ON d AND M PROBLEMS FOR NEWTONIAN POTENTIALS IN EUCLIDEAN n SPACE

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Dedicated to the memory of Albert Baernstein II, Ronald Gariepy, and Walter Hayman

Abstract. In this paper we first make and discuss a conjecture concerning Newtonian potentials in Euclidean n space which have all their mass on the unit sphere about the origin, and are normalized to be one at the origin. The conjecture essentially divides these potentials into subclasses whose criteria for membership is that a given member have its maximum on the closed unit ball at most M and its minimum at least d. It then lists the extremal potential in each subclass which is conjectured to solve certain extremal problems. In Theorem 1.1 we show existence of these extremal potentials. In Theorem 1.2 we prove an integral inequality on spheres about the origin, involving so called extremal potentials, which lends credence to the conjecture.

Contents

1. Introduction

Let $n \geq 2$ be a positive integer, $x = (x_1, x_2, \dots, x_n)$ a point in Euclidean n space, $\mathbb{R}^n$, and let $|x|$ denote the norm of x. Put $B(x,r) = \{x : |x| < r\}$ when $r > 0$. For fixed $n \geq 2$, let μ be a positive Borel measure on $\mathbb{S}^{n-1} = \{x : |x| = 1\}$ with $\mu(\mathbb{S}^{n-1}) = 1$. Next let $\mathcal{H}^{n-1}$ denote Hausdorff $n-1$ measure and let Φ be a non decreasing convex

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function on $\mathbb{R}$. Finally let $\mathcal{F}$ denote the family of potentials p satisfying

$$
\begin{aligned}
&(a) \quad p(x) = \int_{\mathbb{S}^{n-1}} |x-y|^{2-n} d\mu(y), x \in \mathbb{R}^n, \text{ when } n > 2, \\
(1.1) \qquad & \\
&(b) \quad p(x) = 2\int_{\mathbb{S}^{n-1}} \log \tfrac{1}{|x-y|} \, d\mu(y), x \in \mathbb{R}^2.
\end{aligned}
$$

Theorems 1.1 and 1.2 in this paper are based on my efforts to prove the following conjecture:

Conjecture 1 : *If $n \geq 3$, there is a 1 - 1 map from*

$$\{(\xi_1, \xi_2) : 0 \leq \xi_1 < \xi_2 \leq \pi\} \to \{(d, M) : 2^{2-n} \leq d < 1, 1 < M \leq \infty\}$$

for which there exists a potential $P(\cdot, d, M) \in \mathcal{F}$, satisfying

$$
(1.2) \qquad
\begin{array}{ll}
(a) & P(\cdot, d, M) \equiv M \text{ on } E_1 = \{x \in \mathbb{S}^{n-1} : \cos \xi_1 \leq x_1 \leq 1\}, \\
(b) & P(\cdot, d, M) \equiv d \text{ on } E_2 = \{x \in \mathbb{S}^{n-1} : -1 \leq x_1 \leq \cos \xi_2\}, \\
(c) & P(\cdot, d, M) \geq d \text{ in } \bar{B}(0,1), \text{ and } P(\cdot, d, M) \leq M \text{ in } \mathbb{R}^n, \\
(d) & P(0, d, M) = 1 \text{ and } P(\cdot, d, M) \text{ is harmonic in } \mathbb{R}^n \setminus (E_1 \cup E_2).
\end{array}
$$

Given $d = d(\xi_1, \xi_2), M = M(\xi_1, \xi_2)$, let

$$\mathcal{F}_d^M = \{p \in \mathcal{F} \text{ with } d \leq p \text{ in } \bar{B}(0,1) \text{ and } p \leq M \text{ in } \mathbb{R}^n \} \, .$$

If $0 < r < \infty$, and $p \in \mathcal{F}_d^M$, then

$$(1.3) \qquad \int_{\mathbb{S}^{n-1}} \Phi(p(ry)) d\mathcal{H}^{n-1}y \leq \int_{\mathbb{S}^{n-1}} \Phi(P(ry, d, M)) d\mathcal{H}^{n-1}y \, .$$

The analog of Conjecture 1 is true in $\mathbb{R}^2$. To briefly outline its proof we use complex notation. So $i = \sqrt{-1}, z = x + iy, \bar{z} = x - iy, e^{i\theta} = \cos\theta + i \sin\theta$, and $B(z_0, \rho) = \{z : |z - z_0| < \rho\}$. Let U denote the class of univalent (i.e, 1-1 and analytic) functions f satisfying $f(0) = 0, f'(0) = 1$, and for which $D = f(B(0,1))$ is starlike with respect to 0 (so each line segment connecting 0 to a point in D is also contained in D). If $f \in U$, then using the fact that $f(B(0,r))$ is also starlike, with respect to 0, one can show Arg $f(re^{i\theta})$ (i.e the principal argument of f) on $\partial B(0,r)$ is non decreasing as a function of θ so if $z = re^{i\theta} \in B(0,1)$, then

$$(1.4) \qquad -i\frac{d}{d\theta} \log f(re^{i\theta}) = \frac{d}{d\theta} [\text{Arg } f(re^{i\theta}) - i \log |f(re^{i\theta}|\,] = zf'(z)/f(z)$$

and thus Re $(zf'(z)/f(z)) \geq 0$, when $z \in B(0,1)$.

From (1.4) and the Poisson integral formula for $B(0,1)$ it follows (see [D]) that

$$(1.5) \qquad \frac{zf'(z)}{f(z)} = \int_{\mathbb{S}^1} \frac{1 + e^{-i\theta}z}{1 - e^{-i\theta}z} d\nu(e^{i\theta}), \;\; z \in B(0,1),$$

where ν is a positive Borel measure on $\mathbb{S}^1$ with $\nu(\mathbb{S}^1) = 1$. Note that if f is sufficiently smooth on $\bar{B}(0,1)$, then

$$\frac{d}{d\theta}\mathrm{Arg}\, f(e^{i\theta}) = 2\pi d\nu(e^{i\theta})/d\theta. \tag{1.6}$$

(1.6) implies that if $I \subset \partial B(0,1)$ is an arc and $f(I)$ lies on a ray through 0, then $\nu(I) \equiv 0$. Dividing (1.4), (1.5) by z and integrating we get

$$\log(f(z)/z) = -2\int_{\mathbb{S}^1} \log(1 - e^{-i\theta}z)\, d\nu(e^{i\theta}) \tag{1.7}$$

where log is the principal logarithm. From (1.7) we see that if $\nu = \mu$, then $p(z) = \log|f(z)/z|, z \in B(0,1)$, in (1.1) (b). Given $\tilde{d}, 1/4 \leq \tilde{d} < 1$, and $1 < \tilde{M} \leq \infty$, let $U_{\tilde{d}}^{\tilde{M}}$ denote starlike univalent functions f in U satisfying $\tilde{M} \geq |f(z)/z| \geq \tilde{d}$ in $B(0,1)$. Thus if $d = \log\tilde{d}$ and $M = \log\tilde{M}$, then for $n = 2$,

$$\mathcal{F}_d^M = \{p(z) = \log|f(z)/z| : f \in U_{\tilde{d}}^{\tilde{M}}\}$$

Using this fact one sees that the analogue of $P(\cdot, d, M)$ in Conjecture 1 for $n = 2$ is $P(z,d,M) = \log|G(z,\tilde{d},\tilde{M})/z|, z \in B(0,1)$, where $G(\cdot,\tilde{d},\tilde{M})$ maps $B(0,1)$ onto $D = D(\tilde{d},\tilde{M})$. If $1/4 < \tilde{d} < 1 < \tilde{M} < \infty$, then D is the bounded keyhole domain described as follows. For some $\tau = \tau(\tilde{d},\tilde{M}), 0 < \tau < \pi, \partial D$ is the union of the arcs :

$$\{\tilde{M}e^{i\theta} : -\tau \leq \theta \leq \tau\}, \{\tilde{d}e^{i\theta} : \tau \leq \theta \leq 2\pi - \tau\},$$

and the line segments, $[\tilde{d}e^{i\tau}, \tilde{M}e^{i\tau}], [\tilde{d}e^{-i\tau}, \tilde{M}e^{-i\tau}]$.

In [BL1] we showed for fixed $\tilde{d},\tilde{M}$, that $L(z,\tilde{d}.\tilde{M}) = \log(G(z,\tilde{d},\tilde{M})/z)$ maps $B(0,1)$ univalently onto a convex domain containing 0 and if $f \in U_{\tilde{d}}^{\tilde{M}}$, then $l(z) = \log(f(z)/z)$ is subordinate to $L(z,\tilde{d},\tilde{M})$. That is, $L^{-1}\circ l$ maps $B(0,1)$ into $B(0,1)$. This result was in fact a corollary of a much more general subordination theorem for Mocanu convex univalent functions that are bounded above and below in the unit disk. Our proof used a contradiction type argument and the Hadamard - Julia variational formulas to determine the solutions to a certain class of extremal problems. Runge's theorem then gave subordination in the given class of Mocanu convex functions. Conjecture 1 for $n = 2$, follows from the relationship between p, P, and f, G, as well as properties of subordination (see for example [BL2]).

Let e_i denote the point in $\mathbb{R}^n$ with 1 in the i th position and zeroes elsewhere. Conjecture 1 is true in $\mathbb{R}^n, n > 2$, when $\xi_2 = \pi, 0 \leq \xi_1 < \pi$, so $E_2 = \{-e_1\}$, and $1 < M \leq \infty$. It was proved in [GL] . Our proof used a maximum principle for the celebrated Baernstein * function (see [H2] or [BDL]), defined as follows: Given $x \in \mathbb{R}^n \setminus \{0\}$, introduce spherical coordinates, r, θ by $r = |x|, x_1 = r\cos\theta$, $0 \leq \theta \leq \pi$. Let l be a locally integrable real valued function on $A = \{x \in \mathbb{R}^n : r_1 < |x| < r_2\}$, and set

$$l^*(r,\theta) = \sup_{\Lambda} \int_{\Lambda} l(ry)d\mathcal{H}^{n-1}y \tag{1.8}$$

where the supremum is taken over all Borel measurable sets $\Lambda \subset \mathbb{S}^{n-1}$ with

$$\mathcal{H}^{n-1}(\Lambda) = \mathcal{H}^{n-1}(\{y \in \mathbb{S}^{n-1} : y_1 \geq \cos\theta\}).$$

One can show that (1.3) in Conjecture 1 is equivalent to

$$p^*(r,\theta) \leq P^*(r,\theta,d,M), \text{ whenever } 0 < r < \infty,\ 0 \leq \theta \leq \pi. \tag{1.9}$$

For a proof of this equivalence see section 9.2 in [H2].

For $n = 2$, Professor Baernstein in [B] showed that if l is subharmonic in A, then l^* in (1.8) is subharmonic in $I = \{z = re^{i\theta} : 0 < \theta < \pi, r_1 < r < r_2\}$. Moreover if

- (a) $l(re^{i\theta}) = l(re^{-i\theta})$, $0 \leq \theta \leq \pi$,
- (b) $l(re^{i\theta})$ is non increasing on $[0,\pi]$, for fixed $r, r_1 < r < r_2$,
- (c) l is harmonic in A,

then l^* is harmonic in I. In $\mathbb{R}^n, n > 2$, $l^*(r,\theta)$ need not be subharmonic in I even when l is harmonic in A. Instead we proved (see also [BT]) the following maximum principle: Suppose l is subharmonic and L is harmonic in

$$\Omega = \bigcup_{r_1<r<r_2} \{x : |x| = r, x_1 > \cos(\theta(r))\}, 0 < \theta(r) \leq \pi, r \in (r_1,r_2),$$

with $L = L(r,\theta)$ symmetric about the x_1 axis and $L(r,\cdot)$ non increasing on $[0,\theta(r)), r_1 < r < r_2$. Then either $l^* \leq L^*$ in Ω or

$$(l^* - L^*)(x) < \sup_{y\in\Omega}(l^* - L^*)(y), \text{ whenever } x \in \Omega.$$

To briefly outline the proof of Conjecture 1 in [GL], for given $\xi_2 = \pi$, $0 < \xi_1 < \pi$, existence of $P = P(\cdot,d,M)$, satisfying (1.2), can be deduced from a working knowledge of such tools for harmonic functions as (a) Wiener's criteria for solutions to the Dirichlet problem, (b) the maximum principle for harmonic functions, (c) the Riesz representation formula for superharmonic functions, and (d) invariance of the Laplacian under reflection about planes containing zero (see the proof of Theorem 1.1 for more elaborate details). Let $\Omega = \{x : -x \in \mathbb{R}^n \setminus (E_1 \cup [0,\infty])\}$, $L(x) = -P(-x,d,M)$, and $l(x) = -p(-x)$, for $x \in \mathbb{R}^n$. If $\sup_{x\in\Omega}(l^* - L^*)(x) > 0$, then from subharmonicity of $-p$, harmonicity of P in Ω, decay of both potentials at ∞, and the above maximum principle, it follows that there exists $y \in \partial\Omega \cap \mathbb{S}^{n-1}$ with spherical coordinates $|y| = 1, \hat\theta$, $\pi - \xi_1 \leq \hat\theta \leq \pi$, and $l^*(1,\hat\theta) - L^*(1,\hat\theta) > 0$. Since $l \geq -M$ and $L = -M$ on $-E_1 = \{x : -x \in E_1\}$, we then obtain

$$0 < (l^* - L^*)(1,\hat\theta) \leq l^*(1,\pi) - L^*(1,\pi) = 0.$$

From this contradiction we conclude that $l^* \leq L^*$ in $\mathbb{R}^n$. Next using $l^*(r,\pi) = L^*(r,\pi), 0 \leq r < \infty$, and whenever $q \in \{l, L\}$, that

$$(-q)^*(r,\theta) = q^*(r,\pi-\theta) - q^*(r,\pi), 0 < r < \infty, 0 \leq \theta \leq \pi,$$

we get $p^* \leq P^*$ in $\mathbb{R}^n$, which as mentioned in (1.9) implies (1.3).

We have not been able to prove (1.3) in Conjecture 1 for any other values of ξ_1, ξ_2. However in [L] we used a mass moving method in [S] to show that if $\xi_1 = 0, 0 < \xi_2 < \pi$, and $p \in \mathcal{F}_d^{\infty}$, then for $0 < r \leq 1$,

$$\max_{x \in B(0,r)} p(x) \leq P(r, 0, d, \infty) \text{ and } \min_{x \in B(0,r)} p(x) \geq P(r, \pi, d, \infty). \tag{1.10}$$

Moreover in this paper we prove the first part of our conjecture:

Theorem 1.1. *If $n \geq 3$, there is a 1-1 map from*

$$\{(\xi_1, \xi_2) : 0 \leq \xi_1 < \xi_2 \leq \pi\} \rightarrow \{(d, M) : 2^{2-n} \leq d < 1 < M \leq \infty\},$$

for which there exists a potential $P = P(\cdot, d, M) \in \mathcal{F}$ satisfying (1.2).

Also we prove,

Theorem 1.2. *Given $P(\cdot, d, M)$ as in Theorem 1.1. If $0 < \xi_1 < \pi, \xi_2 = \pi$, and $p \in \mathcal{F}_d^M$, then*

$$\int_{\mathbb{S}^{n-1}} \Phi(p(ry)) d\mathcal{H}^{n-1} y \leq \int_{\mathbb{S}^{n-1}} \Phi(P(ry, d, \infty)) d\mathcal{H}^{n-1} y. \tag{1.11}$$

As for the plan of this paper, in section 2 we set the stage for the proof of Theorems 1.1 and 1.2 by stating and/or proving several definitions and lemmas. In section 3 we prove Theorem 1.1. In section 4 we prove Proposition 4.1, a rather tedious calculation of mixed partials for a certain function. In section 5, Proposition 4.1 is used to prove Theorem 1.2. After each theorem we make remarks and queries.

2. Notation, Definitions, and Basic Lemmas

Throughout this paper $c(a_1, \dots, a_m)$ denotes a positive constant ≥ 1, depending only on $a_1, \dots, a_n$, and n. Also $A \approx B$ means A/B is bounded above and below by positive constants whose dependence will be stated. As in section 1 let dx denote Lebesgue measure on $\mathbb{R}^n, \bar{F}$ the closure of F, $d(x, F)$ the distance from x to the set F, e_i the point in $\mathbb{R}^n$ with 1 in the i th position and zeroes elsewhere, $\langle \cdot, \cdot \rangle$ the inner product in $\mathbb{R}^n$, $B(x, r) = \{y \in \mathbb{R}^n : |y - x| < r\}$, $\mathcal{H}^k =$ Hausdorff k measure, in $\mathbb{R}^n, 0 < k \leq n$.

Definition 2.1. *If O is an open set in $\mathbb{R}^n, n \geq 3$, and $F \subset O$ is compact, then the Newtonian capacity of F, denoted $C(F)$, is defined to be*

$$C(F) = \inf \int_{\mathbb{R}^n} |\nabla \phi|^2 dx, \tag{2.1}$$

where $\nabla \phi$ denotes the gradient of ϕ and the infimum is taken over all $\phi \in C_0^{\infty}(\mathbb{R}^n)$ with $\phi \equiv 1$ on F.

Remark 2.2. *Recall that a bounded open set $G \subset \mathbb{R}^n$ is said to be a Dirichlet domain if for any continuous real valued function q defined on ∂G, there exists a harmonic function Q in G with*

$$\lim_{x \rightarrow y} Q(x) = q(y), \text{ whenever } y \in \partial G.$$

Wiener's criteria *for a bounded open set G to be a Dirichlet domain states: If*

$$\int_0^1 r^{1-n} C(B_r(y)\cap\partial G)dr = \infty \text{ for all } y\in\partial G, \tag{2.2}$$

then G is a Dirichlet domain.

Given $\xi_1,\xi_2, 0<\xi_1<\xi_2<\pi$. Let (as in Theorem 1.1),

$$E_1=\{x\in\mathbb{S}^{n-1}: x_1\geq\cos\xi_1\}, E_2=\{x\in\mathbb{S}^{n-1}: x_1\leq\cos\xi_2\}.$$

Put $\Omega(\xi_1,\xi_2)=\mathbb{R}^n\setminus(E_1\cup E_2)$. For $i=1,2$, one can show (see [H1]) that for $i=1,2$,

$$r^{n-2}\leq c(\xi_1,\xi_2)\, C(B(x,r)\cap E_i) \text{ whenever } x\in E_i \text{ and } 0<r\leq\min(\xi_1,\pi-\xi_2).$$

Using Wiener's criteria and the boundary maximum principle for harmonic functions it follows that given $R>2$ there exists $\omega_{i,R}$ harmonic in $B(0,R)\cap\Omega(\xi_1,\xi_2)$ with continuous boundary values $\omega_{i,R}=1$ on E_i and $\omega_{i,R}\equiv 0$ on $E_j, j\neq i$. Using the boundary maximum principle for harmonic functions we find that if $R_1<R_2$, then $|\omega_{i,R_1}-\omega_{i,R_2}|\leq R_1^{2-n}$. From this fact we deduce that $\lim_{R\to\infty}\omega_{i,R}=\omega_i$ uniformly on compact subsets of $\mathbb{R}^n$ and $\omega_i, i=1,2,$ is harmonic in $\Omega(\xi_1,\xi_2)$ with continuous boundary values 1 on E_i and 0 on $E_j, j\neq i$. Also $\omega_i(x)\to 0$ as $|x|\to\infty$ and ω_i is superharmonic in an open set containing E_i, as well as, subharmonic in an open set containing $E_j, j\neq i$. From these observations and the Riesz representation theorem for sub -super harmonic functions (see [H1]), it follows that there exists finite positive Borel measures $\nu_{i,j}, i,j=1,2,$ with the support of $\nu_{i,1}\subset E_i$ while the support of $\nu_{i,2}$ is contained in $E_j, j\neq i$. Moreover

$$\omega_i(x)=\int_{E_i}|x-y|^{2-n}d\nu_{i,1}(y)-\int_{E_j}|x-y|^{2-n}d\nu_{i,2}(y), x\in\mathbb{R}^n, \tag{2.3}$$

(once again $E_j\neq E_i$).

Next for $i=1,2$, let

$$\lambda_i(\theta_1)=(\nu_{i,1}+\nu_{i,2})(\{x\in\mathbb{S}^{n-1}: x_1\geq\cos\theta_1\})$$

and note that ω_i has boundary values that are symmetric about the x_1 axis. Thus from the boundary maximum principle for harmonic functions and invariance of the Laplacian under rotations we have : $\omega_i(x)=\omega_i(r,\theta)$ for $i=1,2$, whenever $r=|x|, x_1=r\cos\theta$. Using this fact and arguing as in section 2 of [L] we get the Lebesgue - Stieltjes integral :

$$\omega_i(x)=\omega_i(r,\theta)=c_n\int_0^\pi h(r,\theta,\theta_1)d\lambda_i(\theta_1), x\in\mathbb{R}^n. \tag{2.4}$$

Here

$$h(r,\theta,\theta_1)=\int_0^\pi(1+r^2-2r\psi(\theta,\theta_1,\phi))^{1-n/2}\sin^{n-3}\phi\, d\phi \tag{2.5}$$

with

$$\psi(\theta,\theta_1,\phi)=\cos\theta\cos\theta_1+\cos\phi\sin\theta\sin\theta_1 \tag{2.6}$$

and c_n is chosen so that $c_n h(0,\theta,\theta_1)\equiv 1$. With this notation we prove,

Lemma 2.3. *Given* $r > 0, \frac{\partial \omega_1(r,\theta)}{\partial\theta} < 0$, *and* $\frac{\partial \omega_2(r,\theta)}{\partial\theta} > 0$ *on* $(0,\pi)$ *when* $0 < r < \infty, r \neq 1$, *and these inequalities also hold when* $r = 1, \theta \in (\xi_1, \xi_2)$.

This lemma is essentially trivial in $\mathbb{R}^2$ but perhaps not so obvious in $\mathbb{R}^n, n > 2$, so we give some details.

Proof. To begin the proof of Lemma 2.3 let $\tilde{\Sigma}$ be a plane containing the origin with unit normal, $\mathbf{n}$, satisfying $\langle e_1, \mathbf{n}\rangle = -\sin\theta$ for some $\theta, 0 < \theta < \pi$. Let

$$H_1 = \{x : \langle x, \mathbf{n}\rangle < 0\} \text{ and } H_2 = \{x : \langle x, \mathbf{n}\rangle > 0\}.$$

Clearly H_1 contains e_1 and H_2 contains $-e_1$. If $x \in \bar{H}_1$ let $\tilde{x} \in \bar{H}_2$ be the reflection of x in $\tilde{\Sigma}$ defined by $\tilde{x} = x - 2\langle x, \mathbf{n}\rangle\, \mathbf{n}$. We claim that if

$$\begin{aligned} &(a) \quad \tilde{x} \in \bar{H}_2 \cap E_1 \text{ then } x \in \bar{H}_1 \cap E_1 \\ &\text{ while if} \\ &(b) \quad x \in \bar{H}_1 \cap E_2 \text{ then } \tilde{x} \in \bar{H}_2 \cap E_2. \end{aligned} \tag{2.7}$$

To prove our claim we assume, as we may, that $\langle e_i, \mathbf{n}\rangle = 0, 3 \leq i \leq n$, which is permissible since E_1, E_2 are symmetric about the x_1 axis. Then $\mathbf{n} = -\sin\theta\, e_1 + \cos\theta\, e_2$ for some $0 < \theta < \pi$ and $\hat{e} = \cos\theta\, e_1 + \sin\theta\, e_2 \in \tilde{\Sigma}$. We note that if $x \in H_1 \cap \mathbb{S}^{n-1}$, then $x = \alpha\mathbf{n} + \beta\hat{e} + \gamma e'$ where $\mathbf{n}, \hat{e}, e'$ are orthogonal unit vectors with $\langle e', e_1\rangle = 0$ and $\{\alpha, \beta, \gamma\}$ are any real numbers with $\alpha < 0$ and $\alpha^2 + \beta^2 + \gamma^2 = 1$. Also $\tilde{x} = -\alpha\mathbf{n} + \beta\hat{e} + \gamma e'$ and $H_2 \cap \mathbb{S}^{n-1} = \{\tilde{x} : x \in H_1 \cap \mathbb{S}^{n-1}\}$. Using this notation we see that $\tilde{x}_1 = -|\alpha|\sin\theta + \beta\cos\theta \leq x_1 = |\alpha|\sin\theta + \beta\cos\theta$ with strict inequality unless $\alpha = 0$. Thus (2.7) (a) is true. (2.7) (b) is proved similarly so we omit the details. To finish the proof of Lemma 2.3 we observe from (2.7) (a), the continuous boundary values of ω_1, and Harnack's inequality for positive harmonic functions that either $\omega_1(x) - \omega_1(\tilde{x}) > 0, x \in H_1$, or $\Omega(\xi_1, \xi_2)$ is symmetric about $\tilde{\Sigma}$. However this cannot happen if $\langle \mathbf{n}, e_1\rangle = -\sin\theta, \theta \in (0,\pi)$, as $E_1 \cap \mathbb{R}^2$ is not even symmetric about the line through the origin and $\hat{e}$. Similarly $\omega_2(x) - \omega_2(\tilde{x}) < 0$ in H_1. Lemma 2.3 now follows from these inequalities and the Hopf boundary maximum principle for harmonic functions (see [E]). □

Finally in this section we state

Lemma 2.4. $\dfrac{\omega_1}{1-\omega_2}(x) \to \hat{\gamma}, 0 < \hat{\gamma} \leq 1,$ *as* $x \in \Omega(\xi_1, \xi_2) \to y \in E_2$ *with* $y_1 = \cos\xi_2$.

Proof. we note that

(2.8)
$1 - \omega_2 = \omega_1 + \tilde{\omega}$ where $\tilde{\omega}$ is harmonic in $\Omega(\xi_1, \xi_2)$ with continuous boundary value 0 on $\partial\Omega(\xi_1, \xi_2)$ and $\tilde{\omega}(x) \to 1$ as $x \to \infty$.

Lemma 2.4 follows from (2.8) and essentially a boundary Harnack inequality in [KJ] even though $\Omega(\xi_1.\xi_2) \cap B(y, \rho)$, is not an NTA domain for any $\rho > 0$. To give a few details, if $z \in \mathcal{S}^{n-1}$ and $\rho = (y_1 - z_1)/100 > 0$, then $\omega_1/(1-\omega_2) \approx c(\xi_1, \xi_2)$ in $B(z, \rho) \setminus E_2$ is easily shown using barriers. Applying Harnack's inequality for positive harmonic functions, we then obtain this inequality in $\Omega(\xi_1,\xi_2)\cap\{x : |x_1-y_1|\leq \rho/2\}$. After that one shows for some $c' = c'(\xi_1,\xi_2)\geq 1$ that

$$\operatorname{osc}_{B(y,r/2)} \frac{\omega_1}{1-\omega_2} \leq (1-1/c') \operatorname{osc}_{B(y,r)} \frac{\omega_1}{1-\omega_2} \text{ for } 0<r\leq \rho/2.$$

Here $\operatorname{osc}_{B(y,r/2)}$ denotes oscillation on $B(y,r/2)\cap\Omega(\xi_1,\xi_2)$. An iterative argument then gives Hölder continuity of $\frac{\omega_1}{1-\omega_2}$ in $\Omega(\xi_1,\xi_2)\cap B(y,\rho/2)$. Thus $\hat\gamma$ exists and $\hat\gamma\leq 1$ since $\tilde\omega>0$ in $\Omega(\xi_1,\xi_2)$. □

3. Proof of Theorem 1.1

Proof. In the proof of existence for $P(\cdot,d,M)$ in Theorem 1.1 we assume that $0<\xi_1<\xi_2<\pi$, as existence of $P(\cdot,d,M)$ when $E_1=\{e_1\}$ was proved in [L] and if $E_2=\{-e_1\}$, $P(\cdot,d,M)=M\omega_1$.

To begin the proof of this theorem let $\gamma=\inf_{\theta\in(\xi_1,\xi_2)}\frac{\omega_1(1,\theta)}{1-\omega_2(1,\theta)}<1$. We assert that $\gamma=\hat\gamma$ where $\hat\gamma$ is as in Lemma 2.4. Indeed since $\omega_1<1-\omega_2$ in $\Omega(\xi_1,\xi_2)$ and $\omega_1, 1-\omega_2$, have continuous boundary value 0 when $\xi_2\leq\theta\leq\pi$, we see from (2.8) that either

$$1>\gamma=\frac{\omega_1(1,\theta_0)}{(1-\omega_2)(1,\theta_0)} \text{ for some } \theta_0\in(\xi_1,\xi_2)\text{and/or } \gamma=\hat\gamma. \tag{3.1}$$

To show the first possibility cannot occur, observe that it implies $\gamma\omega_2(1,\theta_0)+\omega_1(1,\theta_0)=\gamma$. Also, $\gamma\omega_2+\omega_1$ is harmonic in $\Omega(\xi_1,\xi_2)$ with continuous boundary value γ on E_2 and continuous boundary value 1 on E_1. Using the fact that γ is the minimum and 1 the maximum of $\gamma\omega_2+\omega_1$ on $\mathbb{S}^{n-1}$ we can essentially repeat the proof of Lemma 2.3 to arrive at

$$\frac{\partial(\gamma\omega_2+\omega_1)(r,\theta)}{\partial\theta}<0 \text{ when } 0<\theta<\pi, r\neq 0,1, \text{ and for } r=1,\theta\in(\xi_1,\xi_2). \tag{3.2}$$

Thus $(\gamma\omega_2+\omega_1)(1,\cdot)$ is strictly decreasing on $[\theta_0,\xi_2)$, a contradiction to (3.1).

Let $V(x)=a(\gamma\omega_2+\omega_1)$, where $a>0$ is chosen so that $V(0)=1$. We shall show that $V=P(\cdot,d,M)$ where $M=a, d=a\gamma$ to complete the proof of Theorem 1.1 except for showing the map is 1-1. For this purpose we note from (2.3) that

$$V(x)=\int_{\mathbb{S}^{n-1}}|x-y|^{2-n}d\sigma(y), x\in\mathbb{R}^n, \tag{3.3}$$

where σ is a signed measure with support in $\mathbb{S}^{n-1}$ and of finite total variation. Moreover since $V\leq M$ in $\mathbb{R}^n$ with $V\equiv M$ on E_1, we see that V is superharmonic in an open set containing E_1 and consequently (see [H1]), $\sigma|_{E_1}$ is a positive Borel measure. It remains to prove $\sigma|_{E_2}$ is a positive Borel measure in order to conclude existence of $P(\cdot,d,M)$ in Theorem 1.1. For this purpose let $\tau\in\mathbb{S}^{n-1}$, with $\tau_1=\cos\theta$. Let $x=r\tau, 0<r<1$. Differentiating (3.3) we obtain the Poisson integral (see [H1])

$$r^{2-n/2}\,\frac{\partial(r^{n/2-1}V(r\tau))}{\partial r}=(n/2-1)\int_{\mathbb{S}^{n-1}}\frac{1-r^2}{|r\tau-y|^n}d\sigma(y). \tag{3.4}$$

From properties of the Poisson integral (see [H1]), and $V(x) = V(r,\theta)$, we deduce from (3.4) that

$$\lim_{r\longrightarrow 1} \frac{\partial (r^{n/2-1} V(r,\theta))}{\partial r} = \delta_n (n/2-1) \frac{d\sigma}{d\mathcal{H}^{n-1}} = g(\theta) \tag{3.5}$$

for $\mathcal{H}^1$ almost every $\theta \in (0,\pi)$ where $\delta_n^{-1} = \mathcal{H}^{n-1}(\mathbb{S}^{n-1})$. Moreover,

$$g(\theta) = 0 = (n/2-1)V(1,\theta) + \frac{\partial V}{\partial r}(1,\theta), \xi_1 < \theta < \xi_2, \tag{3.6}$$

since V is harmonic in $\Omega(\xi_1,\xi_2)$. Also since $\partial\Omega(\xi_1,\xi_2)$ is smooth at points $y \in E_1 \cup E_2$ with $y \neq \cos(\xi_1), \cos(\xi_2)$, it follows from Schauder type arguments (see [E]) that g is infinitely differentiable on $[0,\pi]\setminus\{\xi_1,\xi_2\}$. From this observation and (3.2) we deduce that

$$\frac{d - V(r,\theta_1)}{1-r} \leq \frac{d-V(r,\theta_2)}{1-r} \quad \text{for } \xi_2 \leq \theta_1 < \theta_2 \leq \pi \text{ and } r<1. \tag{3.7}$$

From (3.7), (3.5), and the mean value theorem from calculus we arrive at

$$\liminf_{r\longrightarrow 1} \left[\frac{d-V(r,\xi_2)}{1-r}\right] \leq -d\,(n/2-1) + g(\theta) \text{ for } \theta \in (\xi_2,\pi].$$

Thus to show that $\sigma|_{E_2} \geq 0$, it suffices to show

$$\liminf_{r\longrightarrow 1} \left[\frac{d-V(r,\xi_2)}{1-r}\right] \geq -d\,(n/2-1). \tag{3.8}$$

To prove (3.8) we need some boundary Harnack inequalities in [DS1] (see also [DS2]). To set the stage for these inequalities put $y = (y_1, y')$ where

$$y' = (y_2,\dots,y_n) \text{ and } y = -\frac{x-e_1}{|x-e_1|^2} - e_1/2 = T(x) \text{ for } x \in \mathbb{R}^n. \tag{3.9}$$

Then T maps

$$\begin{aligned}
&(a) \quad B(0,1) \text{ onto } \{y : y_1 > 0\},\\
&(b) \quad (\mathbb{R}^n \cup \infty) \setminus \bar{B}(0,1) \text{ onto } \{y : y_1 < 0\},\\
&(c) \quad \mathbb{S}^{n-1} \text{ onto } \{y : y_1 = 0\} \cup \infty.\\
&(d) \quad E_1 \text{ onto } \{y : y_1 = 0, |y'| \geq (1/2)\cot(\xi_1/2)\} = F_1.\\
&(e) \quad E_2 \text{ onto } \{y : y_1 = 0, |y'| \leq (1/2)\cot(\xi_2/2)\} = F_2,
\end{aligned} \tag{3.10}$$

We note that

$$x = T^{-1}(y) = e_1 - \frac{y+e_1/2}{|y+e_1/2|^2} \text{ when } y \in \mathbb{R}^n \cup \infty.$$

Using this note, the Kelvin transformation (see [H1]), and translation invariance of harmonic functions we find that if $\hat{u}$ is harmonic at x, then

$$\hat{v}(y) = |y+e_1/2|^{2-n}\ \hat{u}(T^{-1}(y)),$$

is harmonic at $y \in \mathbb{R}^n$ (i.e, in a neighborhood of y). From this deduction we conclude for fixed $\xi_1, \xi_2, 0 < \xi_1 < \xi_2 < \pi$, that if $w_i(y) = |y + e_1/2|^{2-n}\omega_i(x), i = 1, 2$, then

$$
\begin{aligned}
&(a) \quad w_1, w_2 \text{ are continuous on } \mathbb{R}^n \text{ and harmonic in } \mathbb{R}^n \setminus (F_1 \cup F_2),\\
&(b) \quad w_1 \equiv 0 \text{ on } F_2 \text{ and } w_1(y) = |y + e_1/2|^{2-n}, y \in F_1,\\
&(c) \quad w_2 \equiv 0 \text{ on } F_1 \text{ and } w_2(y) = |y + e_1/2|^{2-n},\ y \in F_2,\\
&(d) \quad w_1(y), w_2(y) \to 0 \text{ as } y \to \infty,\\
&(e) \quad w_i(\pm\, e_1/2) = \omega_i(0), i = 1, 2,\\
&(f) \quad w_i(y_1, y') = w_i(-y_1, y'), y \in \mathbb{R}^n, i = 1, 2,\\
&(g) \quad w_i(y) = w_i(y_1, |y'|) = w_i(y_1, \rho), y \in \mathbb{R}^n, i = 1, 2.
\end{aligned} \tag{3.11}
$$

From (3.9) we see that if $r = |x|, x_1 = r\cos\xi_2$, then

$$
\begin{aligned}
&(a) \quad y_1 = \frac{(1/2)(1 - r^2)}{1 + r^2 - 2r\cos\xi_2} \text{ and } \rho = |y'| = \frac{r\sin\xi_2}{1 + r^2 - 2r\cos\xi_2},\\
&(b) \quad 1 - r^2 = \frac{2y_1}{|y + e_1/2|^2}.
\end{aligned} \tag{3.12}
$$

Let $w_3(y) = |y + e_1/2|^{2-n} - w_2(y), y \in \mathbb{R}^n$. Then $w_3 = w_3(y_1, \rho)$ is harmonic in $\mathbb{R}^n \setminus (F_1 \cup F_2 \cup \{-e_1/2\})$ with continuous boundary values $\equiv 0$ on F_2 and $\equiv |y + e_1/2|^{2-n}$ on F_1. We now are in a position to use a boundary Harnack inequality proved in Theorem 3.3 of [DS1] tailored to our situation. Let $\hat\rho = \rho - (1/2)\cot(\xi_2/2)$, and set $s = s(y) = \sqrt{\hat\rho^2 + y_1^2}$. Next put

$$W_0(y) = \frac{1}{\sqrt{2}}\sqrt{s + \hat\rho}, \text{ whenever } y \in \mathbb{R}^n.$$

Note that $s(y)$ denotes the distance from y to $\partial' F_2$, and $\hat\rho$ denotes the signed distance from y' to $\partial' F_2$. Here $\partial' F_2$ denotes the boundary of F_2 relative to $\mathbb{R}^{n-1}$.

Theorem A Given k a positive integer and $i = 1, 3$, there exists $a_{i,j}(\hat\rho), b_j(\hat\rho), j = 1, 2, \ldots$, infinitely differentiable whenever

$$|\hat\rho| < \hat\lambda = \min[(1/10)\cot(\xi_2/2), (1/10)\cot(\xi_1/2) - (1/10)\cot(\xi_2/2)]$$

with $a_{i,0} > 0$ for $i = 1, 3$, and for which as $(y_1, \rho) \to (0, (1/2)\cot(\xi_2/2))$

$$\begin{aligned} &(a) \quad w_1(y_1, \rho) = W_0(y)\,(\sum_{j=0}^{k} a_{1,j}(\hat\rho) s^j\,) + O(W_0(y) s^{k+1/2}) \\ (3.13) \qquad &(b) \quad w_3(y_1, \rho) = W_0(y)(\sum_{j=0}^{k} a_{3,j}(\hat\rho) s^j) + y_1 \sum_{0 \le j \le (k-1)/2} b_j(\hat\rho) y_1^{2j} \\ &\qquad + \ O(W_0(y)\, s^{k+1/2}). \end{aligned}$$

Constants in the big O terms depend on k, n, ξ_2, ξ_1, and the $C^{k+3/2}$ norms of $a_{i,j}(\hat\rho), b_j(\hat\rho)$, for $i = 1, 3, 1 \le j \le k$, on $[0, \hat\lambda]$. We note that w_1, w_3 in (3.13) $(a), (b)$ have slightly different expansions since w_1, is even in y_1 while w_3 is not. From Lemma 2.4, $\lambda = d/M$, (3.10), and (3.11), we find that as $(y_1, \rho) \to (0, (1/2)\cot(\xi_2/2))$ through coordinates of points not in F_2,

$$\frac{w_1(y_1, \rho)}{w_3(y_1, \rho)} \to d/M.$$

which in view of (3.13) $(a), (b)$ implies that

$$a_{1,0}(0) = (d/M) a_{3,0}(0). \tag{3.14}$$

Let $y = (y_1, \rho)$, be as in (3.12). From (3.12) (a) we find that as $r \to 1$,

$$y_1 \approx 1 - r,\ \rho = \frac{r \sin \xi_2}{(1-r)^2 + 4r \sin^2(\xi_2/2)} = O(y_1^2) + (1/2)\cot(\xi_2/2), \tag{3.15}$$

where constants depend only on ξ_1, ξ_2, n, provided $r \ge 1/2$. From (3.11) (f), (3.13) $(a), (b)$, (3.14), and $w_2(y) = |y + e_1/2|^{2-n} - w_3(y)$, we find that if $\bar y = (-y_1, \rho)$, $y_0 = (y + \bar y)/2$, then

$$\begin{aligned} & U(y) = M w_1(y) + d w_2(y) = \\ (3.16) \qquad & (d/2)\,(|y + e_1/2|^{2-n} + |\bar y + e_1/2|^{2-n}) + O(y_1^{3/2}) \\ & = U(y_0) + O(y_1)^{3/2} = d|y_0 + e_1/2|^{2-n} + O(y_1)^{3/2} \text{ as } y_1 \to 0. \end{aligned}$$

Finally we prove (3.8) and thus finish the proof of Theorem 1 up to showing the map is 1-1. Let $\tilde U(y) = |y + e_1/2|^{n-2} U(y)$. From (3.12) (b) , (3.11) (f), (3.16), we see that

$$\begin{aligned} & \liminf_{r \to 1_-} \frac{d - V(r, \xi_2)}{1 - r} = 2 \liminf_{r \to 1_-} \frac{d - V(r, \xi_2)}{1 - r^2} \\ (3.17) \qquad & = -\limsup_{y_1 \to 0} \left(|y + e_1/2|^2 \frac{[\tilde U(y) - \tilde U(y_0)]}{2y_1} \right) = \\ & -(1/2)|y_0 + e_1/2|^2 U(y_0) \tfrac{\partial}{\partial y_1}(|y + e_1/2|^{n-2})(y_0) = -\frac{(n-2)d}{2}. \end{aligned}$$

From (3.17) we conclude (3.8).

Finally showing the map $(\xi_1,\xi_2)\to(d,M)$ is 1-1 follows easily from the maximum principle for harmonic functions. Indeed suppose $(\xi_1,\xi_2),(\xi_1',\xi_2')$ are both mapped into (d,M). Define E_1,E_2 and E_1',E_2' relative to ξ_1,ξ_2, and ξ_1',ξ_2', respectively. Let P,P' be the corresponding potentials in Theorem 1.1. Then either $E_1\subset E_1'$ or vice versa and likewise for E_2,E_2'. If $E_1'\subset E_1$, suppose first that $E_2'\subset E_2$ Then using the fact that P is superharmonic and P' is harmonic in $\mathbb{R}^n\setminus(E_1'\cup E_2')$ as well as that both potentials have the same boundary values, we obtain from the maximum principle for harmonic functions that $P'(0)=1<P(0)=1$ unless $E_1=E_1',E_2=E_2'$. If $E_2\subset E_2'$, we compare boundary values of P,P' on $E_1\cup E_2'$. Using the maximum principle for harmonic functions and (1.2) (c) we once again get a contradiction to (1.2) (d) unless $P=P'$. Interchanging the roles of P,P' we obtain $P=P'$ in all cases. The proof of Theorem 1.1 is now complete. □

Remark 3.1. *More sophisticated arguments in* [DS2] *yield that $P(\cdot,d,M)$ is $C^{1,1/2}$ in an open neighborhood of $\{x\in\mathbb{S}^{n-1}:x_1=\cos\xi_2\}$. Fix $\xi_2\in(0,\pi)$ and let ξ_1 vary in $(0,\xi_2)$. Then from* (2.8)*, the maximum principle for harmonic functions, and $\hat\gamma=d/M$ in Lemma 2.4 we see that*

(3.18) $$d/M \text{ is increasing as a function of } \xi_1 \text{ to say } d'/M'.$$

Using this fact it is easily seen from Theorem 1 and the maximum principle for harmonic functions that M decreases as a function of ξ_1 so $M'<M$. Moreover $d',M'\to1$ as $\xi_1\to\xi_2$. In view of our conjecture and $\mathbb{R}^2$ results it appears likely that

(3.19) $$d \text{ also increases as a function of } \xi_1.$$

However so far we have not been able to prove this.

In $\mathbb{R}^2$ we can prove (3.19), without relying on [BL1], as follows: From (1.4), (1.5), and a Schwarz- Christoffel type argument we have

(3.20)
$$\frac{\partial P(r,\theta,d,M)}{\partial\theta}=\frac{iz\,\partial P(z,d,M)}{\partial z}=\operatorname{Re}\left[i\left(\frac{1+z^2-2az}{1+z^2-2bz}\right)^{1/2}\right]\text{ for } z\in\bar B(0,1)\backslash\{\xi_1,\xi_2\}.$$

where $b=\cos\xi_1,a=\cos\xi_2$. If $\xi_1<\xi_1'<\xi_2=\xi_2'$ and $P(z,d',M')$ denotes the extremal potential corresponding to ξ_1',ξ_2', then (3.20) holds for this potential with b replaced by $b'=\cos\xi_1'<b$. Moreover

(3.21) $$d'-d=\int_0^1\left[\left(\frac{1+r^2+2ar}{1+r^2+2b'r}\right)^{1/2}-\left(\frac{1+r^2+2ar}{1+r^2+2br}\right)^{1/2}\right]dr/r>0.$$

Also from (3.20) for $P(\cdot,d,M),P(\cdot,d',M')$, we see that if $\xi_1'\le\theta_0<\xi_2$, then

(3.22)
$$\begin{aligned}&P(1,\theta_0,d',M')-P(1,\theta_0,d,M)>\\&P(1,\theta_0,d',M')-P(1,\theta_0,d,M)+d-d'=\\&\int_{\theta_0}^{\xi_2}\left[\left(\frac{\cos\theta-\cos\xi_2}{\cos\xi_1'-\cos\theta}\right)^{1/2}-\left(\frac{\cos\theta-\cos\xi_2}{\cos\xi_1-\cos\theta}\right)^{1/2}\right]d\theta>0\end{aligned}$$

Using (3.21), (3.22), we can show that

$$\max_{\tau\in[0,\pi]} \int_0^\tau [P(1,\theta,d',M') - P(1,\theta,d,M)]d\theta = 0. \tag{3.23}$$

Indeed from $M' < M, \xi_1 < \xi_1'$, the fact that $P(\cdot,d,M)$ is strictly decreasing on (ξ_1,ξ_2) and the first derivative test for maxima, we find first that the maximum in (3.23) cannot occur at some $\tau \in (0,\xi_1')$. Moreover by the same reasoning and (3.21), (3.22), this maximum cannot occur when $\tau \in [\xi_1',\pi)$. Finally the integral in (3.23) $= 0$ for $\tau = 0,\pi$. From (3.23), (1.9), as well as Baernstein's Theorem (mentioned after (1.9)), one can conclude that (1.3) in $\mathbb{R}^2$ is valid for $p = P(\cdot,d',M')$.

In view of the above $\mathbb{R}^2$ results, one wonders if in $\mathbb{R}^n, n \geq 3$, it is true that for $\theta \in (\xi_1,\xi_2)$,

$$\frac{\partial^2 P(1,\theta,d,M)}{\partial\theta\,\partial\xi_1} < 0 \text{ and } \frac{\partial^2 P(1,\theta,d,M)}{\partial\theta\,\partial\xi_2} > 0. \tag{3.24}$$

Remark 3.2. *Another question of interest to us, is to what extent does Theorem 1.1 generalize to other PDE' s ? For example can one replace harmonic in Theorem 1.1 by p - harmonic when $1 < p < n$. To be more specific, if $1 < p < n$, does there exist a super p-harmonic function $u > 0$ on $\mathbb{R}^n$ with $u(x)\to 0$ as $x\to\infty$, satisfying (a) $-$ (d) of Theorem 1.1 for some choice of $(\xi_1,\xi_2), 0 \leq \xi_1 < \xi_2 \leq \pi$, and $(d,M), 0 < d < 1 < M \leq \infty$ (see* [HKM] *for relevant definitions).*

4. Proof of Proposition 4.1

Recall from (2.5), (2.6) that

$$\begin{aligned} h(r,\theta,\theta_1) &= \int_0^\pi \left(1 + r^2 - 2r\psi(\theta,\theta_1,\phi)\right)^{1-n/2} (\sin\phi)^{n-3}\, d\phi \text{ with} \\ \psi(\theta,\theta_1,\phi) &= \cos\theta\cos\theta_1 + \cos\phi\sin\theta\sin\theta_1. \end{aligned} \tag{4.1}$$

Theorem 1.2 is an easy consequence of the following proposition.

Proposition 4.1. $\frac{\partial^2 h(1,\theta,\theta_1)}{\partial\theta\,\partial\theta_1} < 0$ *whenever* $\theta_1 \neq \theta, \theta, \theta_1 \in [0,\pi]$.

Proof. Since $h(1,\theta,\theta_1)$ is symmetric in θ,θ_1, we assume as we may that $0 \leq \theta_1 < \theta \leq \pi$. Differentiating (4.1) we obtain

$$\begin{aligned} \frac{\partial h(1,\theta,\theta_1)}{\partial\theta} &= (n-2)2^{-n/2} \int_0^\pi \frac{\partial\psi}{\partial\theta}(1-\psi)^{-n/2}(\sin\phi)^{n-3} d\phi \\ &= (n-2)2^{-n/2} \int_0^\pi [\cos\phi\cos\theta\sin\theta_1 - \sin\theta\cos\theta_1] f(\theta,\theta_1,\phi) d\phi \end{aligned} \tag{4.2}$$

where

$$f(\theta,\theta_1,\phi) = (1 - \psi(\theta,\theta_1,\phi))^{-n/2}(\sin\phi)^{n-3}\,.$$

From(4.1) we see that (4.2) can be rewritten as

(4.3)
$$\frac{\partial h(1,\theta,\theta_1)}{\partial\theta} = -(n/2-1)\cot\theta\, h(1,\theta,\theta_1) + \frac{(n-2)2^{-n/2}(\cos\theta-\cos\theta_1)}{\sin\theta}\int_0^\pi f(\theta,\theta_1,\phi)d\phi.$$

Taking second partials in (4.3) we have

(4.4)
$$\sin\theta\frac{\partial^2 h(1,\theta,\theta_1)}{\partial\theta\,\partial\theta_1} = -(n/2-1)\cos\theta\,\frac{\partial h(1,\theta,\theta_1)}{\partial\theta_1} + (n-2)2^{-n/2}\sin\theta_1\int_0^\pi f(\theta,\theta_1,\phi)d\phi$$
$$+(n-2)2^{-n/2}(\cos\theta-\cos\theta_1)\int_0^\pi \frac{\partial f(\theta,\theta_1,\phi)}{\partial\theta_1}d\phi = J_1+J_2+J_3$$

Since h is symmetric in θ,θ_1 we can interchange θ,θ_1 in (4.2), (4.3), to get

(4.5)
$$J_1 = (n/2-1)^2\cos\theta\cot\theta_1 h(1,\theta,\theta_1) - \frac{(n-2)^2 2^{-(n+2)/2}\cos\theta(\cos\theta_1-\cos\theta)}{\sin\theta_1}\int_0^\pi f(\theta,\theta_1,\phi)d\phi.$$

Moreover, as in (4.3)

(4.6)
$$J_3 = -(n/2)(n-2)2^{-n/2}(\cos\theta-\cos\theta_1)\cot\theta_1\int_0^\pi f(\theta,\theta_1,\phi)d\phi$$
$$-(n-2)(n/2)2^{-n/2}\frac{(\cos\theta_1-\cos\theta)^2}{\sin\theta_1}\int_0^\pi (1-\psi)^{-1}f(\theta,\theta_1,\phi)d\phi = J_4+J_5$$

Adding J_2 in (4.4) to J_4+J_5 in (4.6) we get

(4.7)
$$J_2+J_4+J_5 = [(n-2)2^{-n/2}\sin\theta_1 - (n/2)(n-2)2^{-n/2}(\cos\theta-\cos\theta_1)\cot\theta_1]\int_0^\pi f(\theta,\theta_1,\phi)d\phi$$
$$-(n-2)(n/2)2^{-n/2}\frac{(\cos\theta_1-\cos\theta)^2}{\sin\theta_1}\int_0^\pi (1-\psi)^{-1}f(\theta,\theta_1,\phi)d\phi$$

Finally we arrive at

(4.8)
$$(n/2-1)^{-1}2^{n/2-1}(J_1+J_2+J_4+J_5) = (n/2-1)\cos\theta\cot\theta_1\int_0^\pi(1-\psi)f(\theta,\theta_1,\phi)d\phi$$
$$+\left[(n/2-1)\frac{(\cos\theta_1-\cos\theta)^2}{\sin\theta_1} + \frac{1-\cos\theta\cos\theta_1}{\sin\theta_1}\right]\int_0^\pi f(\theta,\theta_1,\phi)d\phi$$
$$-(n/2)\frac{(\cos\theta_1-\cos\theta)^2}{\sin\theta_1}\int_0^\pi(1-\psi)^{-1}f(\theta,\theta_1,\phi)d\phi$$

From (4.4) - (4.8) we conclude that to finish the proof of Proposition 4.1 we need to show that the right hand side of (4.8) is negative. To do this we let $a=\sin\theta\sin\theta_1, b=$

$1-\cos\theta\cos\theta_1$ and use the beta function to calculate in terms of our previous notation:
(4.9)

$$A = b^{n/2-1}\int_0^\pi (1-\psi)^{1-n/2}(\sin\phi)^{n-3}d\phi = \left(\sum_{l=0}^\infty \frac{(n/2-1)_{2l}\,\Gamma(n/2-1)\,\Gamma(l+1/2)}{(2l)!\;\Gamma(n/2+l-1/2)}\left(\frac{a}{b}\right)^{2l}\right)$$

where Γ is the Gamma function and $(\lambda)_k = \lambda(\lambda+1)\dots(\lambda+k-1)$ when k is a positive integer with $(\lambda)_0 = 1 = (0)!$.
(4.10)

$$B = b^{n/2}\int_0^\pi (1-\psi)^{-n/2}(\sin\phi)^{n-3}d\phi = \left(\sum_{l=0}^\infty \frac{(n/2)_{2l}\,\Gamma(n/2-1)\,\Gamma(l+1/2)}{(2l)!\;\Gamma(n/2+l-1/2)}\left(\frac{a}{b}\right)^{2l}\right)$$

(4.11)

$$C = b^{n/2+1}\int_0^\pi (1-\psi)^{-(n/2+1)}(\sin\phi)^{n-3}d\phi = \left(\sum_{l=0}^\infty \frac{(n/2+1)_{2l}\,\Gamma(n/2-1)\,\Gamma(l+1/2)}{(2l)!\;\Gamma(n/2+l-1/2)}\left(\frac{a}{b}\right)^{2l}\right)$$

In terms of this notation, $0\le a\le 1, a<b\le 2$, and (4.4) - (4.11), we get at $(1,\theta,\theta_1)$,

$$\begin{aligned}
&(n/2-1)^{-1}(2b)^{n/2-1}\sin\theta_1\sin\theta\frac{\partial^2 h(1,\theta,\theta_1)}{\partial\theta\,\partial\theta_1}\\
&= (n/2-1)^{-1}2^{n/2-1}b^{n/2-1}(\sin\theta_1)(J_1+J_2+J_4+J_5)\\
&= (n/2-1)(1-b)A + b^{-1}\left[(n/2-1)\,(b^2-a^2)+b\right]\,B\\
&-b^{-2}(n/2)(b^2-a^2)C\\
&= b(n/2-1)(B-A)+(n/2-1)(A-C)+(B-C)\\
&-(n/2-1)(a^2/b)\,B+(n/2)(a^2/b^2)C = \frac{\Gamma(n/2-1)\Gamma(1/2)}{\Gamma(n/2-1/2)}D
\end{aligned} \tag{4.12}$$

if $n=2m$, the $l=0$ term in the sum for D is

$$T_0^2 = -(m-1)(a^2/b) + m(a^2/b^2). \tag{4.13}$$

Let $A_l, B_l, C_l, l = 1,2,\dots$ be the l th nonzero coefficient multiplying $(a/b)^{2l}$ in $A.B,C$. If again $n=2m$, then
(4.14)

$$(B-A)_1 = (\frac{m+1}{m-1}-1)A_1 = \frac{2}{m-1}A_1,\ \ (C-A)_1 = (\frac{(m+1)(m+2)}{m(m-1)}-1)A_1 = \frac{4m+2}{(m-1)m}A_1,$$

$$(C-B)_1 = \frac{2(m+1)}{(m-1)m}A_1.$$

Also

$$\frac{\Gamma(m-1/2)}{\Gamma(1/2)\Gamma(m-1)}A_1 = \frac{(m-1)m}{2(2m-1)}. \tag{4.15}$$

Using (4.14)- (4.15) we deduce that the part of the sum in (4.12) involving A_1, B_1, C_1, in D is

(4.16)

$$\left[2b - 4 - (2/m) - 2\frac{m+1}{(m-1)m} - (m+1)a^2/b + \frac{(m+1)(m+2)}{m-1}a^2/b^2\right]\frac{(m-1)\,m\,(a/b)^2}{2(2m-1)} =$$

$$\left(\frac{m(m-1)}{2m-1}a^2/b - \frac{4m^2}{4m-2}a^2/b^2\right) + \left(-\frac{(m-1)m(m+1)}{4m-2}a^4/b^3 + \frac{m(m+1)(m+2)}{4m-2}a^4/b^4\right)$$

$$= T_1^1 \qquad\qquad + \qquad\qquad T_1^2$$

From (4.13), (4.16), we get

$$T_0^2 + T_1^1 = -\frac{(m-1)^2}{(2m-1)}a^2/b - \frac{2m}{4m-2}\,a^2/b^2 < 0 \tag{4.17}$$

Fix l a positive integer and once again set $n = 2m$. As in the case $l = 1$ we see from (4.9) - (4.11) that

$$\begin{aligned}
(B-A)_l &= (\frac{m+2l-1}{m-1} - 1)A_l = \frac{2l}{m-1}A_l,\\
(C-A)_l &= [\frac{(m+2l-1)(m+2l)}{(m-1)m} - 1]A_l = \frac{4lm + 2l(2l-1)}{(m-1)m}A_l\\
(C-B)_l &= \left[\frac{(m+2l-1)(m+2l)}{(m-1)m} - \frac{m+2l-1}{m-1}\right]A_l = \frac{2l(m+2l-1)}{(m-1)m}A_l
\end{aligned} \tag{4.18}$$

Also

$$\frac{\Gamma(m-1/2)}{\Gamma(1/2)\Gamma(m-1)}A_l = \frac{(m-1)_{2l}(l-1/2)(l-3/2)\dots(1/2)}{(2l)!(m+l-3/2)(m+l-5/2)\dots(m-1/2)}\,. \tag{4.19}$$

Using (4.18), (4.12), we see that the terms in D involving $(a/b)^{2l}$ times A_l, B_l, C_l are ,

(4.20)

$$\left[[2lb - 4l - (2l/m)(2l-1) - \tfrac{2l(m+2l-1)}{m(m-1)}\right]A_l\,(a/b)^{2l} +$$

$$\left[-(a^2/b)(m+2l-1) + \frac{(m+2l-1)(m+2l)}{m-1}(a^2/b^2)\right]A_l\,(a/b)^{2l} = T_l^1 + T_l^2$$

We claim that

$$\sum_{k=0}^{l} T_k^1 + \sum_{k=0}^{l-1} T_k^2 \le 0 \text{ for } l = 1, 2, \dots \tag{4.21}$$

where $T_0^1 = 0$. Note from (4.17) that (4.21) is true when $l = 1$. Proceeding by induction assume (4.21) is true for some positive integer l. We need to show that

$$\begin{aligned}&\left[[2(l+1)b - 4(l+1) - \frac{2(l+1)(2l+1)}{m} - \frac{2(l+1)(m+2l+1)}{m(m-1)}\right] A_{l+1}\,(a/b)^2 \\ &+ \left[-(m+2l-1)\frac{a^2}{b} + \frac{(m+2l-1)(m+2l)}{m-1}(a/b)^2\right] A_l < 0\end{aligned} \tag{4.22}$$

Now

$$A_{l+1}/A_l = \frac{(m+2l)(m+2l-1)(l+1/2)}{(2l+2)(2l+1)(m+l-1/2)} = \frac{(m+2l)(m+2l-1)}{4(l+1)(m+l-1/2)} \tag{4.23}$$

From (4.23) we get for the a^2/b term in (4.22)

$$2(l+1)b\,A_{l+1}\,(a/b)^2 - (m+2l-1)(a^2/b)A_l = -\frac{(m-1)(m+2l-1)}{2(m+l-1/2)}\,(a^2/b)\,A_l < 0. \tag{4.24}$$

Next we observe from (4.23) for the $A_{l+1}(a/b)^2$ term in (4.22) that

$$\begin{aligned}&\left(-4(l+1) - \frac{2(l+1)(2l+1)}{m} - \frac{2(l+1)(m+2l+1)}{m(m-1)}\right) A_{l+1}(a/b)^2 \\ &= -\frac{4(l+1)(m+l)}{m-1}A_{l+1}(a/b)^2 = -\frac{(m+l)(m+2l)(m+2l-1)}{(m-1)(m+l-1/2)}A_l(a/b)^2\end{aligned} \tag{4.25}$$

Hence adding the $(a/b)^2 A_l$ term in (4.22) to the right hand side of (4.25),

$$\begin{aligned}&-\frac{(m+l)(m+2l)(m+2l-1)}{(m-1)(m+l-1/2)}A_l(a/b)^2 + \frac{(m+2l-1)(m+2l)}{m-1}A_l\,(a/b)^2 \\ &= -\frac{(m+2l-1)(m+2l)}{2(m-1)(m+l-1/2)}A_l(a/b)^2 < 0.\end{aligned} \tag{4.26}$$

Finally adding right hand sides of (4.26), (4.24), we find from (4.22) and induction that claim (4.21) is true. From the definition of A, B, C we see that these functions have absolutely convergent series involving powers of $(a/b)^{2l}$ and thereupon that (4.21) converges to $\frac{\Gamma(n/2-1)\Gamma(1/2)}{\Gamma(n/2-1/2)}D$. So $D < 0$ and it follows from (4.12) that Proposition 4.1 is valid when $n \geq 3$ is a positive integer. □

5. Proof of Theorem 1.2

Proof. Recall from section 1 that if $\xi_2 = \pi, 0 < \xi_1 < \pi$, and $P = P(\cdot, d, M)$ is the corresponding extremal potential satisfying $(a) - (d)$ of Theorem 1.1, then P is harmonic in $\mathbb{R}^n \setminus E_1$, so

$$d = P(-e_1, d, M), \text{ and (1.3) holds whenever } p \in \mathcal{F}_d^M. \tag{5.1}$$

Thus to prove Theorem 1.2 we show for d as in (5.1) and $1 < M < \infty$, that

$$\int_{\mathbb{S}^{n-1}} \Phi(P(ry, d, M))\, d\mathcal{H}^{n-1} \leq \int_{\mathbb{S}^{n-1}} \Phi(P(ry, d, \infty))\, d\mathcal{H}^{n-1}, 0 < r < \infty. \tag{5.2}$$

To do so we first note for any $p \in \mathcal{F}$ that $2^{2-n} \leq p$ in $B(0,1)$ as follows from the minimum principle for potentials and the fact that any two points on $\mathbb{S}^{n-1}$ are at most distance two apart. Also in [L] we showed the existence of $P(\cdot, d, \infty)$ in Theorem 1.2 whenever $2^{2-n} \leq d < 1$. Thus $P(\cdot, d, \infty)$ exists when d is as in (5.1).

The proof of (5.2) is by contradiction. For ease of writing we put $P = P(\cdot, d, M)$, $P' = P(\cdot, d, \infty)$, and write in spherical coordinates, $P(r, \theta), P'(r, \theta)$, which is permissible, since both functions are symmetric about the x_1 axis. Also for fixed $\xi_1 \in (0, \pi)$, let E_1, be as defined in Theorem 1.1 relative to P while $\xi_2' \in (0, \pi), E_2'$, are defined relative to P'. As in (2.3)-(2.6), and from Theorem 1.1 we deduce the existence of positive Borel measures, ν, μ, on $[0, \pi]$ with total mass c_n^{-1} corresponding to P, P', respectively. ν has its support in $[0, \xi_1]$ while μ has its support in $\{0\} \cup [\xi_2', \pi]$. Moreover
(5.3)

$$(a) \quad P(1, \theta) = c_n \int_0^{\xi_1} h(1, \theta, \theta_1)\, d\nu(\theta_1) \text{ and}$$

$$(b) \quad P'(1, \theta) = c_n\, \alpha\, h(1, \theta, 0) + c_n \int_{\xi_2'}^{\pi} h(1, \theta, \theta_1)\, d\mu(\theta_1) \text{ with } \alpha = \mu(\{e_1\}) > 0.$$

We note from Theorem 1.2 that $P(r, \cdot), P'(r, \cdot)$, are continuous in the extended sense and non increasing on $[0, \pi]$ whenever $0 < r < \infty$. From this fact, (1.8), (1.9), and $\mathrm{d}\mathcal{H}^{n-1} = c_n (\sin\theta)^{n-2} d\theta$, we deduce that to prove (5.2) it suffices to show

$$\int_0^\tau P(r, \theta) \sin^{n-2}\theta d\theta \leq \int_0^\tau P'(r, \theta) \sin^{n-2}\theta d\theta \tag{5.4}$$

whenever $0 < r < \infty, 0 \leq \tau \leq \pi$. Moreover from the Baernstein maximum principle (see the discussion after (1.8)) , we need only prove (5.4) when $r = 1$. To do this observe that if (5.4) is false for some $\hat\theta \in (0, \pi)$, when $r = 1$, then there exists $\bar\theta \in (0, \pi)$ with

$$0 < \max_{\tau \in [0,\pi]} \int_0^\tau (P - P')(1, \theta) \sin^{n-2}\theta d\theta = \int_0^{\bar\theta} (P - P')(1, \theta) \sin^{n-2}\theta d\theta. \tag{5.5}$$

Since P' is strictly decreasing on $[0, \xi_2']$ with $P' \equiv d < P$ on $[\xi_2', \pi)$, $P \equiv M$ on $[0, \xi_1]$, and the integrals in (5.4) are equal when $\tau = \pi$, we see from the first derivative test in calculus, that $\xi_1 < \xi_2'$ and we may assume

$$\bar\theta \in [\xi_1, \xi_2'], \text{ with } P(1, \bar\theta) = P'(1, \bar\theta). \tag{5.6}$$

To get a contradiction we note from Proposition 4.1 and (5.3) (a) that if $\theta \in (\xi_1, \xi_2')$,

$$-\frac{\partial P}{\partial \theta}(1, \theta) = -c_n \int_0^{\xi_1} \frac{\partial h}{\partial \theta}(1, \theta, \theta_1) d\mu(\theta_1) \geq -c_n \frac{\partial h}{\partial \theta}(1, \theta, 0). \tag{5.7}$$

On the other hand from Proposition 4.1 we have for $\theta, \theta_1 \in [0, \pi], \theta_1 > \theta$,

$$\frac{\partial h}{\partial \theta}(1, \theta, \theta_1) > \frac{\partial h}{\partial \theta}(1, \theta, \pi) = (n/2 - 1)\sin\theta(1 + \cos\theta)^{-n/2} > 0. \tag{5.8}$$

Using (5.8) amd (5.3)(b) we see for $\theta \in (\xi_1, \xi_2')$, that

$$\frac{\partial P'}{\partial \theta}(1, \theta) > c_n\, \alpha\, \frac{\partial h}{\partial \theta}(1, \theta, 0) = -c_n \alpha (n/2 - 1)\sin\theta(1 - \cos\theta)^{-n/2}. \tag{5.9}$$

Combining (5.7) and (5.9) we get

$$\frac{\partial P'}{\partial \theta}(1, \theta) - \frac{\partial P}{\partial \theta}(1, \theta) > c_n(\alpha - 1)\frac{\partial h}{\partial \theta}(1, \theta, 0) > 0 \text{ on } (\xi_1, \xi_2'), \tag{5.10}$$

since $0 < \alpha < 1$. Thus $P' - P$ is increasing on $(\xi_1, \xi_2']$, so

$$P'(1, \theta) - P(1, \theta) < P'(1, \xi_2) - P(1, \xi_2) = d - P(1, \xi_2) < 0, \tag{5.11}$$

since P is strictly decreasing on $[\xi_1, \pi]$ with $P(1, \pi) = d$. Letting $\theta \to \bar{\theta}$ we arrive at a contradiction to (5.5). From this contradiction we conclude Theorem 1.2. □

Remark 5.1. *Conjecture 1 when $E_2 \neq \{-e_1\}$ or even an analogue of (1.10) when $E_1 \neq \{e_1\}$, seems difficult. If $E_2 \neq \{-e_1\}$, the main problem is that the proposed extremal potential, $P(\cdot, d, M)$ must have mass at points on $\mathbb{S}^{n-1}$ where it assumes both its maximum and minimum values on $\bar{B}(0,1)$. This splitting of the mass seems to rule out an immediate proof of Conjecture 1 using Baernstein's $*$ function. A simpler problem in view of (1.9) is to show for $p \in \mathcal{F}_d^M$:*

$$\int_{\{y \in \mathbb{S}^{n-1} : y_1 \geq \cos\theta_0\}} p(ry)\, d\mathcal{H}^{n-1} y \leq \int_{\{y \in \mathbb{S}^{n-1} : y_1 \geq \cos\theta_0\}} P(ry, d, M) d\mathcal{H}^{n-1} y \tag{5.12}$$

whenever $0 \leq \theta_0 \leq \pi, 0 < r < \infty$, Note that a positive answer to (5.12) would imply an analogue of (1.10) when $E_1 \neq \{e_1\}$ and also (3.19).

To indicate our efforts in trying to prove (5.12), we first observe that it suffices to prove (5.12) when p is symmetric about the x_1 axis, so $p(x) = p(r, \theta)$ when $|x| = r, x_1 = r\cos\theta$. Also, using the Baernstein * function, as mentioned earlier, we need only prove (5.12) when $r = 1$. Next as in (2.4), we see that

$$p(1, \theta) = c_n \int_0^\pi h(1, \theta, \theta_1) d\sigma(\theta_1) \tag{5.13}$$

where h is as in (2.5) and σ is a positive Borel measure on $[0, \pi]$ with $\sigma([0, \pi]) = c_n^{-1}$. Given $\theta_0 \in (0, \pi)$, it follows from the Fubini theorem and (5.13) that

$$\int_0^{\theta_0} p(1, \theta)\sin^{n-2}\theta d\theta = \int_0^\pi k(1, \theta_1) d\sigma(\theta_1) \tag{5.14}$$

where

$$k(1, \theta_1) = c_n \int_0^{\theta_0} h(r, \theta, \theta_1)\sin^{n-2}\theta d\theta \tag{5.15}$$

We note that $k(1,\cdot)$ is continuous on $[0,\pi]$ (in fact Hölder continuous) so from a theorem on weak convergence of measures, we deduce that

$$\sup_{p\in\mathcal{F}_d^M}\int_0^{\theta_0} p(1,\theta)\sin^{n-2}\theta d\theta = \int_0^{\theta_0}\tilde{P}(1,\theta)\sin^{n-2}\theta d\theta \tag{5.16}$$

for some $\tilde{P}\in\mathcal{F}_d^M$. As for $\tilde{P}$, we can prove when $M=\infty$:

Lemma 5.2. *Let $\tilde{\nu}$ be the positive Borel measure on $[0,\pi]$ corresponding to $\tilde{P}$ in (5.16). Then*

$$\tilde{\nu}\{\theta_1:\theta_0<\theta_1\le\pi \text{ and } \tilde{P}(1,\theta_1)\neq d\}=0 \tag{5.17}$$

Proof. The proof of (5.17) is by contradiction and essentially given in Lemma 4 of [L]. To briefly outline the argument, if (5.17) is false, then using lower semi-continuity of $\tilde{P}$, one can assume there exists $\tau_i, 1\le i\le 3$, with $\theta_0<\tau_1<\tau_2<\tau_3\le\pi$, $\tilde{P}(1,\cdot)>d$, on $[\tau_1,\tau_3]$, and $\tilde{\nu}([\tau_1,\tau_2])=\tilde{\nu}([\tau_2,\tau_3])$. For ϵ_0 small, $\epsilon\in[-\epsilon_0,\epsilon_0]\setminus\{0\}$, and $\tau_1<\theta_1<\tau_3$ define $\tilde{g}(\theta_1,\epsilon)$ by

$$\begin{array}{ll}(a) & 0<\tilde{g}(\theta_1,\epsilon)<\pi\\ (b) & (1+\epsilon)[(1+\cos\tilde{g}(\theta_1,\epsilon))^{1-2/n}+B]=(1+\cos\theta_1)^{1-2/n}+B\end{array} \tag{5.18}$$

where $B>0$ is to be chosen. One can verify from (5.18) that $\tilde{g}(\theta_1,\epsilon)$ is an increasing function of θ_1 on $[\tau_1,\tau_3]$. For $\epsilon\in(0,\epsilon_0]$, let

$$\begin{aligned}\tilde{P}_\epsilon(r,\theta)&=c_n(1+\epsilon)\int_{\tau_1}^{\tau_2}h(r,\theta,\tilde{g}(\theta_1,\epsilon))d\nu(\theta_1)\\ &+c_n(1-\epsilon)\int_{\tau_2}^{\tau_3}h(r,\theta,\tilde{g}(\theta_1,-\epsilon))d\nu(\theta_1)\\ &+c_n\int_{[0,\pi]\setminus[\tau_1,\tau_3]}h(r,\theta,\theta_1)d\nu(\theta_1).\end{aligned} \tag{5.19}$$

Next since $\tilde{g}(\theta_1,\pm\epsilon)$ is an increasing function of θ_1 on $[\tau_1,\tau_3]$ it follows that $\tilde{P}_\epsilon$ is a potential symmetric about the x_1 axis. A lengthy calculation then gives for B sufficiently large that $\tilde{P}_\epsilon\in\mathcal{F}_d^\infty$ and

$$\frac{\partial\tilde{P}_\epsilon(r,\theta)}{\partial\epsilon}>0 \text{ when } 0<\epsilon<\epsilon_0, 0<r<\infty, \text{ and } \theta\in[0,\theta_0]. \tag{5.20}$$

From (5.20) we obtain

$$\int_0^{\theta_0}\tilde{P}(r,\theta)(\sin\theta)^{n-2}d\theta<\int_0^{\theta_0}\tilde{P}_\epsilon(r,\theta)(\sin\theta)^{n-2}d\theta \tag{5.21}$$

which contradicts (5.16) and so Lemma 1 is true. $\square$

Remark 5.3. *Using (5.17) of Lemma 5.2, one can get as in [L] that for some $\hat{\theta}_0\in[\theta_0,\pi]$,*

$$\tilde{P}(1,\theta)=d \text{ on } [\hat{\theta}_0,\pi]. \tag{5.22}$$

To use the above 'mass moving ' argument further in proving (5.12) appears difficult since from the mean value property for harmonic functions,

$$\int_0^\pi \frac{\partial \tilde{P}_\epsilon(1,\theta)}{\partial \epsilon}(\sin\theta)^{n-2}d\theta = c_n^{-1}\frac{\partial \tilde{P}_\epsilon(0,0)}{\partial \epsilon} = 0.$$

Thus moving mass inside $(0,\theta_0)$, as above, would create points in this interval where $\tilde{P}_\epsilon < \tilde{P}$ and so perhaps not imply (5.21).

The lengthy calculation mentioned above was to show that

$$\frac{\partial}{\partial\theta_1}\left[\frac{\frac{\partial h(1,\theta,\theta_1)}{\partial\theta_1}}{\frac{\partial h(1,\pi,\theta_1)}{\partial\theta_1}}\right] > 0 \tag{5.23}$$

whenever $\theta,\theta_1,\in(0,\pi)$ and $\theta\neq\theta_1$. Originally in [L], after many months of trying, I had just proved (5.23) for $n=3$ so (1.10) was just valid in $\mathbb{R}^3$. Still I submitted my paper to the Proc. of LMS and after a few months received a handwritten report from the referee (Walter Hayman) to the effect that I should try using the substitution,

$$\frac{\sin\theta\sin\theta_1}{1-\cos\theta\cos\theta_1} = \frac{2t}{1+t^2} \text{ with } t = \frac{\tan(\theta/2)}{\tan(\theta_1/2)}$$

to simplify my calculations. Using this observation I was eventually able to prove (5.23) and after that use the above contradiction argument to get (1.10) in $\mathbb{R}^n, n\geq 3$. Note that Proposition 4.1 is in the same spirit as (5.23).

References

[B] A. Baernstein II, *Proof of Edrei's Spread Conjecture,* Proc. London Math. Soc. (3) **26** (1973), 418-434.

[BDL] A. Baernstein II, D. Drasin, and R. Laugeson, *Symmetrization in Analysis,* Cambridge University Press, 2019.

[BT] A. Baernstein II and B. A. Taylor, *Spherical Rearrangements, Subharmonic Functions, and $*$ Functions in n Space,* Duke Math. J. **43** (2) (1976), 245-268.

[BL1] R. Barnard and J. Lewis, *Subordination Theorems for Some Classes of Starlike Functions,* Pac. J. Math. **56** (1975), 333-366.

[BL2] R. Barnard and J. Lewis, *Coefficient Bounds for Some Classes of Starlike Functions,* Pac. J. Math. **56** (1975), 325-331.

[DS1] D. De Silva and O. Savin, *C^∞ Regularity of Certain Thin Free Boundaries,* Indiana Univ. Math. J. **64** (2015), no. 5, 1575 - 1608.

[DS2] D. De Silva and O. Savin, *Boundary Harnack Estimates in Slit Domains and Applications to Thin Free Boundary Problems,* Rev. Mat. Ibero **32** (2016). no. 3, 891 - 912.

[D] P. Duren, *Univalent Functions,* Grundlehren der Mathematischen Wissenschaften **259**, Springer, New York, 2010

[E] L.C. Evans, *Partial Differential Equations,* Graduate Studies in Mathematics **19**, second edition, American Mathematics Society, 2010.

[GL] R. Gariepy and J. Lewis, *A Maximum Principle with Applications to Subharmonic Functions in N-Space,* Arkiv for Mathematik, **12** (1974), 253-266.

[H1] W. K. Hayman and P.B. Kennedy, *Subharmonic Functions, Vol. 1,* London Mathematical Society Monographs **9**, Academic Press, 1976.

[H2] W. K. Hayman, *Subharmonic Functions, Vol. 2,* London Mathematical Society Monographs **20**, Academic Press, 1989.

[HKM] J. Heinonen, T. Kilpeläinen, and O. Martio,Nonlinear Potential Theory Degenerate Elliptic Equations, Dover Publications, 2006.

[KJ] C. Kenig and D. Jerison, *Boundary Behavior of Harmonic Functions in Non-tangentially Accessible Domains* **46** (1982), 80-147

[L] J. Lewis, *An n-Space Analogue of a Theorem of Suffridge,* Proc. London Math. Soc. **30** (1975), 444-464.

[S] T. Suffridge, *A Coefficient Problem for a Class of Univalent Functions,* Michigan Math. J. **16** (1962), 33-42.

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