, \tau) \text{ is } C^{\infty} \text{ in } \Omega \cap B(w, 3r'/2) \tag{4.11}$$

whenever $\tau \in [\xi', \hat{\xi}]$. From (4.1) we also observe that

$$c^{-1} u(\cdot, \tau_1) \leq \frac{u(\cdot, \tau_2) - u(\cdot, \tau_1)}{\tau_2 - \tau_1} = v - u \leq c\, u(\cdot, \tau_1) \tag{4.12}$$

on $\partial(\Omega \cap B(w, 2r'))$ whenever $\tau_1, \tau_2 \in [0, 1]$. From the boundary maximum principle for p harmonic functions this inequality also holds in $\Omega \cap B(w, 2r')$. Using (4.1) and (4.12), for fixed $x \in \Omega \cap B(w, s/4)$, we have that $\tau \rightarrow u(x, \tau), \tau \in [0, 1]$, is Lipschitz with norm $\leq cu(x)$. Thus $u_\tau(x, \cdot)$ exists almost everywhere in [0,1]. Let (x_ν) be a dense sequence of $B(w, s/4)$ and let W be the set of all $\tau \in [0, 1]$ for which $u_\tau(x_m, \cdot)$ exists, in the sense of difference quotients, whenever $x_m \in (x_\nu)$. Then $H^1([0, 1] \setminus W) = 0$ where H^1 is linear Lebesgue or Hausdorff one measure. To continue we let, for $\tau, t \in [\xi', \hat{\xi}]$,

$$U(x) \;=\; U(x, \tau, t) = \frac{u(x, t) - u(x, \tau)}{t - \tau}$$

and define, whenever $x \in \Omega \cap B(w, s/4)$,

$$\begin{aligned} A_{ij}(x) &= A_{ij}(x, \tau, t) = \int_0^1 a_{ij}[\lambda \nabla u(x, t) + (1 - \lambda) \nabla u(x, \tau)]\, d\lambda, 1 \leq i, j \leq n, \\ a_{ij}(\eta) &= |\eta|^{p-4}\, [(p - 2)\eta_i\, \eta_j \,+\, \delta_{ij}\, |\eta|^2] \text{ for } \eta \in \mathbf{R}^n \setminus \{0\}. \end{aligned} \tag{4.13}$$

We note that

$$\begin{aligned} &(i) \qquad c^{-1}|\xi|^2 \Big(|\nabla u(x, t)| + |\nabla u(x, \tau)| \Big)^{p-2} \leq \sum_{i,j=1}^{n} A_{ij}(x) \xi_i \xi_j \text{ for } \xi \in \mathbf{R}^n \setminus \{0\}, \\ &(ii) \qquad \sum_{i,j=1}^{n} |A_{ij}(x)| \leq c \Big(|\nabla u(x, t)| + |\nabla u(x, \tau)| \Big)^{p-2}. \end{aligned} \tag{4.14}$$

In (4.14) the constant c depends only on p, n. Using Lemma 2.5, the assumption that $|\nabla u(\cdot, \hat{\tau})|$ satisfies the uniform non-degeneracy condition (1.9) in $\Omega \cap B(w, s/4)$, with constant $\hat{\delta}$ independent of $\tau \in [\xi', \hat{\xi}]$, and (4.11) we see that U is a solution, whenever $x \in \Omega \cap B(w, s/8)$, to the following locally uniformly elliptic PDE with C^{∞} coefficients

$$\tilde{L} U(x) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} [A_{ij}(x)\, U_{x_j}] = 0. \tag{4.15}$$

From (4.11)-(4.15) we see that U is a bounded, with bound independent of $\tau, t \in [\xi', \hat{\xi}]$, C^∞ solution to $\tilde{L}$ in $\Omega \cap B(w, s/8)$. Moreover, $U \equiv 0$ on $\Delta(w, s/4)$. Let $U \equiv 0$ on $B(w, s/4) \setminus \Omega$. Using (4.12) with τ_1, τ_2 replaced by τ, t, Lemma 2.4 for $u(\cdot, \tau)$, and standard Hölder continuity estimates in $\Omega \cap B(w, s/8)$ we see that U is Hölder continuous in $B(w, s/8)$ with constant and exponent independent of $\tau, t \in [\xi', \hat{\xi}]$. Since $(u(x,t) - u(x,\tau)) = (t - \tau)(v(x) - u(x))$ on $\partial(\Omega \cap B(w, 2r'))$, it follows, from the maximum principle for p harmonic functions, that $u(x,t) \rightarrow u(x,\tau)$ uniformly in the closure of $\Omega \cap B(w, s/4)$ as $t \rightarrow \tau$. Using these facts and Schauder type estimates for the operator $\tilde{L}$ it follows that there exists a subsequence of $\{U(\cdot, \tau, t)\}$, $\{U(\cdot, \tau, t_k)\}$, with $t_k \rightarrow \tau$ as $k \rightarrow \infty$, which converges uniformly, in $B(w, s/8)$, to a function $f = f(\cdot, \tau)$ with $f \in C^\infty(\Omega \cap B(w, s/8))$. Furthermore, f is a solution to the equation

$$L\zeta = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} [\, b_{ij}(x) \zeta_{x_j}(x) \,] = 0, \tag{4.16}$$

in $\Omega \cap B(w, s/8)$ where, at $x \in \Omega \cap B(w, s/8)$ and for $1 \leq i, j \leq n$,

$$b_{ij}(x) = |\nabla u(x,\tau)|^{p-4} [(p-2) u_{x_i}(x,\tau) u_{x_j}(x,\tau) + \delta_{ij} |\nabla u|^2(x,\tau)]. \tag{4.17}$$

δ_{ij} is the Kronecker δ. Moreover, we note that

$$\begin{array}{ll} (i) & f \text{ is a solution to (4.16)-(4.17) in } \Omega \cap B(w, s/8), \\ (ii) & f \text{ is continuous in } B(w, s/8) \text{ with } f \equiv 0 \text{ on } B(w, s/8) \setminus \Omega, \\ (iii) & f(x_m, \tau) = u_\tau(x_m, \tau) \text{ whenever } x_m \in \Omega \cap B(w, s/8), \tau \in W, \\ (iv) & c^{-1} \leq f(\cdot, \tau)/u(\cdot, \tau) \leq c \text{ on } \Omega \cap B(w, s/8). \end{array} \tag{4.18}$$

For $x_m \in (x_\nu)$, $x_m \in \Omega \cap B(w, s/8)$ we see, using (4.18) and the fact that $H^1([0,1] \setminus W) = 0$,

$$\log\left(\frac{u(x_m, \hat{\xi})}{u(x_m, \xi')}\right) = \int_{\xi'}^{\hat{\xi}} \frac{f(x_m, \tau)}{u(x_m, \tau)} \, d\tau. \tag{4.19}$$

We now observe that it follows, from the assumption that $|\nabla u(\cdot, \hat{\tau})|$ satisfies (1.9), (1.10) in $\Omega \cap B(w, s/4)$ with constants independent of $\tau \in [\xi', \hat{\xi}]$ and from (4.18), that Lemma 2.8 applies to $h_1(\cdot) = f(\cdot, \tau)$, $h_2(\cdot) = u(\cdot, \tau)$. Furthermore, all constants are independent of τ. Therefore, using Lemma 2.8 and (4.18)(iii) we conclude that there exist $\tilde{c}_2, \hat{c}_2, \alpha$, depending only on $p, n, M, \hat{\delta}$ and $\hat{\delta}'$, such that

$$\left| \log\left(\frac{u(x_m, \hat{\xi})}{u(x_m, \xi')}\right) - \log\left(\frac{u(x_k, \hat{\xi})}{u(x_k, \xi')}\right) \right| \leq \int_{\xi'}^{\hat{\xi}} \left| \frac{u_\tau(x_m, \tau)}{u(x_m, \tau)} - \frac{u_\tau(x_k, \tau)}{u(x_k, \tau)} \right| d\tau \, \leq \, \hat{c}_2 \left(\frac{|x_m - x_k|}{s} \right)^\alpha \tag{4.20}$$

whenever $x_m, x_k \in (x_\nu)$ and $x_m, x_k \in \Omega \cap B(w, s/\tilde{c}_2)$. As (x_ν) is a dense sequence in $\Omega \cap B(w, s/\tilde{c}_2)$ it follows, from (4.20) and continuity, that the conclusion of Lemma 4.4 is valid whenever $x, y \in B(w, s/c_2)$. $\Box$

4.2 Proof of Theorem 2

In this section we prove Theorem 2 by an induction argument making iterative use of Lemma 4.3 and Lemma 4.4 and in the following we will use the notation of Theorem 2. To start with we let $w' = w + \frac{r}{4}e_n$ and observe, as in the proof of Theorem 1, that if c' is large enough, depending on the Lipschitz constant M, then the domain $\Omega' \subset \Omega$ obtained from drawing all open line segments from points in $\Delta(w, r/c')$ to points in $B(w', r/c')$ is starlike Lipschitz with center w' and Lipschitz constant bounded by $c(M)$. Let $\tilde{r} = \frac{r}{4c'}$ and let u' be the p capacitary function for $D' = \Omega' \setminus \bar{B}(w', \tilde{r}/4)$. Then D' is a starlike Lipschitz ring domain and we note that we can also assume, by choosing $c_1 = c_1(M) > c'$ large enough, that

$$\Delta(w, r/c_1) = \partial D' \cap B(w, r/c_1). \tag{4.21}$$

Furthermore, from Lemma 2.6 and the remark after this lemma we see, for c_1 large enough, that

$$(1.9), (1.10) \text{ hold for } u' \text{ in } D' \cap B(w, r/c_1) \text{ with } \delta, \delta' \text{ depending only on } p, n, M. \tag{4.22}$$

To start the proof of Theorem 2 we note, by the arbitrariness of v in the statement of Theorem 2, that it follows from the triangle inequality that it suffices to prove Theorem 2 with u replaced by u', and r replaced by r/c_1 for c_1 large enough. Thus in the following we write u for u' and assume that (4.22) holds for u. As argued above we can also assume that (4.1) holds with $\hat{c}_1$ having the same dependence as c_1. Based on these arguments we can conclude that Lemma 4.3 and Lemma 4.4 apply to the pair u, v with $\hat{r} = r/(2c_1)$ and $r' = r/(4c_1^2)$.

Proof of Theorem 2. In the following we will first prove Theorem 2 assuming that Ω, ϕ are as in Lemma 2.5 and we then, in the final argument, remove the smoothness assumption on ϕ.

To start with we observe, as in the beginning of the proof of Lemma 4.4, that it follows from (4.1) and the boundary maximum principle for p harmonic functions that (4.12) is valid in $\Omega \cap B(w, 2r')$ for $\tau_1, \tau_2 \in [0, 1]$. Thus for ϵ_0 as in Lemma 4.3 there exists ϵ_0', $0 < \epsilon_0' \leq \epsilon_0$, with the same dependence as ϵ_0, such that if $|\tau_2 - \tau_1| \leq \epsilon_0'$, then

$$1 - \epsilon_0/2 \leq \frac{u(\cdot, \tau_2)}{u(\cdot, \tau_1)} \leq 1 + \epsilon_0/2 \text{ in } \Omega \cap B(w, 2r'). \tag{4.23}$$

Let $\xi_1 = 0 < \xi_2 < ... < \xi_m = 1$ and consider [0,1] as divided into $\{[\xi_k, \xi_{k+1}]\}, 1 \leq k \leq m-1$. We assume that all of these intervals have a length of $\epsilon_0'/2$ with the possible exception of the interval containing $\xi_m = 1$ which is of length $\leq \epsilon_0'/2$. We let $r_1 = r'$ and we note that it follows, from our choice of ϵ_0' and the fact that $u(\cdot, \xi_1) = u$, that the hypotheses of Lemma 4.3 are satisfied whenever $\hat{\tau} \in [\xi_1, \xi_2]$ with $\hat{L} = 1$ and $s = r_1$. Thus, combining Lemma 4.3 and Lemma 4.4, we see that,

$$\left|\log\left(\frac{u(x,\xi_2)}{u(x)}\right) - \log\left(\frac{u(y,\xi_2)}{u(y)}\right)\right| = \left|\log\left(\frac{u(x,\xi_2)}{u(x,\xi_1)}\right) - \log\left(\frac{u(y,\xi_2)}{u(y,\xi_1)}\right)\right| \leq \hat{c}_2\left(\frac{|x-y|}{r_1}\right)^{\alpha} \tag{4.24}$$

whenever $x, y \in \Omega \cap B(w, r_2)$ and where we have introduced $r_2 = r_1/\tilde{c}_2$. We can now continue by induction and to do this we assume that we have shown, for some $2 \leq k \leq m$, that

$$\left|\log\left(\frac{u(x,\xi_k)}{u(x)}\right) - \log\left(\frac{u(y,\xi_k)}{u(y)}\right)\right| \leq (k-1)\hat{c}_2\left(\frac{|x-y|}{r_{k-1}}\right)^{\alpha} \tag{4.25}$$

whenever $x, y \in \Omega \cap B(w, r_k)$. Here $\hat{c}_2, \alpha$ are as in Lemma 4.4, and $r_k \le r_{k-1} \le \hat{c} r_k$ for some $\hat{c} = \hat{c}(p, n, M) > 1$. If $k = m$ we quit but assuming $k < m$, we choose, using the induction hypothesis in (4.25), σ and $r'_k \le r_k$ so that

$$\left| \frac{u(x, \xi_k)}{u(x)} - \frac{u(y, \xi_k)}{u(y)} \right| \le \sigma \frac{u(x, \xi_k)}{u(x)}$$

whenever $x, y \in \Omega \cap B(w, r'_k)$. We fix a $x \in \Omega \cap B(w, r'_k)$ and choose $\sigma > 0$ small enough to ensure that if $y \in \Omega \cap B(w, r'_k)$ and if $\tau \in [\xi_k, \xi_{k+1}]$, then

$$(1 - \epsilon_0) \frac{u(x, \xi_k)}{u(x)} \le \frac{u(y, \tau)}{u(y)} \le (1 + \epsilon_0) \frac{u(x, \xi_k)}{u(x)}. \tag{4.26}$$

To estimate the magnitude of σ we observe that if $\tau \in [\xi_k, \xi_{k+1}]$ then

$$\frac{u(y, \tau)}{u(y)} = \frac{u(y, \tau)}{u(y, \xi_k)} \cdot \frac{u(y, \xi_k)}{u(y)} \le (1 + \epsilon_0/2)(1 + \sigma) \frac{u(x, \xi_k)}{u(x)}.$$

Thus if $\sigma = \epsilon_0/4$ and if ϵ_0 is small enough then the right hand inequality in (4.26) is valid. A similar argument gives the left hand inequality in (4.26) when $\sigma = \epsilon_0/4$ and ϵ_0 is small enough. Also since α is independent of $k, k \le 2/\epsilon'_0$, and $\epsilon'_0 = \epsilon'_0(p, n, M)$, we deduce from (4.25) that one can take $r'_k = r_k/\tilde{c}$ for $\tilde{c} = \tilde{c}(p, n, M)$ large enough. From (4.26) we find that we can apply Lemma 4.3 with $\hat{L} = \frac{u(x, \xi_k)}{u(x)}$, $s = r'_k$, and $\hat{\tau}$ replaced by an arbitrary $\tau \in [\xi_k, \xi_{k+1}]$. Hence, first applying Lemma 4.3 we see that the assumptions of Lemma 4.4 are fulfilled and secondly we can conclude that the conclusion of Lemma 4.4 is valid for all $\tau \in [\xi_k, \xi_{k+1}]$ with $s = r'_k$. As

$$\begin{aligned} \left| \log\left(\frac{u(x, \xi_{k+1})}{u(x)} \right) - \log\left(\frac{u(y, \xi_{k+1})}{u(y)} \right) \right| &\le \left| \log\left(\frac{u(x, \xi_{k+1})}{u(x, \xi_k)} \right) - \log\left(\frac{u(y, \xi_{k+1})}{u(y, \xi_k)} \right) \right| \\ &+ \left| \log\left(\frac{u(x, \xi_k)}{u(x)} \right) - \log\left(\frac{u(y, \xi_k)}{u(y)} \right) \right| \end{aligned}$$

we can, defining $r_{k+1} = r'_k/\tilde{c}_2 = r_k/(\tilde{c}_2 \tilde{c})$, make use of the induction hypothesis to conclude that if $x, y \in \Omega \cap B(w, r_{k+1})$, then

$$\begin{aligned} \left| \log\left(\frac{u(x, \xi_{k+1})}{u(x)} \right) - \log\left(\frac{u(y, \xi_{k+1})}{u(y)} \right) \right| &\le \hat{c}_2 \left(\frac{|x - y|}{r_k} \right)^\alpha + (k-1) \hat{c}_2 \left(\frac{|x - y|}{r_{k-1}} \right)^\alpha \\ &\le k \hat{c}_2 \left(\frac{|x - y|}{r_k} \right)^\alpha. \end{aligned} \tag{4.27}$$

From (4.27) and induction we conclude that (4.25) is valid for all $k \in \{2, .., m\}$. In particular, applying (4.25) with $k = m$ we see that Theorem 2 is valid.

Finally we remove the assumption that ϕ is infinitely differentiable. To do this we put $u \equiv 0 \equiv v$ in $B(w, 2r) \setminus \Omega$ and note that both functions are now continuous in $\bar{B}(w, 2r)$. Let Ω_ϵ be a bounded Lipschitz domain defined as in Theorem 2 with ϕ replaced by $\phi_\epsilon \in C^\infty(\mathbf{R}^{n-1})$ where $\phi - \epsilon < \phi_\epsilon < \phi$ in $\{x' \in \mathbf{R}^{n-1} : |x' - w'| < 2r\}$ and $|||\nabla \phi_\epsilon|||_\infty \le |||\nabla \phi|||_\infty$. ϕ_ϵ can be constructed by convoluting $\phi - \epsilon/2$ with a suitable approximation of the identity. Let u_ϵ, v_ϵ, be the positive p harmonic functions in $\Omega_\epsilon \cap B(w, 2r)$ which satisfy $u_\epsilon = u, v_\epsilon = v$ on $\partial[B(w, 2r) \cap \Omega_\epsilon]$. From our choice of ϕ_ϵ we have $u_\epsilon = 0 = v_\epsilon$ on $B(w, 2r) \cap \partial\Omega_\epsilon$. From basic estimates, using

Lemma 2.1, it follows that u_ϵ, v_ϵ converge to u, v, uniformly on compact subsets of $\Omega \cap B(w, 2r)$, as $\epsilon \to 0$. Since c_2 in Theorem 2 depends only on p, n and the Lipschitz norm of $\nabla \phi_\epsilon$, it follows that we can prove Theorem 2 for u_ϵ, v_ϵ, and then take limits to get Theorem 2 for u, v. The proof of Theorem 2 is therefore complete. $\Box$

For use in the proof of Theorems 3 and 4 we note the following consequence of the argument outlined above.

Lemma 4.28. *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain. Given $p, 1 < p < \infty, w \in \partial\Omega$, $0 < r < r_0$, suppose that u is a positive p harmonic function in $\Omega \cap B(w, 2r)$. Assume also that u is continuous in $\bar{\Omega} \cap \bar{B}(w, 2r)$ and $u = 0$ on $\Delta(w, 2r)$. Then there exist $\xi \in \partial B(0, 1)$ and $c_3, \delta_+ > 1$, both which only depend on p, n and M, such that*

$$\delta_+^{-1} \frac{u(x)}{d(x, \partial\Omega)} \leq \langle \nabla u(x), \xi \rangle \leq |\nabla u(x)| \leq \delta_+ \frac{u(x)}{d(x, \partial\Omega)}$$

whenever $x \in \Omega \cap B(w, r/c_3)$. Moreover, ξ can be chosen independently of u.

5 Deformation of Martin functions and proof of Theorem 3

In this section we develop an iterative deformation technique for minimal positive p harmonic functions in Lipschitz domains which eventually will be used to prove Theorem 3. The technique is similar to the one developed in the previous section. Throughout this section we assume that $\Omega \subset \mathbf{R}^n$ is a bounded Lipschitz domain with constant M, $w \in \partial\Omega$, and that p, $1 < p < \infty$, is given. We let u and v be minimal positive p harmonic functions relative to $w \in \partial\Omega$ and we note, using Lemma 2.2, that u, v have Hölder continuous extensions to $\mathbf{R}^n \setminus \{w\}$ defined by putting $u = 0 = v$ on $\mathbf{R}^n \setminus \Omega$. Using Theorem 1, Harnack's inequality and the maximum principle for p harmonic functions we can also conclude that

$$a < \liminf_{x \to w} u(x)/v(x) \leq \limsup_{x \to w} u(x)/v(x) < ca$$

for some $a > 0$ and for a constant c depending only on p, n and M. Hence if we redefine $u = u/a$, then we can assume that u/v is bounded from below and above in $\Omega \setminus \{w\}$ by c^{-1} and c respectively. In analogy with (4.1) we can therefore assume, for technical reasons, that

$$u \leq v/2 \leq \hat{c}_1 u \tag{5.1}$$

in $\Omega \setminus \{w\}$ for a constant $\hat{c}_1$. To define the deformations we let $0 < r' << \hat{r}$. We define $u(\cdot, \tau), 0 \leq \tau \leq 1$, to be p harmonic in $\Omega \cap [B(w, 2\hat{r}) \setminus B(w, r'/2)$ and such that, for $0 \leq \tau \leq 1$,

$$u(\cdot, \tau) = \tau v(\cdot) + (1 - \tau) u(\cdot) \text{ on } \partial[\Omega \cap B(w, 2\hat{r}) \setminus B(w, r'/2)]. \tag{5.2}$$

5.1 Deformation of minimal positive p harmonic functions

We here state versions of Lemma 4.3 and Lemma 4.4, namely Lemma 5.4 and Lemma 5.5 stated below, adapted to the new set of deformations defined through (5.2). The proof of

Theorem 3 will be based on these lemmas. Recall from section 1 that $\tilde\Omega(w,\tilde\eta) \subset \Omega$ is said to be a non-tangential approach region, at $w \in \partial\Omega$, if $d(x,\partial\Omega) > \tilde\eta|x-w|$ for all $x \in \tilde\Omega(w,\tilde\eta)$. From Lemma 4.28 we see that if $0 < r' << \hat r$ and if $\tilde u$ is a positive p harmonic function in $\Omega \cap [B(w,2\hat r) \setminus B(w,r'/2)]$ with continuous boundary values 0 on $\partial\Omega \cap [B(w,2\hat r) \setminus B(w,r'/2)]$, then there exist $\tilde\eta > 0$ and $c > 1$, depending only on p, n, M, such that

$$
\begin{array}{ll}
(a) & \tilde u \text{ satisfies (1.9) with } u' \text{ replaced by } \tilde u \text{ in } [\Omega \setminus \tilde\Omega(w,\tilde\eta)] \cap [B(w,\hat r) \setminus B(w,r')] \\
& \text{with a constant } \delta = \delta(p,n,M), \\
(b) & \text{if } y \in [\Omega \setminus \tilde\Omega(w,\tilde\eta)] \cap [B(w,\hat r) \setminus B(w,r')], \text{ there exists } x \in \partial\Omega \text{ with } y \in B(x,|x-w|/c) \\
& \text{and } \xi = \xi(x), \delta' = \delta'(p,n,M), \text{ for which (1.10) holds on } \Omega \cap B(x,2|x-w|/c).
\end{array} \tag{5.3}
$$

Lemma 5.4. *Assume that $\Omega \subset \mathbf{R}^n$ is a bounded Lipschitz domain, $w \in \partial\Omega$ and that p, $1 < p < \infty$, is given. Let $r' << \hat r$ and suppose $\hat u, \hat v$ are positive p harmonic functions in $\Omega \cap [B(w,2\hat r) \setminus B(w,r'/2)]$, continuous on the closure of $\Omega \cap [B(w,2\hat r) \setminus B(w,r'/2)]$ with $\hat u = 0 = \hat v$ on $\partial\Omega \cap [B(w,2\hat r) \setminus B(w,r'/2)]$. Let $\hat u(\cdot,\tau), 0 \le \tau \le 1$, be the p harmonic function in $\Omega \cap [B(w,2\hat r) \setminus B(w,r'/2)]$ with boundary values $\tau\hat v + (1-\tau)\hat u$. Let $b > 2$ be given and suppose also that $\hat u$ satisfies (1.9) for some $\tilde\delta > 0$ in $\tilde\Omega(w,\tilde\eta/2) \cap [B(w,bs) \setminus B(w,s/b)]$, where $2r' < s/b < bs < \min(r_0/2, \hat r/2)$ and $\tilde\eta$ is as in (5.3). Then there exists $\epsilon_0 \in (0,1/4)$, depending only on p, n, M such that if*

$$(1-\epsilon_0)\hat L \le \frac{\hat u(\cdot,\hat\tau)}{\hat u(\cdot)} \le (1+\epsilon_0)\hat L$$

in $\Omega \cap [B(w,bs) \setminus B(w,s/b)]$ for some $\hat\tau \in [0,1]$, $\hat L, 0 < \hat L < \infty$, then

$$\hat\delta^{-1}\frac{\hat u(x,\hat\tau)}{d(x,\partial\Omega)} \le |\nabla \hat u(x,\hat\tau)| \le \hat\delta\frac{\hat u(x,\hat\tau)}{d(x,\partial\Omega)}$$

for all $x \in \Omega \cap [B(w,b^{1/2}s) \setminus B(w,s/b^{1/2})]$ and some $\hat\delta = \hat\delta(p,n,M)$.

Lemma 5.5. *Let $\Omega \subset \mathbf{R}^n$, p, w, b, $\hat r$, r', s, $\hat u$, $\hat v$, and $\hat u(\cdot,\hat\tau)$ be as in Lemma 5.4. Let $l, i \in \mathbf{Z}_+$, $i \le l$, and let $s_k, 1 \le k \le l$, be a decreasing sequence of positive numbers such that $2br' < s_k < s_{k-1}/b < \frac{1}{2b^2}\min(r_0,\hat r)$ for $2 \le k \le i$. Suppose also that $\Omega \subset B(w,\hat r)$ and that (5.1) holds with u, v replaced by $\hat u, \hat v$. Put*

$$
\begin{aligned}
\tilde M(s,\xi',\hat\xi) &= \sup_{\Omega\setminus B(w,s)} \log\left(\frac{\hat u(y,\hat\xi)}{\hat u(y,\xi')}\right), \\
\tilde m(s,\xi',\hat\xi) &= \inf_{\Omega\setminus B(w,s)} \log\left(\frac{\hat u(y,\hat\xi)}{\hat u(y,\xi')}\right).
\end{aligned}
$$

Let $\xi', \hat\xi \in [0,1]$, $\xi' < \hat\xi$, and assume that the conclusion of Lemma 5.4 holds for $\hat u(\cdot,\hat\tau)$ on $\Omega \cap [B(w,b^{1/2}s) \setminus B(w,s/b^{1/2})]$ whenever $\hat\tau \in [\xi',\hat\xi]$ and $s \in \{s_k\}_1^i$, with constant $\hat\delta$ independent of $\hat\tau$. If $b = b(p,n,M,\hat\delta)$ is large enough, then there exists $\theta = \theta(p,n,M,\hat\delta)$, $\theta \in (0,1)$, such that

$$\tilde M(s_1,\xi',\hat\xi) - \tilde m(s_1,\xi',\hat\xi) \le c\theta^i.$$

Moreover, the constant c depends only on the constant in (5.1).

Proof of Lemma 5.4. The proof of Lemma 5.4 is similar to the proof of Lemma 4.3. In this case though, we see from (5.3) that Lemma 5.4 holds when $x \in [\Omega \setminus \tilde{\Omega}(w, \tilde{\eta})] \cap [B(w, b^{1/2}s) \setminus B(w, s/b^{1/2})]$ with constant $\delta = \delta(p, n, M)$. Hence it only remains to prove the conclusion of the lemma for points in $\tilde{\Omega}(w, \tilde{\eta}) \cap [B(w, b^{1/2}s) \setminus B(w, s/b^{1/2})]$. We therefore let $x \in \tilde{\Omega}(w, \tilde{\eta}) \cap [B(w, b^{1/2}s) \setminus B(w, s/b^{1/2})]$ and note that there exists $\gamma = \gamma(n, M, b) > 0$ such that $B(x, \gamma d(x, \partial\Omega)) \cap B(w, s/b) = \emptyset$ and such that $B(x, \gamma d(x, \partial\Omega)) \subset \tilde{\Omega}(w, \tilde{\eta}/2) \cap B(w, bs)$. Using Lemma 2.4 and the Harnack inequality we see that,

$$|\nabla \hat{u}(z_1, \hat{\tau}) - \nabla \hat{u}(z_2, \hat{\tau})| \leq ct^\sigma \max_{B(x, t\gamma d(x,\partial\Omega))} |\nabla \hat{u}(\cdot, \hat{\tau})| \leq c^2 t^\sigma \, \hat{u}(x, \hat{\tau})/d(x, \partial\Omega) \tag{5.6}$$

whenever $z_1, z_2 \in \bar{B}(x, \gamma t d(x, \partial\Omega))$ and $0 \leq t \leq 1/2$. Here c depends only on p, n and b. Based on (5.6) we see, as in the proof of Lemma 4.3, that we only have to prove bounds from below for the gradient of $\hat{u}(\cdot, \hat{\tau})$. To do this we suppose that there exists a point $x \in \tilde{\Omega}(w, \tilde{\eta}) \cap [B(w, b^{1/2}s) \setminus B(w, s/b^{1/2})]$ such that, for $\zeta > 0$ to be chosen,

$$|\nabla \hat{u}(x, \hat{\tau})| \leq \zeta \, \hat{u}(x, \hat{\tau})/d(x, \partial\Omega). \tag{5.7}$$

From (5.6) with $z = z_1, x = z_2$ and (5.7) we deduce

$$|\nabla \hat{u}(z, \hat{\tau})| \leq [\zeta + c^2 t^\sigma] \, \hat{u}(x, \hat{\tau})/d(x, \partial\Omega) \tag{5.8}$$

for $z \in B(x, t\gamma d(x, \partial\Omega))$. Integrating, it follows that if $y \in \partial B(x, \gamma t d(x, \partial\Omega))$, with $|x - y| = \gamma t d(x, \partial\Omega)$, $t = \zeta^{1/\sigma}$, then

$$|\hat{u}(y, \hat{\tau}) - \hat{u}(x, \hat{\tau})| \leq c' \zeta^{1+1/\sigma} \, \hat{u}(x, \hat{\tau}). \tag{5.9}$$

Constants in (5.8),(5.9) depend only on p, n and b. On the other hand (5.6) also holds with $\hat{u}(\cdot, \hat{\tau})$ replaced by $\hat{u}$. Let $\lambda = \frac{\nabla \hat{u}(x)}{|\nabla \hat{u}(x)|}$. Then we see, using (5.6), (5.3) and the uniform non-degeneracy assumption on $|\nabla \hat{u}|$, that

$$\langle \nabla \hat{u}(z), \lambda \rangle \geq (1 - c\,\zeta) |\nabla \hat{u}(x)| \text{ in } B(x, \gamma \zeta^{1/\sigma}).$$

Integrating we get, for $y = x + \zeta^{1/\sigma} \gamma d(x, \partial\Omega) \lambda$, that

$$c^* (\hat{u}(y) - \hat{u}(x)) \geq \zeta^{1/\sigma} \hat{u}(x) \tag{5.10}$$

with a constant c^* depending only on $p, n, M, \tilde{\delta}$ and b. Using this value of y in (5.9) we can now repeat the argument following (4.9) to get

$$(1 - \epsilon_0) \hat{L} \;\; \leq \;\; \frac{\hat{u}(y, \hat{\tau})}{\hat{u}(y)} < \; (1 - \epsilon_0) \hat{L}$$

provided $1/\tilde{c} \geq \zeta^{1/\sigma} \geq \tilde{c}\, \epsilon_0$ for some large $\tilde{c}$. With $\tilde{c}$ now fixed we put $\epsilon_0 = 1/\tilde{c}^2$ and assume that the hypotheses of Lemma 5.4 hold for this ϵ_0. Then, in order to avoid the contradiction in the above display, at y, it must be true, for all $x \in \tilde{\Omega}(w, \tilde{\eta}) \cap [B(w, b^{1/2}s) \setminus B(w, s/b^{1/2})]$, that

$$|\nabla \hat{u}(x, \hat{\tau})| \geq \frac{\hat{u}(x, \hat{\tau})}{\tilde{c}^\sigma \, d(x, \partial\Omega)}.$$

Lemma 5.4 is therefore proved. □

Proof of Lemma 5.5. The proof of Lemma 5.5 follows the same lines as the proof of Lemma 4.4 and in this case we start by assuming that

$$\partial\Omega \cap B(w, 2bs_1) \text{ is the graph of a } C^\infty \text{ function } \phi. \tag{5.11}$$

From the hypotheses on $\hat{u}(\cdot, \hat{\tau})$ and Lemma 2.5 we see, for $\tau \in [\xi', \hat{\xi}]$, that

$$\hat{u}(\cdot, \tau) \text{ is } C^\infty \text{ in } \Omega \cap [B(w, b^{1/2}s) \setminus B(w, s/b^{1/2})] \text{ whenever } s \in \{s_k\}_1^i. \tag{5.12}$$

Note that the assumption $\Omega \subset B(w, \hat{r})$ implies $\Omega \cap [B(w, 2\hat{r}) \setminus B(w, r'/2)] = \Omega \setminus B(w, r'/2)$. Hence, using this assumption and (5.1) we observe, for $\tau_1, \tau_2 \in [0, 1]$, that

$$c^{-1}\hat{u}(\cdot, \tau_1) \leq \frac{\hat{u}(\cdot, \tau_2) - \hat{u}(\cdot, \tau_1)}{\tau_2 - \tau_1} = \hat{v} - \hat{u} \leq c\,\hat{u}(\cdot, \tau_1) \tag{5.13}$$

on $\Omega \cap \partial B(w, r'/2)$. Therefore, using the maximum principle for p harmonic functions it follows that this inequality also holds in $\Omega \setminus \bar{B}(w, r'/2)$. More generally, if λ is a real number, $t \geq r'/2$, and

$$g = \lambda\hat{u}(\cdot, \tau_2) \pm \frac{\hat{u}(\cdot, \tau_2) - \hat{u}(\cdot, \tau_1)}{\tau_2 - \tau_1} \geq 0 \text{ on } \Omega \cap \partial B(w, t), \text{ then } g \geq 0 \text{ on } \Omega \setminus \bar{B}(w, t), \tag{5.14}$$

as we see using the maximum principle for p harmonic functions and the assumption that $\Omega \subset B(w, \hat{r})$. We can now repeat the proof of Lemma 4.4 for all points $x \in \Omega \cap [B(w, b^{1/2}s) \setminus B(w, s/b^{1/2})]$ whenever $s \in \{s_k\}_1^i$. In particular, using (5.1) and (5.13), for fixed $x \in \Omega \setminus \bar{B}(w, r'/2)$, we have that $\tau \to \hat{u}(x, \tau), \tau \in [0, 1]$, is Lipschitz with norm $\leq c\hat{u}(x)$. Thus $\hat{u}_\tau(x, \cdot)$ exists almost everywhere in [0,1]. Let (x_ν) be a dense sequence of $\Omega \setminus \bar{B}(w, r'/2)$ and let again W be the set of all $\tau \in [0, 1]$ for which $\hat{u}_\tau(x_m, \cdot)$ exists, in the sense of difference quotients, whenever $x_m \in (x_\nu)$. Repeating the arguments in (4.13)-(4.19) we see that

$$\log\left(\frac{\hat{u}(x_m, \hat{\xi})}{\hat{u}(x_m, \xi')}\right) = \int_{\xi'}^{\hat{\xi}} \frac{f(x_m, \tau)}{\hat{u}(x_m, \tau)}\, d\tau$$

whenever $x_m \in (x_\nu)$, $x_m \in \Omega \setminus \bar{B}(w, r'/2)$, and where f has the following properties,

$$\begin{array}{rl} (i) & f \text{ is a solution to (4.16)-(4.17) in } \Omega \cap [B(w, b^{1/2}s) \setminus B(w, s/b^{1/2})], s \in \{s_k\}_1^i, \\ (ii) & f(x_m, \tau) \to 0 \text{ as } x_m \to \partial\Omega \setminus \bar{B}(w, r'/2), \tau \in W, \\ (iii) & f(x_m, \tau) = \hat{u}_\tau(x_m, \tau) \text{ whenever } x_m \in (x_\nu), \tau \in W, \\ (iv) & c^{-1} \leq f(x_m, \tau)/\hat{u}(x_m, \tau) \leq c \text{ whenever } x_m \in (x_\nu), \tau \in W, \\ (v) & \text{if } \hat{g} = \lambda\hat{u}(\cdot, \tau) \pm f(\cdot, \tau) \geq 0 \text{ on } \Omega \cap \partial B(w, b^{1/2}s), \\ & \text{then } \hat{g} \geq 0 \text{ on } \Omega \setminus \bar{B}(w, b^{1/2}s). \end{array} \tag{5.15}$$

Note that (v) (which does not appear in (4.18)) follows from (5.14) and $(i) - (iv)$. From (5.15) we deduce,

$$\log\left(\frac{\hat{u}(x_m, \hat{\xi})}{\hat{u}(x_m, \xi')}\right) - \log\left(\frac{\hat{u}(x_k, \hat{\xi})}{\hat{u}(x_k, \xi')}\right) = \int_{\xi'}^{\hat{\xi}} \left(\frac{f(x_m, \tau)}{\hat{u}(x_m, \tau)} - \frac{f(x_k, \tau)}{\hat{u}(x_k, \tau)}\right) d\tau. \tag{5.16}$$

To estimate the integrand in (5.16) we put

$$\mathrm{Osc}(t) = \sup_{\Omega\setminus\bar B(w,t)} \frac{f(\cdot,\tau)}{u(\cdot,\tau)} - \inf_{\Omega\setminus\bar B(w,t)} \frac{f(\cdot,\tau)}{u(\cdot,\tau)} \text{ whenever } s/b^{1/2} \leq t \leq b^{1/2}s \text{ and } s \in \{s_k\}_1^i.$$

We note from (5.15) (v) that $\mathrm{Osc}(\cdot)$ is non-increasing as a function of t on its domain. From (5.3), the conclusion of Lemma 5.4, and (5.15) we see that Lemma 2.12 can be applied with $\hat b = b^{1/4}, t = s_k$ and $h_1(\cdot) = f(\cdot,\tau)$, $h_2(\cdot) = \hat u(\cdot,\tau)$ whenever $\tau \in [\xi',\hat\xi]$. Doing this we see that there exists $\theta = \theta(p,n,M,\hat\delta)$, $\theta \in (0,1)$, such that

$$\mathrm{Osc}(s_{k-1}/b^{1/2}) \leq \ \mathrm{Osc}(b^{1/2}s_k) \leq \theta\ \mathrm{Osc}(s_k/b^{1/2})$$

for $2 \leq k \leq i$. Iterating this inequality for $k = 2,\ldots,i$, we deduce from (5.16) that

$$\left|\log\left(\frac{\hat u(x_m,\hat\xi)}{\hat u(x_m,\xi')}\right) - \log\left(\frac{\hat u(x_k,\hat\xi)}{\hat u(x_k,\xi')}\right)\right| \leq \int_{\xi'}^{\hat\xi} \left|\frac{f(x_m,\tau)}{\hat u(x_m,\tau)} - \frac{f(x_k,\tau)}{\hat u(x_k,\tau)}\right| d\tau \leq c\theta^i \tag{5.17}$$

whenever $x_k, x_m \in \Omega\cap\partial B(w,s_1)$. Since $u(\cdot,\hat\xi), u(\cdot,\xi')$ are positive and continuous in Ω, it first follows that (5.17) holds for $x_k, x_m \in \Omega\cap\partial B(w,s_1)$ and thereupon, from the maximum principle for p harmonic functions, that Lemma 5.5 is true when (5.11) holds.

The smoothness assumption on ϕ is removed in the same way as it was removed in the proof of Theorem 2. Indeed, let ϕ_ϵ be as defined below (4.27) and let $\alpha \in C_0^\infty(\mathbf{R}^{n-1})$ with $\alpha = 1$ on $\{x' \in \mathbf{R}^{n-1} : (x',\phi(x')) \in B(w,bs_1)\}$ while $\alpha \equiv 0$ on the complement of $\{x' \in \mathbf{R}^{n-1} : (x',\phi(x')) \in B(w,3bs_1/2)\}$. Define Ω_ϵ by

$$\begin{aligned}
\Omega_\epsilon \cap B(w,2bs_1) &= \{y = (y',y_n) \in \mathbf{R}^n : y_n > [(1-\alpha)\phi + \alpha\phi_\epsilon](y')\} \cap B(w,2bs_1),\\
\Omega_\epsilon \setminus B(w,2bs_1) &= \Omega \setminus B(w,2bs_1).
\end{aligned}$$

Put $\hat u = \hat v = 0$ outside of Ω and let $\hat u_\epsilon, \hat v_\epsilon$ be positive p harmonic functions in $\Omega_\epsilon \cap [B(w,2\hat r) \setminus B(w,r'/2)]$ with continuous boundary values $\hat u_\epsilon = \hat u, \hat v_\epsilon = \hat v$. From Lemmas (2.2)-(2.4) we see that $\hat u_\epsilon, \hat v_\epsilon \to \hat u, \hat v$ uniformly in the closure of $\Omega_\epsilon\cap[B(w,\hat r)\setminus B(w,r')]$ and $\nabla\hat u_\epsilon$ converges uniformly to $\nabla\hat u$ on $\Omega(w,\tilde\eta/2) \cap [B(w,\hat r) \setminus B(w,r')]$. Also, $\nabla\hat u_\epsilon(\cdot,\cdot)$ converges uniformly to $\nabla\hat u(\cdot,\cdot)$ on $(\Omega(w,\tilde\eta/2) \times [\xi',\hat\xi]) \cap [B(w,\hat r) \setminus B(w,r')]$. Using these facts and (5.3) for $\hat u_\epsilon(\cdot), \hat u_\epsilon(\cdot,\hat\tau)$, we see, for ϵ sufficiently small, that (1.9) holds for $\hat u_\epsilon, \hat u_\epsilon(\cdot,\hat\tau)$ in $\Omega_\epsilon\cap[B(w,b^{1/2}s)\setminus B(w,s/b^{1/2})]$ whenever $s \in \{s_k\}_1^i$ and $\hat\tau \in [\xi',\hat\xi]$, with constants depending only on p,n,M and $\hat\delta$. Applying Lemma 5.5 to $\hat u_\epsilon, \hat v_\epsilon$ in Ω_ϵ and letting $\epsilon\to 0$ we conclude that Lemma 5.5 holds for $\hat u, \hat v$. This completes the proof of Lemma 5.5. □

5.2 Proof of Theorem 3

We now prove Theorem 3. Recall that u and v are minimal positive p harmonic functions, for given p, $1 < p < \infty$, relative to $w \in \partial\Omega$. To start the proof of Theorem 3 we assume that there exists a non-increasing sequence of positive numbers $\{\rho_l\}$ and $\tilde b > 1$ such that $\lim_{l\to\infty} \rho_l = 0$ and such that

$$\delta^{-1}\frac{u(x)}{d(x,\partial\Omega)} \leq |\nabla u(x)| \leq \delta\frac{u(x)}{d(x,\partial\Omega)} \tag{5.18}$$

for all x in $\cup_l \Omega(w, \tilde{\eta}/2) \cap [B(w, \tilde{b}\rho_l) \setminus B(w, \rho_l/\tilde{b})]$. From (1.18) and (5.18) we see that (5.18) holds in $\cup_l \Omega \cap [B(w, \tilde{b}\rho_l) \setminus B(w, \rho_l/\tilde{b})]$ with δ replaced by $\hat{\delta} = \hat{\delta}(p, n, M, \delta)$. We also choose $\hat{r}$ so large that $\Omega \subset B(w, \hat{r})$. We shall show, for given $\hat{\epsilon} > 0$, that there exists $\rho, 2r' < \rho < \min(\hat{\epsilon}, \hat{r})$, such that

$$\left|\log\left(\frac{u(x)}{v(x)}\right) - \log\left(\frac{u(y)}{v(y)}\right)\right| \leq \hat{\epsilon} \text{ when } x, y \in \Omega \setminus B(w, \rho). \tag{5.19}$$

Since $\hat{\epsilon}$ is arbitrary it will then follow that $v = \lambda u$ for some $\lambda \in (0, \infty)$.

(5.19) will be proved using Lemma 5.4 and Lemma 5.5 in a fashion similar to the proof of Theorem 2. To begin, we note that we can without loss of generality assume that $\tilde{b}\rho_{l+1} < \rho_l/\tilde{b}$ for $l = 1, 2, \ldots$, and that $2r' < \rho_l/\tilde{b} < \tilde{b}\rho_l < \frac{1}{2}\min(r_0, \hat{r})$ for all l under consideration. We also assume that the technical condition in (5.1) is fulfilled and we define $u(\cdot, \tau), \tau \in [0, 1]$, to be the p harmonic functions in $\Omega \setminus B(w, r'/2)$ with boundary values as in (5.2). Using the same argument as in the proof of Theorem 2 (see (4.23)) it follows, for ϵ_0 as in Lemma 5.4, that there exists ϵ_0', $0 < \epsilon_0' \leq \epsilon_0$, with the same dependence as ϵ_0, such that if $|\tau_2 - \tau_1| \leq \epsilon_0'$, then

$$1 - \epsilon_0/2 \leq \frac{u(\cdot, \tau_2)}{u(\cdot, \tau_1)} \leq 1 + \epsilon_0/2 \text{ in } \Omega \setminus B(w, r'). \tag{5.20}$$

Again we let $\xi_1 = 0 < \xi_2 < \ldots < \xi_m = 1$ be the subdivision of [0,1] into intervals $\{[\xi_k, \xi_{k+1}]\}, 1 \leq k \leq m-1$, of length $\epsilon_0'/2$ or less. Using the assumption on u in (5.18) and combining Lemmas 5.4, 5.5 we see, for $l > i$, using $s_1 = \rho_{l-i}, s_i = \rho_l$, in Lemma 5.5 that if $\tilde{b}$ is large enough there exists $\theta = \theta(p, n, M, \delta) \in (0, 1)$ such that

$$\tilde{M}(\rho_{l-i}, \xi_1, \xi_2) - \tilde{m}(\rho_{l-i}, \xi_1, \xi_2) \leq c\theta^i. \tag{5.21}$$

We shall prove that for given i and $k \in \{2, \ldots, m\}$ there exists $l_k = l_k(p, n, M, \delta) \in \mathbf{Z}_+$ such that if $l > l_k + i$, and $\rho_l/\tilde{b} > 2r'$, then

$$\tilde{M}(\rho_{l-l_k-i}, \xi_1, \xi_k) - \tilde{m}(\rho_{l-l_k-i}, \xi_1, \xi_k) \leq (k-1)c_-\,\theta^i \tag{5.22}$$

for a constant c_- which is independent of l and i. Since $m \leq 2/\epsilon_0'$ and ϵ_0' is independent of l, i, we can then use (5.22) for $k = m$ and the fact that $u(\cdot, \xi_m) = v, u(\cdot, \xi_1) = u$ to conclude from (5.22), for l, i large and r' sufficiently small, that (5.19) holds. Thus to complete the proof of Theorem 3 it suffices to prove (5.22). Using (5.20) and (5.21) we see that (5.22) is true for $k = 2$ with $l_k = l_2 = 1$. To proceed by induction, we assume that we have established (5.22) for some k, $2 \leq k \leq m$, and if $k = m$ then we quit. Assuming $k < m$ we define $l_k' > l_k$ by putting $i = l_k' - l_k$ in (5.22) where i is chosen so large that, for $l > l_k'$, we have

$$\left|\frac{u(x, \xi_k)}{u(x)} - \frac{u(y, \xi_k)}{u(y)}\right| \leq \sigma\frac{u(x, \xi_k)}{u(x)}$$

whenever $x, y \in \Omega \setminus B(w, \rho_{l-l_k'})$. We fix a $x \in \Omega \setminus B(w, \rho_{l-l_k'})$ and choose $\sigma > 0$ small enough to ensure that if $y \in \Omega \setminus B(w, \rho_{l-l_k'})$ and if $\tau \in [\xi_k, \xi_{k+1}]$, then

$$(1 - \epsilon_0)\frac{u(x, \xi_k)}{u(x)} \leq \frac{u(y, \tau)}{u(y)} \leq (1 + \epsilon_0)\frac{u(x, \xi_k)}{u(x)}. \tag{5.23}$$

To estimate the magnitude of σ we observe, as in the proof of Theorem 2, that if $\tau \in [\xi_k, \xi_{k+1}]$ then

$$\frac{u(y, \tau)}{u(y)} = \frac{u(y, \tau)}{u(y, \xi_k)} \cdot \frac{u(y, \xi_k)}{u(y)} \leq (1 + \epsilon_0/2)(1 + \sigma)\frac{u(x, \xi_k)}{u(x)}.$$

Thus if $\sigma = \epsilon_0/4$ and if ϵ_0 is small enough then the right hand inequality in (5.23) is valid. A similar argument gives the left hand inequality in (5.23) when $\sigma = \epsilon_0/4$ and ϵ_0 is small enough. $l_k' - l_k$ can be chosen to depend only on ϵ_0 and θ and hence on p, n, M, δ only, assuming $\tilde{b}$ is large. From (5.23) we find that we can apply Lemma 5.4 in $\Omega \cap [B(w, \tilde{b}\rho_{l-\tilde{l}}) \setminus B(w, \rho_{l-\tilde{l}}/\tilde{b})]$ with $\hat{\tau}$ replaced by an arbitrary $\tau \in [\xi_k, \xi_{k+1}]$ provided $\tilde{l}$ is such that $\rho_{l-\tilde{l}}/\tilde{b} > \rho_{l-l_k'}$ and $\hat{L} = \frac{u(x,\xi_k)}{u(x)}$. Since the sets in the set $\{B(w, b\rho_j) \setminus B(w, \rho_j/b)\}_1^\infty$ are disjoint, it follows that if $l_{k+1} = l_k' + 1$, then the assumptions of Lemma 5.5 are fulfilled in $\Omega \cap [B(w, \tilde{b}^{1/2}\rho_{l-l_{k+1}}) \setminus B(w, \rho_{l-l_{k+1}}/\tilde{b}^{1/2})]$ for $\tau \in [\xi_k, \xi_{k+1}]$ whenever $l > l_{k+1}$. Applying Lemma 5.5 with $s_1 = \rho_{l-l_{k+1}-i+1}$, $s_i = \rho_{l-l_{k+1}}$, we get

$$\tilde{M}(\rho_{l-l_{k+1}-i}, \xi_k, \xi_{k+1}) - \tilde{m}(\rho_{l-l_{k+1}-i}, \xi_k, \xi_{k+1}) \leq c_-\theta^i \tag{5.24}$$

provided $2r' < \rho_{l-l_{k+1}-i}$. Finally we note that if $x, y \in \Omega \setminus B(w, \rho_{l-l_{k+1}-i})$ then

$$\left| \log \frac{u(x, \xi_{k+1})}{u(x)} - \log \frac{u(y, \xi_{k+1})}{u(y)} \right|$$

is dominated by

$$\begin{aligned} &\tilde{M}(\rho_{l-l_{k+1}-i}, \xi_k, \xi_{k+1}) - \tilde{m}(\rho_{l-l_{k+1}-i}, \xi_k, \xi_{k+1}) \\ &+ \tilde{M}(\rho_{l-l_{k+1}-i}, \xi_1, \xi_k) - \tilde{m}(\rho_{l-l_{k+1}-i}, \xi_1, \xi_k) \leq kc_-\theta^i \end{aligned}$$

as we see from (5.24), (5.22) and the fact that by construction $l_{k+1} > l_k$. We can therefore first conclude, for $l > l_{k+1} + i$, that

$$\tilde{M}(\rho_{l-l_{k+1}-i}, \xi_1, \xi_{k+1}) - \tilde{m}(\rho_{l-l_{k+1}-i}, \xi_1, \xi_{k+1}) \leq kc_-\theta^i$$

and then, by induction, that (5.22) is valid for all $k \in \{2, .., m\}$. The proof of Theorem 3 is now complete. □

Finally in this section we observe that our proof implies the following corollary.

Corollary 5.25 *Let $p, \Omega, \hat{u}, \hat{v}, r', \hat{r}$ be as in Lemma 5.5 and suppose that (1.9) holds for $\hat{u}$ in $\Omega \setminus B(w, r')$. There exists $c = c(p, n, M, \delta) > 1$ and $a = a(p, n, M, \delta), 0 < a \leq 1/2$, such that*

$$\left| \log \frac{\hat{u}(y_1)}{\hat{v}(y_1)} - \log \frac{\hat{u}(y_2)}{\hat{v}(y_2)} \right| \leq c \left(\frac{r}{\min(|y_1 - w|, |y_2 - w|)} \right)^a$$

whenever $r \geq cr'$ and $y_1, y_2 \in \Omega \setminus B(w, r)$. Moreover, there exists $\hat{\delta} = \hat{\delta}(p, n, M, \delta) > 1$ such that (1.9) holds for $\hat{v}$ in $\Omega \setminus B(w, cr)$.

Proof: The display in Corollary 5.25 follows from Lemmas 5.4, 5.5, and iteration, as in the proof of Theorem 3. The uniform non-degeneracy of $|\nabla v|$ in $\Omega \setminus B(w, cr)$ is a consequence of the above display and the same argument as in the proof of Lemma 5.4. □

6 Proof of Theorem 4

In this section we use Theorem 3 to prove Theorem 4. Throughout the section we assume that $\Omega \subset \mathbf{R}^n$ is a bounded Lipschitz domain and that $w \in \partial\Omega$. To prove Theorem 4 we have to construct a minimal positive p harmonic function in Ω relative to $w \in \partial\Omega$ satisfying the criteria stated in Theorem 3. We supply such constructions when Ω is convex and when $\partial\Omega$ is C^1. We treat each case separately.

To start with we assume that Ω is a bounded convex domain. In this case we fix $p, 1 < p < \infty$, and we let $\{w_m\}_1^\infty$ be a sequence of points in Ω, tending non-tangentially to w, and we put

$$\begin{aligned} s_m &= d(w_m, \partial\Omega)/8,\ m = 1, 2, \ldots, \\ D_m &= \Omega \setminus \bar{B}(w_m, s_m), \\ u_m &= \text{the } p \text{ capacitary function for } D_m. \end{aligned}$$

We claim there exists $c > 1$, depending only on p, n, such that

$$u_m(x) \leq c\langle w_m - x, \nabla u_m \rangle \text{ whenever } x \in D_m. \tag{6.1}$$

Indeed, suppose for the moment that

$$\partial\Omega \text{ is } C^\infty. \tag{6.2}$$

Using Lemmas 2.5 and 2.6 we see that $u_m(x)$ as well as $\langle w_m - x, \nabla u_m(x)\rangle$ both satisfy the partial differential equation in (1.6), (1.7) in D_m. Moreover, from barrier type estimates and Lemma 2.5 one easily sees that $c|\nabla u_m| \geq s_m^{-1}$ on $\partial B(w_m, s_m)$ where c depends only on p, n. Also from Lemma 2.5 and Lemma 2.6 we have $\langle w_m - x, \nabla u_m\rangle > 0$ in $\bar{D}_m$. Comparing the boundary values of $u_m(x)$ and $\langle w_m - x, \nabla u_m\rangle$, we get (6.1) when (6.2) holds. If (6.2) is not true we can choose a sequence $\{\Omega_j'\}$ of convex domains with C^∞ boundaries which converges to Ω in the sense of Hausdorff distance. Let $\{\psi_j\}$ be the corresponding p capacitary functions for $\Omega_j' \setminus \bar{B}(w_m, s_m)$ and put $\psi_j = 0$ in $\mathbf{R}^n \setminus \Omega_j'$. Using Lemmas 2.2-2.4 we see that $\{\psi_j\}$ converges, uniformly on D_m, to u_m and $\{\nabla\psi_j\}$ converges to ∇u_m, uniformly on compact subsets of D_m. Applying (6.1) with u_m replaced by ψ_j, $j = 1, 2, \ldots$, using uniform convergence and the fact that the constant in (6.1) depends only on p, n, we get (6.1) for u_m.

Fix $\hat{x} \in \Omega$ and consider the sequence $u_m/u_m(\hat{x}), m = 1, 2, \ldots$ If we let $m \to \infty$, then from Lemmas 2.2-2.4 we see that a subsequence of this sequence converges, uniformly on compact subsets of Ω, to a minimal positive p harmonic function u relative to w in Ω and $u(\hat{x}) = 1$. Also (6.1) implies that

$$u(x) \leq c\langle w - x, \nabla u(x)\rangle \tag{6.3}$$

for all $x \in \Omega$ where c is the constant in (6.1). Let $\tilde{\Omega}(w, \tilde{\eta})$ be the non-tangential approach region in (5.3) defined with respect to w and Ω. Using (6.3) we see that (1.9) holds for u in $\tilde{\Omega}(w, \tilde{\eta})$ with a constant $\delta = \delta(p, n, M)$. Applying Theorem 3 we can therefore conclude that Theorem 4 is true when Ω is a bounded convex domain.

We now move on to the case of C^1-domains and in the following we will simply let u be an arbitrary minimal positive p harmonic function in Ω relative to $w \in \partial\Omega$ and we will prove, for u, the existence of $\{\rho_l\}$ and $\tilde{b}$ as in the statement of Theorem 3. To do this we let, for simplicity, $w = 0$ and we note that we can assume, after a rotation if necessary, that there exists $r_1 = r_1(\sigma)$, for given $\sigma > 0$ (sufficiently small), such that if $0 < r \leq r_1$, then

$$B(0, nr) \cap \{y : y_n \geq \sigma r\} \subset \Omega, \quad B(0, nr) \cap \{y : y_n \leq -\sigma r\} \subset \mathbf{R}^n \setminus \Omega. \tag{6.4}$$

Extend u to a continuous function in $\mathbf{R}^n \setminus \{0\}$ by putting $u \equiv 0$ on $\mathbf{R}^n \setminus (\Omega \cup \{0\})$. Let

$$Q = \{y : |y_i| < r, 1 \leq i \leq n-1\} \cap \{y : \sigma r < y_n < r\} \setminus B(0, \sqrt{\sigma}\, r)$$

and let v_1 be the p harmonic function in Q with the following continuous boundary values,

$$\begin{aligned} v_1(y) &= u(y) \qquad \text{if } y \in \partial Q \cap \{y : 2\sigma r \leq y_n\}, \\ v_1(y) &= \frac{(y_n - \sigma r)}{\sigma r} u(y) \qquad \text{if } y \in \partial Q \cap \{y : \sigma r \leq y_n < 2\sigma r\}. \end{aligned}$$

Comparing boundary values and using the maximum principle for p harmonic functions, we deduce

$$v_1 \leq u \text{ in Q}. \tag{6.5}$$

Let $\sigma(\epsilon) = \exp(-1/\epsilon)$. To complete the proof of Theorem 4 we will make use of the following lemmas.

Lemma 6.6. *Let $0 < \epsilon \leq \hat{\epsilon}$, let $\sigma = \sigma(\epsilon)$ be as above and let $\tilde{\eta}$ be as in (5.3). If $\hat{\epsilon}$ is small enough, then there exists $\hat{\theta} = \hat{\theta}(p, n, M), 0 < \hat{\theta} \leq 1/2$, such that if $\hat{\rho} = \sigma^{1/2 - \hat{\theta}} r$, then*

$$1 \leq u(y)/v_1(y) \leq 1 + \epsilon$$

whenever $y \in \tilde{\Omega}(0, \tilde{\eta}/16) \cap [B(0, \hat{\rho}) \setminus B(0, 4\sqrt{\sigma} r)]$.

Lemma 6.7. *Let $v_1, \epsilon, \hat{\epsilon}, \hat{\theta}, r, \sigma$, be as in Lemma 6.6 and let $\tilde{\eta}$ be as in (5.3). If $\hat{\epsilon}$ is small enough, then there exist $\theta = \theta(p, n, M), 0 < \theta \leq \hat{\theta}/10$, and $\hat{\delta} = \hat{\delta}(p, n, M) > 1$ such that if $\rho = \sigma^{1/2 - 4\theta} r, b = \sigma^{-\theta}$, then*

$$\hat{\delta}^{-1} \frac{v_1(x)}{d(x, \partial\Omega)} \leq |\nabla v_1(x)| \leq \hat{\delta} \frac{v_1(x)}{d(x, \partial\Omega)}$$

whenever $x \in \tilde{\Omega}(0, \tilde{\eta}/4) \cap [B(0, b\rho) \setminus B(0, \rho/b)]$ and $0 < \epsilon \leq \hat{\epsilon}$.

Lemmas 6.6 and 6.7 are proved below but here we indicate how the proof of Theorem 4 in the case of C^1-domains follows from these lemmas. Indeed, using Lemmas 6.6, 6.7 and arguing as in the proof of Lemma 5.4 we see for $\hat{\epsilon}$ sufficiently small and fixed, $0 < \epsilon \leq \hat{\epsilon}$, that there exists $\tilde{\delta} > 1$, depending only on p, n, M, such that

$$\tilde{\delta}^{-1} \frac{u(x)}{d(x, \partial\Omega)} \leq |\nabla u(x)| \leq \tilde{\delta} \frac{u(x)}{d(x, \partial\Omega)} \text{ in } \tilde{\Omega}(0, \tilde{\eta}/2) \cap [B(0, b^{1/2}\rho) \setminus B(0, \rho/b^{1/2})]. \tag{6.8}$$

Fix ϵ so small that $b^{1/2} = \sigma^{-\theta/2} > \tilde{b}(p, n, M, \tilde{\delta}) = \tilde{b}(p, n, M)$, where $\tilde{b}$ is as in Theorem 3. Choosing $\rho \in \{\rho_l\}$ with $\lim_{l \longrightarrow \infty} \rho_l = 0$ we conclude from (6.8) and Theorem 3 that Theorem 4 is valid in the C^1-case.

Proof of Lemma 6.6. From (6.5) we observe that it suffices to prove the righthand side inequality stated in Lemma 6.6. We note that if $y \in \partial Q$ and $u(y) \neq v_1(y)$, then y lies within $4\sigma r$ of a point in $\partial\Omega$. Also, $\max_{\partial B(0,t)} u$ is nonincreasing as a function of $t > 0$ as we see from the maximum principle for p harmonic functions. Using these notes and Lemmas 2.1 - 2.3, we see that

$$u \leq v_1 + c\sigma^{\alpha/2} u(\sqrt{\sigma} e_n) \tag{6.9}$$

on ∂Q, where α is the exponent of Hölder continuity in Lemma 2.2. By the maximum principle for p harmonic functions this inequality also holds in Q.

Using the interior Harnack inequality and Lemmas 2.1-2.3 we also find that there exist $\beta = \beta(p, n, M) \geq 1$ and $c = c(p, n, M) > 1$ such that

$$\max\{\psi(z), \psi(y)\} \leq c\,(d(z, \partial Q)/d(y, \partial Q))^{\beta} \min\{\psi(z), \psi(y)\} \tag{6.10}$$

whenever $z \in Q$, $y \in Q \cap B(z, 4d(z, \partial Q))$ and $\psi = u$ or v_1. Also from Lemmas 2.1 - 2.3 applied to v_1 we deduce that

$$v_1(2\sqrt{\sigma}\, re_n) \geq c^{-1}\, u(\sqrt{\sigma}\, re_n). \tag{6.11}$$

Let $\hat{\rho}, \hat{\theta}$ be as in Lemma 6.6. From (6.9) - (6.11) we see that if $y \in \tilde{\Omega}(0, \tilde{\eta}/16) \cap [B(0, \hat{\rho}) \setminus B(0, 4\sqrt{\sigma}\, r)]$, then

$$u(y) \leq v_1(y) + c\sigma^{\alpha/2} u(\sqrt{\sigma} e_n) \leq (1 + c^2 \sigma^{\alpha/2 - \hat{\theta}\beta}) v_1(y) \leq (1 + \epsilon) v_1(y) \tag{6.12}$$

provided $\hat{\epsilon}$ is small enough and $\hat{\theta}\beta = \alpha/4$. Thus Lemma 6.6 is true. $\Box$

Proof of Lemma 6.7. Using Lemmas 2.1-2.3 and Harnack's inequality we note that there exist $\gamma = \gamma(p, n, M) > 0, 0 < \gamma \leq 1/2$, and $c = c(p, n, M) > 1$ such that

$$u(x) \leq c(s/t)^{\gamma} u(se_n) \tag{6.13}$$

provided $x \in \mathbf{R}^n \setminus B(0, t), t \geq s$, and $se_n \in \Omega$ with $d(se_n, \partial\Omega) \geq c^{-1}s$. Using (6.13) with $t = r, s = \sqrt{\sigma} r$, we find that

$$v_1 \,\leq\, c\,\sigma^{\gamma/2}\, u(\sqrt{\sigma} re_n) \text{ on } \partial Q \setminus \bar{B}(0, \sqrt{\sigma}\, r\,), \tag{6.14}$$

where c depends only on p, n, M. Let $\tilde{v}$ be the p harmonic function in Q with continuous boundary values $\tilde{v} = 0$ on $\partial Q \setminus \bar{B}(0, \sqrt{\sigma}\, r)$ and $\tilde{v} = v_1$ on $\partial Q \cap \partial B(0, \sqrt{\sigma} r)$. From the maximum principle for p harmonic functions and (6.14) it follows that

$$0 \,\leq\, \tilde{v} \,\leq\, v_1 \,\leq\, \tilde{v} + c\sigma^{\gamma/2} u(\sqrt{\sigma} re_n) \text{ in } Q. \tag{6.15}$$

From Lemmas 2.1 - 2.3 we observe that

$$\tilde{v}(2\sqrt{\sigma} re_n) \geq c^{-1}\, v_1(\sqrt{\sigma} re_n) = c^{-1} u(\sqrt{\sigma} re_n). \tag{6.16}$$

Using (6.15), (6.16), and (6.10) applied to $\psi = \tilde{v}$ we obtain for $\rho = \sigma^{1/2 - 4\theta} r, \theta$ small, $b = \sigma^{-\theta}$, and $\hat{b} = 8b^2$, that

$$\tilde{v} \leq v_1 \leq (1 + c\sigma^{\gamma/2 - 6\theta\beta})\tilde{v} \leq (1 + \epsilon)\tilde{v} \tag{6.17}$$

on $\tilde{\Omega}(0, \tilde{\eta}/8) \cap [B(0, \hat{b}\rho) \setminus B(0, \rho/\hat{b})]$, provided $\hat{\epsilon}$ is small enough and $\theta = \min\{\gamma/(24\beta), \hat{\theta}/10\}$.

Next let v be the p harmonic function in

$$Q' = \{y : |y_i| < r, 1 \leq i \leq n-1\} \cap \{y : \sigma r < y_n < r\} \setminus \bar{B}(2\sqrt{\sigma}\, re_n, \sqrt{\sigma}\, r\,)$$

with continuous boundary values $v = 0$ on $\partial Q' \setminus \bar{B}(2\sqrt{\sigma} e_n,\, \sqrt{\sigma}\, r)$ while $v = 1$ on $\partial B(2\sqrt{\sigma}\, re_n, \sqrt{\sigma}\, r\,)$. Arguing as in the proof of (6.1) we see that

$$v(x) \leq c\langle 2\sqrt{\sigma}\, re_n - x, \nabla v(x)\rangle \tag{6.18}$$

when $x \in Q'$ where $c = c(p, n)$. Clearly this inequality implies that there exists $c = c(p, n, \eta) \geq 1$, for given $\eta, 0 < \eta \leq 1/2$, such that

$$c^{-1} \frac{v(x)}{d(x, \partial Q')} \leq |\nabla v(x)| \leq c \frac{v(x)}{d(x, \partial Q')} \tag{6.19}$$

in $\tilde{Q}'(0, \eta) \setminus B(0, 10\sqrt{\sigma}\, r\,)$ where $\tilde{Q}'(0, \eta)$ is the non-tangential approach region defined relative to $0, \eta, Q'$ as in section 1. Using Lemma 4.28 and (6.19) for suitable $\eta = \eta(p, n)$ we conclude that (6.19) actually holds in $Q' \setminus B(0, 10\sqrt{\sigma} r)$. We now use (6.19) and Corollary 5.25 applied to $v, \tilde{v}$ with Ω, r', replaced by $Q', 10\sqrt{\sigma} r$, in order to get, for some $a = a(p, n) > 0$ and $c = c(p, n) > 1$, that

$$\left| \log \left(\frac{\tilde{v}(x)}{v(x)} \right) - \log \left(\frac{\tilde{v}(y)}{v(y)} \right) \right| \leq c \left(\frac{\sqrt{\sigma}\, r}{\min(|x|, |y|)} \right)^a \tag{6.20}$$

whenever $x, y \in Q' \cap B(0, r/4) \setminus B(0, c\sqrt{\sigma} r)$. Using (6.20), (6.19), and arguing as in the proof of Lemma 5.4 it then follows that there exists $\kappa = \kappa(p, n) > 20$ such that

$$c^{-1} \frac{\tilde{v}(x)}{d(x, \partial\Omega)} \leq |\nabla \tilde{v}(x)| \leq c \frac{\tilde{v}(x)}{d(x, \partial\Omega)} \tag{6.21}$$

in $\tilde{\Omega}(0, \tilde{\eta}/8) \cap [B(0, r/2) \setminus B(0, \kappa\sqrt{\sigma}\, r)]$. Finally, note that if $0 \leq \epsilon \leq \hat{\epsilon}$ and if $\hat{\epsilon}$ is sufficiently small, then $r/2 > b^2\rho > \rho/b^2 > \kappa\sqrt{\sigma}\, r$. Hence, if $\hat{\epsilon}$ is small enough then we can use (6.21), (6.17) and the same argument as in Lemma 5.4 to conclude that Lemma 6.7 is valid. The proof of Theorem 4 is now complete in the C^1-case.

To complete the proof of Theorem 4 we observe that if $\partial\Omega$ has a tangent plane at $0 \in \partial\Omega$, then Lemmas 6.6, 6.7 are valid for $r, \hat{\epsilon}$, sufficiently small. Using the argument in the paragraph below Lemma 6.7 we get that u is unique up to constant multiples. $\Box$

Finally in this section we note, without proof, that Theorem 3 and Lemmas 6.6, 6.7 imply a stronger result. In order to state this result let $h(E, F)$ be the Hausdorff distance between E, F defined as the infimum of the set of all λ such that every point of E lies within λ of a point of F and vice versa. Let $w \in \partial\Omega$ and put

$$f(r) = \inf_P h\left(\partial\Omega \cap B(w, r), P \cap B(w, r)\right)$$

where P is an $(n-1)$-dimensional plane containing w. Then the following is true.

Corollary 6.22. *There exists $\zeta = \zeta(p, n, M) > 0$ such that if $\liminf_{r \to 0} r^{-1} f(r) \leq \zeta(p, n, M)$, then minimal positive p harmonic functions relative to w are unique up to constants.*

7 Proof of Theorem 5 and Closing Remarks

In this section we first prove Theorem 5. To this end we let Ω be a bounded Lipschitz domain and we let p, $1 < p < \infty$, be given. Recall from section 1, in case of the Martin boundary problem for the exterior of Ω, that $u > 0$ is said to be a minimal positive p harmonic function relative to $w \in \partial\Omega$ provided u is p harmonic in $\mathbf{R}^n \setminus \bar{\Omega}$ with continuous boundary value zero on $\partial\Omega \setminus \{w\}$ and $\lim_{x \to \infty} u(x) = 0$.

7.1 Proof of Theorem 5

If $\partial\Omega$ is C^1, then the proof of Theorem 5 is essentially the same as the proof of Theorem 4. To briefly outline the proof, one first proves analogues of Lemmas 5.4, 5.5 for $\mathbf{R}^n \setminus \bar{\Omega}$. In fact the proof of Lemma 5.4 is unchanged if Ω is replaced by $\mathbf{R}^n \setminus \bar{\Omega}$. Also the proof of Lemma 5.5 follows with minor changes if one assumes that $\hat{u}, \hat{v}$ are positive p harmonic functions in $\mathbf{R}^n \setminus [\bar{\Omega} \cup \bar{B}(w, r'/2)]$ with continuous boundary values 0 on $(\partial\Omega \cup \{\infty\}) \cap [\mathbf{R}^n \setminus \bar{B}(w, r'/2)]$. Indeed, in this case one can define $u^*(\cdot, \tau), \tau \in [0,1]$, to be the p harmonic function in $(\mathbf{R}^n \setminus \bar{\Omega}) \cap [B(w, 2\hat{r}) \setminus \bar{B}(w, r'/2)]$ with boundary values $\tau\hat{v} + (1-\tau)\hat{u}$ and one can then let $\hat{r} \to \infty$ to get $\hat{u}(\cdot, \tau)$ satisfying (5.13) in $\mathbf{R}^n \setminus [\bar{\Omega} \cup \bar{B}(w, r'/2)]$ and (5.14) with Ω replaced by $\mathbf{R}^n \setminus \bar{\Omega}$. We then deduce (5.15), (5.16) with Ω replaced by $\mathbf{R}^n \setminus \bar{\Omega}$ and next that the function $\mathrm{Osc}(\cdot)$ defined below (5.16) is non-increasing as a function of t on its domain. This fact, (5.16), Lemma 2.12, and iteration, imply (5.17) and Lemma 5.5, as previously explained. The proof of Theorem 3 with Ω replaced by $\mathbf{R}^n \setminus \bar{\Omega}$, follows from Lemmas 5.4, 5.5 without any essential modifications. Finally, in the case when $\partial\Omega$ is C^1 the construction of $\{\rho_l\}$ and u, satisfying (1.9) in $(\mathbf{R}^n \setminus \bar{\Omega}) \cap \bigcup_l B(w, \tilde{b}\rho_l) \setminus B(w, \rho_l/\tilde{b})$, is no different than the corresponding construction in Theorem 4.

Next we consider the case when Ω is convex. We assume, as we may, that $w = 0 \in \partial\Omega$. Also, using convexity of Ω we assume, as we may, that there exists a Lipschitz function $\phi : \mathbf{R}^{n-1} \to \mathbf{R}$ with Lipschitz norm $\leq M$ and the property that, after a rotation if necessary,

$$
\begin{aligned}
&(a) \quad \phi(x') \leq \phi(0) = 0, x' \in \mathbf{R}^{n-1}, \\
&(b) \quad \partial\Omega \cap B(0, 4r_0) = \{(x', \phi(x')) : x' \in \mathbf{R}^{n-1}\} \cap B(0, 4r_0), \\
&(c) \quad (\mathbf{R}^n \setminus \bar{\Omega}) \cap B(0, 4r_0) = \{(x', x_n) : x_n > \phi(x'), x' \in \mathbf{R}^{n-1}\} \cap B(0, 4r_0).
\end{aligned} \tag{7.1}
$$

Given $0 < \sigma$ and $0 < r < r_0\sigma^4$, let $G(\sigma, r)$ be the union of all open line segments drawn from re_n to points in $\partial\Omega \cap B(0, r/\sigma^4)$. We shall need the following lemma.

Lemma 7.2 *Given $\sigma, 0 < \sigma < 10^{-10}$, there exists $r_+(\sigma), 0 < r_+ < \sigma^4 r_0$, such that if $0 < r < r_+$, then $G(\sigma, r) \subset \mathbf{R}^n \setminus \bar{\Omega}$.*

Proof: Note from (7.1) that if $\theta_0 = \theta_0(M, n)$ is small enough, then

$$K \subset \Omega, \text{ where } K = \{y \in B(0, r_0) \setminus \{0\} : -\langle y, e_n \rangle \geq (1-\theta_0)|y|\}.$$

Let $\omega_0 \in \partial B(0,1)$ and let $l = \{se_n + t\omega_0, 0 < t < \infty\}$ be the ray drawn from se_n with direction ω_0. If $0 < s < r_0\sigma^4$, and $G(\sigma, s) \not\subset \mathbf{R}^n \setminus \bar{\Omega}$, then there exists $\omega_0 \in \partial B(0,1)$ and $0 < t_1 < t_2$ such that $P_i = se_n + t_i\omega_0 \in \partial\Omega \setminus K$ for $i = 1, 2$. Let $\hat{O}$ be the interior of the set obtained from drawing all line segments from P_1 to points in K. From convexity of Ω we see that the line segment $l_1 \subset l$ connecting P_2 to infinity cannot intersect Ω. In this case we claim there exists $0 < \zeta = \zeta(M, n, \sigma)$ such that if $\omega \in \partial B(0,1) \cap B(\omega_0, \zeta)$, then any line segment l' of the form $\{\tau e_n + t\omega, 0 < t < \infty\}$, for $0 < \tau < \sigma^5 s$, can intersect $\partial\Omega \cap B(0, \tau/\sigma^4)$ in at most one point. Indeed, if l' intersects $\partial\Omega \cap B(0, \tau/\sigma^4)$ in two points, then there is a line segment $l'' \subset l'$ connecting one of these points to ∞ which does not intersect Ω. On the other hand it is easily seen, for $\zeta = \zeta(M, n, \sigma)$ small enough, that l'' must intersect $\hat{O}$, which is a contradiction. Thus our claim is true. From our claim we conclude that if a ray drawn from se_n in a given direction, ω_0, intersects $\partial\Omega \cap B(0, s/\sigma^4)$ in two points, then every ray drawn from τe_n in the direction ω

intersects $\partial\Omega \cap B(0, \tau/\sigma^4)$ in at most one point for $\omega \in \partial B(0,1) \cap B(\omega_0, \zeta)$. Let E be the set of all points $\omega_0 \in \partial B(0,1)$ for which there are two intersections as just described. Using a well known covering lemma we can cover E by balls, $B(\omega_i, \zeta), 1 \leq i \leq N$, with $\omega_i \in E$, in such a way that the balls $B(\omega_i, \zeta/10), 1 \leq i \leq N$, are pairwise disjoint. It follows that $N \leq c\zeta^{-n}$ so that for $r_+(\sigma)$ small enough, we must have $G(\sigma, r) \subset \mathbf{R}^n \setminus \bar{\Omega}$. □

To continue the proof of Theorem 5, fix $r, 0 < r < \sigma^2 r_+(\sigma)$, and let K^* be the cylinder defined by $K^* = \{(x', x_n) : |x'| < r/\sigma^2, x_n < r/\sigma^2\}$. Put $H = H(\sigma, r) = K^* \cap (\mathbf{R}^n \setminus \bar{\Omega})$ and observe from (7.1) that if $y \in \partial H \cap K^*$, then $y \in B(0, r/\sigma^4) \cap \partial\Omega$ for $\sigma = \sigma(M, n) > 0$ small enough. Thus by Lemma 7.2 the open line segment from re_n to y is in H. From this observation we see that

$$H \text{ is open and starlike with respect to } re_n. \tag{7.3}$$

Next for fixed $p, 1 < p < \infty$, let u be a minimal positive p harmonic function in $\mathbf{R}^n \setminus \bar{\Omega}$ relative to 0. If $0 < \eta < 1/2$, let $U(\eta) \subset \mathbf{R}^n \setminus \bar{\Omega}$ be the non-tangential approach region to 0 defined by,

$$U(\eta) = \{x \in (\mathbf{R}^n \setminus \bar{\Omega}) \cap B(0, r_0) : d(x, \partial\Omega) > \tilde{\eta}\,|x|\}.$$

From Theorem 2 we see there exists $\tilde{\eta} = \tilde{\eta}(p, n, M) > 0$, such that if

$$W = (\mathbf{R}^n \setminus \bar{\Omega}) \cap [B(0, r_0) \setminus U(\tilde{\eta})],$$

then

$$\begin{array}{ll} (a) & u \text{ satisfies (1.9) in } W \text{ with a constant } \delta = \delta(p, n, M), \\ (b) & \text{given } y \in W \text{ there exists } z \in \partial\Omega \setminus \{0\} \text{ with } y \in B(z, |z|/c) \text{ and } \xi = \xi(z), \\ & \delta' = \delta'(p, n, M), \text{ for which (1.10) holds on } (\mathbf{R}^n \setminus \bar{\Omega}) \cap B(z, 2|z|/c). \end{array} \tag{7.4}$$

Let v_1 be the p harmonic function in $H \setminus \bar{B}(0, r)$ with continuous boundary values $v_1 \equiv 0$ on $\partial H \setminus \bar{B}(0, r)$ while $v_1 \equiv u$ on $\partial B(0, r) \cap H$. To finish the proof of Theorem 5 we need the following lemmas (compare with Lemmas 6.6, 6.7).

Lemma 7.5. *Given $\epsilon > 0$ small, let v_1 be defined as above. Then there exist $\sigma_0 = \sigma_0(p, n, M, \epsilon)$ and $\theta = \theta(p, n, M), 0 < \theta, \sigma_0 < 10^{-10}$, such that if $0 < \sigma \leq \sigma_0$ and if $\rho = \sigma^{-2\theta} r, b = \sigma^{-\theta}$, then*

$$1 - \epsilon \leq v_1/u \leq 1$$

in $U(\tilde{\eta}/4) \cap [B(0, 4b\rho) \setminus B(0, \rho/(4b))]$.

Lemma 7.6. *Let ϵ, v_1, ρ and b be as in Lemma 7.5. If $\sigma_1 = \sigma(p, n, M)$ is small enough and $0 < \sigma < \sigma_1$, then there exists $\tilde{\delta} = \tilde{\delta}(p, n, M) > 1$ such that*

$$\tilde{\delta}^{-1} \frac{v_1(x)}{d(x, \partial\Omega)} \leq |\nabla v_1(x)| \leq \tilde{\delta} \frac{v_1(x)}{d(x, \partial\Omega)}$$

for $x \in U(\tilde{\eta}/2) \cap [B(0, b\rho) \setminus B(0, \rho/b)]$.

To get Theorem 5 from Lemmas 7.5, 7.6, we observe, as in the the proof of Lemma 5.4, that since $\tilde{\delta}$ depends only on p, n, M we can choose $\epsilon > 0$ sufficiently small and deduce that

$$\hat{\delta}^{-1}\frac{u(x)}{d(x,\partial\Omega)} \leq |\nabla u(x)| \leq \hat{\delta}\frac{u(x)}{d(x,\partial\Omega)}. \tag{7.7}$$

in $U(\tilde{\eta}) \cap [B(0, b^{1/2}\rho) \setminus B(0, \rho/b^{1/2})]$. Here $\hat{\delta} > 1$, depends only on p, n, M. From (7.4) we see that (7.7) holds in $(\mathbf{R}^n \setminus \bar{\Omega}) \cap [B(0, b^{1/2}\rho) \setminus B(0, \rho/b^{1/2})]$.

Choosing b sufficiently large and $\rho \in \{\rho_l\}$ with $\lim_{l\longrightarrow\infty} \rho_l = 0$ we conclude from (7.7) and the analogue of Theorem 3 for $\mathbf{R}^n \setminus \bar{\Omega}$, that Theorem 5 is valid when Ω is convex.

Proof of Lemma 7.5. Comparing boundary values we see that the right hand inequality in Lemma 7.5 is a consequence of the maximum principle for p harmonic functions. To get the left hand inequality extend u to a continuous function on $\mathbf{R}^n \setminus \{0\}$ by putting $u \equiv 0$ on $\bar{\Omega} \setminus \{0\}$. We note from Lemmas 2.1-2.3, the fact that $\lim_{x\longrightarrow\infty} u(x) = 0$ and Harnack's inequality, as in (6.13), that there exist $\gamma > 0, 0 < \gamma \leq 1/2$, and $c = c(p, n, M) \geq 1$ such that

$$u(x) \leq c(s/t)^{\gamma} u(se_n) \tag{7.8}$$

provided $x \in \mathbf{R}^n \setminus B(0, t)$ and $r_0 \geq t \geq s$. Using (7.8) with $t \geq r/\sigma^2, s = r$ and Lemmas 2.1-2.3 we see that

$$v_1 \geq u - c\sigma^{2\gamma} u(re_n) \tag{7.9}$$

on $\partial H \setminus \bar{B}(0, r)$ and hence, by the boundary maximum principle, this inequality also holds in $H \setminus \bar{B}(0, r)$. Now from Harnack's inequality we see, as in (6.10), that for some $\beta = \beta(p, n, M), c = c(p, n, M) \geq 1$,

$$u \geq c^{-1}(r/t)^{\beta} u(re_n) \text{ in } B(0, t) \cap U(\tilde{\eta}/4) \tag{7.10}$$

whenever $r_0 \geq t > 2r$. Let $t = \sigma^{-2\theta} r = \rho$, $b = \sigma^{-\theta}$ and let $\theta = \theta(p, n, M) > 0$ be defined through the relation $3\beta\theta = \gamma$. Then, using (7.9), (7.10) we see that

$$v_1/u \geq 1 - \frac{c\sigma^{2\gamma}\, u(re_n)}{u} \geq 1 - c^2\sigma^{2\gamma - 3\beta\theta} \geq 1 - \epsilon \tag{7.11}$$

in $U(\tilde{\eta}/4) \cap B(0, 4b\rho) \setminus B(0, \rho/(4b))$ provided $\sigma \leq \sigma_0(p, n, M, \epsilon)$. The proof of Lemma 7.5 is now complete. $\Box$.

Proof of Lemma 7.6. Let $\tilde{v}$ be the p harmonic function in $H \setminus B(re_n, r/4)$ with continuous boundary values $\tilde{v} = 0$ on ∂H and $\tilde{v} = 1$ on $\partial B(re_n, r/4)$. Using (7.3) and arguing as in (6.1)-(6.3) we deduce the existence of $c = c(p, n)$ such that

$$\tilde{v}(x) \leq c\, \langle re_n - x, \nabla\tilde{v}(x)\rangle \text{ for } x \in H \setminus \bar{B}(re_n, r/4). \tag{7.12}$$

Clearly (7.12) implies for given $\eta > 0$ that there exists $\tilde{c} = \tilde{c}(p, n, \eta) \geq 1$ such that

$$\tilde{c}^{-1}\frac{\tilde{v}(x)}{d(x,\partial H)} \leq |\nabla\tilde{v}(x)| \leq \tilde{c}\frac{\tilde{v}(x)}{d(x,\partial H)} \tag{7.13}$$

in $\tilde{H}(0, \eta) \setminus \bar{B}(0, 4r)$ where $\tilde{H}(0, \eta)$ denotes the nontangential approach region defined relative to $H, 0, \eta$. Using Lemma 4.28 we see that (7.13) actually holds in $H \setminus B(0, 4r)$ for $\eta = \eta(p, n, M) > 0$

sufficiently small. In view of (7.13) we can now apply Corollary 5.25 to $\tilde{v}, v_1$ in $H \setminus \bar{B}(0, 4r)$ with $r' = 4r$ (see also (6.19), (6.20)). We get for some $c = c(p, n, M) > 1$ and $a = a(p, n, M), 0 < a \leq 1/2$, that

$$\left| \log \left(\frac{\tilde{v}(x)}{v_1(x)} \right) - \log \left(\frac{\tilde{v}(y)}{v_1(y)} \right) \right| \leq c \left(\frac{r}{\min(|x|, |y|)} \right)^a \tag{7.14}$$

whenever $x, y \in B(0, 2r/\sigma) \setminus B(0, cr)$. Using (7.13), (7.14) and arguing as in the proof of Lemma 5.4 it follows that there exists $\hat{\kappa} = \hat{\kappa}(p, n, M)$ such that

$$c^{-1} \frac{v_1(x)}{d(x, \partial H)} \leq |\nabla v_1(x)| \leq c \frac{v_1(x)}{d(x, \partial H)} \tag{7.15}$$

in $H \cap U(\tilde{\eta}/2) \cap [B(0, \frac{r}{\sigma}) \setminus B(0, \hat{\kappa} r)]$. Finally from (7.15) we deduce the existence of $\sigma_1(p, n, M) > 0$ such that Lemma 7.6 is valid. The proof of Lemma 7.6 and Theorem 5 in the convex case is now complete. Moreover, the proof that a Martin boundary function is unique at a point where $\partial\Omega$ has a tangent plane is unchanged. In fact Corollary 6.22, defined relative to $\mathbf{R}^n \setminus \bar{\Omega}$, remains true. The proof of Theorem 5 is therefore complete. $\square$

7.2 Closing Remarks

We note that our arguments in Theorems 4 and 5 actually show that there exists a minimal positive p harmonic function u relative to $w \in \partial\Omega$ and a $\rho > 0$ such that (1.9) holds for u in $\Omega \cap B(w, \rho)$ (Theorem 4) or $(\mathbf{R}^n \setminus \bar{\Omega}) \cap B(w, \rho)$ (Theorem 5). Thus we could have proved a weaker version of Theorem 3. On the other hand the full generality of Theorem 3 is needed in Corollary 6.22. Moreover, Theorem 3 gives us more flexibility in studying the Martin boundary of more general domains such as the complement of a Cantor set. Also, we believe that the conclusions of Theorems 4 and 5 are most likely valid when Ω is a bounded Lipschitz domain in $\mathbf{R}^n, n \geq 3$, and that Theorem 3 (or even more general forms of it) could eventually lead to a proof in the Lipschitz case. In fact we tried to use this idea together with compactness and blow-up type arguments to prove Theorem 4. However we could not rule out the possibility that there exists, for a minimal positive p harmonic function u relative to w in $\partial\Omega$, a snowflake type surface, S, connecting a point in Ω to w with $\nabla u = 0$ on S. Furthermore, the PDE in (1.6), (1.7) degenerates at points where $\nabla u = 0$, and therefore at such points it is not clear how to prove even basic interior estimates for solutions. Finally we end this paper by outlining the proof of the following remark.

Remark 7.16. *The conclusions of Theorems 4 and 5 are valid for bounded Lipschitz domains $\Omega \subset \mathbf{R}^2$.*

To prove Remark 7.16, let $\{w_m\}$ be a sequence of points in Ω with $\lim\limits_{m \to \infty} w_m = w \in \partial\Omega$. Given $p, 1 < p < \infty$, let u_m be the p capacitary function for $\Omega \setminus \bar{B}(w_m, d(w_m, \partial\Omega)/2), m = 1, 2, \ldots$. Then from [L] we have $\nabla u_m \neq 0$ in $\Omega \setminus \bar{B}(w_m, d(w_m, \partial\Omega)/2)$ and ∇u_m is $k = k(p), 0 < k < 1$, quasi-regular in $\Omega \setminus \bar{B}(w_m, d(w_m, \partial\Omega)/2)$ for $m = 1, 2, \ldots$ Let $\tilde{u}_m = u_m/u_m(\hat{x})$ where $\hat{x}$ is a fixed point in Ω. From Lemmas 2.1-2.4 we see there exists a subsequence $\{\tilde{u}_{m_j}\}$ of $\{\tilde{u}_m\}$ such that $\{\tilde{u}_{m_j}\}, \{\nabla\tilde{u}_{m_j}\}$, converges to $u, \nabla u$ uniformly on compact subsets of Ω. Moreover u is p harmonic in Ω with continuous boundary value 0 on $\partial\Omega \setminus \{w\}$. Since $u(\hat{x}) = 1$, it follows that u is a minimal p harmonic function in Ω, relative to w. Now a uniformly convergent sequence of nonzero k quasi-regular mappings on compact subsets of Ω must be either $\equiv 0$ or not vanish anywhere on Ω. Thus

$$\nabla u(x) \neq 0 \text{ when } x \text{ in } \Omega. \tag{7.17}$$

From (7.17) and quasi-regularity of ∇u we see as in [BL, Lemma 2.9 (+)] that $f = \log |\nabla u| + \lambda$, $\lambda =$ constant, is a solution to a divergence form uniformly elliptic PDE in Ω for which positive solutions satisfy a Harnack inequality. Using this fact and Lemmas 2.1-2.3 it follows, as in [BL, Lemma 2.26], that

$$c^{-1}\frac{u(x)}{d(x,\partial\Omega)} \leq |\nabla u(x)| \leq c\frac{u(x)}{d(x,\partial\Omega)} \text{ in } \Omega, \tag{7.18}$$

for some $c = c(p,n,M) \geq 1$. From (7.18) we see that the hypotheses of Theorem 3 are satisfied in Ω. Hence Theorem 4 is true when $\Omega \subset \mathbf{R}^2$. Essentially the same proof also gives Theorem 5.

8 References

[AKSZ] H. Aikawa, T. Kilpeläinen, N. Shanmugalingam and X. Zhong, *Boundary Harnack principle for p harmonic functions in smooth Euclidean domains,* submitted.

[A] A. Ancona, *Principe de Harnack à la frontière et théorème de Fatou pour un opérateur elliptique dans un domain lipschitzien,* Ann. Inst. Fourier (Grenoble) **28** (1978), no. 4, 169-213.

[BL] B. Bennewitz and J. Lewis, *On the dimension of p harmonic measure,* Ann. Acad. Sci. Fenn. **30** (2005), 459-505.

[D] B. Dahlberg, *On estimates of harmonic measure,* Arch Rational Mech. Anal. **65** *(1977), 275-288.*

[JK] D. Jerison and C. Kenig, *Boundary behavior of harmonic functions in non-tangentially accessible domains,* Advances in Math. **46** (1982), 80-147.

[K] J. Kemper, *A boundary Harnack inequality for Lipschitz domains and the principle of positive singularities,* Comm. Pure Appl. Math. **25** (1972), 247-255.

[KP] C. Kenig and J. Pipher, *The Dirichlet problem for elliptic operators with drift term,* Publ. Mat. **45** (2001), no. 1, 199-217.

[L] J. Lewis, *Approximation of Sobolev functions in Jordan domains,* Ark. Mat. **25** (1987), 255-264.

[LN] J. Lewis and K. Nyström, *Boundary behavior for p-harmonic functions in Lipschitz and starlike Lipschitz ring domains,* to appear in Annales Scientifiques de L'Ecole Normale Supérieure.

[Li] G.M. Lieberman, *Boundary regularity for solutions of degenerate elliptic equations,* non-linear Anal. **12** (1988), no. 11, 1203-1219.

[W] J. M. Wu, *Comparisons of kernel functions, boundary Harnack principle and relative Fatou theorem on Lipschitz domains,* Ann. Inst. Fourier (Grenoble) **28** (1978), no. 4, 147-167.