Note on p Harmonic Measure

by John L. Lewis [1]

ABSTRACT: In this paper we continue our study of the dimension of a measure associated with a positive p harmonic function which vanishes on the boundary of a quasicircle. Under various assumptions on the quasicircle, we obtain analogues of well known theorems for the dimension of harmonic measure due to Makarov.

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1. Introduction. Denote points in Euclidean 2 space $\mathbb{R}^2$ by $x = (x_1, x_2)$ and let $\bar{E}, \partial E$, diam E denote the closure, boundary, and diameter of the set $E \subset \mathbb{R}^2$. Let $\langle \cdot, \cdot \rangle$ be the standard inner product on $\mathbb{R}^2$ and put $|x| = \langle x, x \rangle^{1/2}$. Set $B(x,r) = \{y \in \mathbb{R}^2 : |x-y| < r\}$ whenever $x \in \mathbb{R}^2$ and $r > 0$. Given λ a nondecreasing positive function on $(0, r_0), r_0 > 0$, with $\lambda(0) = 0$, define Hausdorff measure corresponding to λ (denoted H^λ) as follows: For fixed $\delta, 0 < \delta < r_0$, and $E \subseteq \mathbb{R}^2$, let $L(\delta) = \{B(x_i, r_i)\}$ be such that $E \subseteq \bigcup B(x_i, r_i)$ and $0 < r_i < \delta, \quad i = 1, 2, ...$ Set

$$\phi_\delta^\lambda(E) = \inf_{L(\delta)} \left(\sum \lambda(r_i) \right).$$

Then

$$H^\lambda(E) = \lim_{\delta \to 0} \phi_\delta^\lambda(E).$$

Let $\alpha(k)$ denote the volume of the unit ball in $\mathbb{R}^k$. For ease of writing, we put $H^k = H^\lambda$ when $\lambda(r) = \alpha(k) r^k, 0 < r \leq 1$. We also write dx in integrals with respect to $H^2 =$ two dimensional Lebesgue measure on $\mathbb{R}^2$. If O is open and $1 \leq q \leq \infty$, let $W^{1,q}(O)$ be the space of equivalence classes of functions u with distributional gradient $\nabla u = (u_{x_1}, u_{x_2})$, both of which are q th power integrable on O. Let

$$\|u\|_{1,q} = \|u\|_q + \|\nabla u\|_q$$

be the norm in $W^{1,q}(O)$ where $\|\cdot\|_q$ denotes the usual Lebesgue q norm in O. Let $C_0^\infty(O)$ be infinitely differentiable functions with compact support in O and let $W_0^{1,q}(O)$ be the closure of $C_0^\infty(O)$ in the norm of $W^{1,q}(O)$. Let Ω be a domain (i.e a open connected set) and suppose that $\partial\Omega$ is bounded and non empty. Let N be a neighborhood of $\partial\Omega$, p fixed, $1 < p < \infty$, and u a positive weak solution to the p Laplace partial differential equation in $\Omega \cap N$. That is, $u \in W^{1,p}(\Omega \cap N), u > 0$, and

$$\int |\nabla u|^{p-2} \langle \nabla u, \nabla \theta \rangle \, dx = 0 \tag{1.1}$$

whenever $\theta \in W_0^{1,p}(\Omega \cap N)$. Observe that if u is smooth and $\nabla u \neq 0$ in $\Omega \cap N$, then $\nabla \cdot (|\nabla u|^{p-2} \nabla u) \equiv 0$, in the classical sense, where $\nabla \cdot$ denotes divergence. We assume that u has zero boundary values on $\partial\Omega$ in the Sobolev sense. More specifically if $\zeta \in C_0^\infty(N)$, then $u\,\zeta \in W_0^{1,p}(\Omega \cap N)$. Extend u to $N \setminus \Omega$ by putting $u \equiv 0$ on $N \setminus \Omega$. Then $u \in W^{1,p}(N)$ and it follows from (1.1), as in [HKM], that there exists a positive finite Borel measure μ on $\mathbb{R}^2$ with support contained in $\partial\Omega$ and the property that

$$\int |\nabla u|^{p-2} \langle \nabla u, \nabla \phi \rangle \, dx = - \int \phi \, d\mu \tag{1.2}$$

whenever $\phi \in C_0^\infty(N)$. We note that if $\partial\Omega$ is sufficiently smooth, then $d\mu = |\nabla u|^{p-1} \, dH^1$.

Given $p, 1 < p < \infty$, the Hausdorff dimension of μ (denoted H-dim μ) is defined as follows:

$$\text{H-dim } \mu = \inf\{k : \text{ there exists } E \text{ Borel} \subset \partial\Omega \text{ with } H^k(E) = 0 \text{ and } \mu(E) = \mu(\partial\Omega)\}.$$

In [BL] we proved the following theorems.

Theorem A. *Let u, μ be as in (1.1), (1.2). If $\partial\Omega$ is a certain snowflake and $1 < p < 2$, then H-dim $\mu > 1$ while if $2 < p < \infty$, then H-dim $\mu < 1$.*

Theorem B. *Let u, μ be as in (1.1), (1.2). If $\partial\Omega$ is a certain self similar Cantor set and $2 < p < \infty$, then H-dim $\mu < 1$.*

Theorem C. *Let u, μ be as in (1.1), (1.2). If $\partial\Omega$ is a k quasicircle, then H-dim $\mu \leq 1$ for $2 \leq p < \infty$, while H-dim $\mu \geq 1$ for $1 < p \leq 2$.*

We point out that Theorems A, B, C, generalized earlier work for $p = 2$ of Carleson - Makarov, among others (see [BL] for references and an outline of earlier work when $p = 2$). In this note we show that Theorem C can be improved upon. We prove,

Theorem 1. *Given $k, 1 < k < \infty$, there exists $\epsilon_0 = \epsilon_0(k) > 0$ such that if $\partial\Omega$ is a k quasicircle, $|p - 2| < \epsilon_0$, and u, μ are as in (1.1), (1.2), then*

(a) *μ is concentrated on a set of σ finite H^1 measure when $2 < p < 2 + \epsilon_0$.*

(b) *There exists $\hat{A} = \hat{A}(k), 0 < \hat{A}(k) < \infty$, such that if $2 - \epsilon_0 < p < 2$, then μ is absolutely continuous with respect to $H^{\hat{\lambda}}$ where*

$$\hat{\lambda}(r) = r \, \exp[\hat{A}\sqrt{\log 1/r \, \log\log\log 1/r}], 0 < r < 10^{-6}.$$

Theorem 2. *Given $p, 1 < p < \infty$, there exists $k_0(p) > 0$ such that if $\partial\Omega$ is a k quasicircle, $0 < k < k_0(p)$, and u, μ are as in (1.1), (1.2), then*

(a) *μ is concentrated on a set of σ finite H^1 measure when $p > 2$.*

(b) *There exists $\tilde{A} = \tilde{A}(p), 0 < \tilde{A}(p) < \infty$, such that if $1 < p < 2$, then μ is absolutely continuous with respect to $H^{\tilde{\lambda}}$ where*

$$\tilde{\lambda}(r) = r \, \exp[\tilde{A}\sqrt{\log 1/r \, \log\log\log 1/r}], 0 < r < 10^{-6}.$$

Theorem 3. *If $\partial\Omega$ is asymptotically conformal, and u, μ, p are as in (1.1), (1.2), then*

(a) *μ is concentrated on a set of σ finite H^1 measure when $p > 2$.*

(b) *There exists $A^+ = A^+(p, \partial\Omega), 0 < A^+(p, \partial\Omega) < \infty$, such that if $1 < p < 2$, then μ is absolutely continuous with respect to H^{λ^+} where*

$$\lambda^+(r) = r \, \exp[A^+\sqrt{\log 1/r \, \log\log\log 1/r}], 0 < r < 10^{-6}.$$

The definition of k quasiconformal and asymptotically conformal will be given in section 2. We remark that Makarov essentially proved $(a), (b)$ of Theorems 1 - 3 when $p = 2$, Ω is simply connected, and $A = 30$ (see [P, Theorem 10.6]). Also, Theorems 1-3 provide a partial answer to question 7 in [BL, section 6]. To outline the proof of (a) in Theorems 1 - 3, we note that Makarov (see [P, Theorem 6.13]) proved a similar result using Plessner's Theorem. Moreover, the main step in proving Plessner's theorem is to show that if G is an open set with $G \subset B(0,1)$, ∂G is locally Lipschitz, and $H^1(\partial G \cap \partial B(0,1)) > 0$, then the harmonic measure of $\partial G \cap \partial B(0,1)$ with respect to some point in G is positive. In our situation we show that if $G \subset \Omega$ and ∂G is a quasicircle with $\mu(\partial G \cap \partial \Omega) > 0$, then also $\omega(\partial G \cap \partial \Omega) > 0$. Here ω is elliptic measure defined with respect to a point in G and the divergence form degenerate ellptic partial differential equation,

$$L\zeta \; = \sum_{i,k=1}^{2} \frac{\partial}{\partial x_i} \left[\, b_{ik}(x) \zeta_{x_k}(x) \, \right] = 0, \tag{1.3}$$

where at $x \in (\Omega \cap N) \cup G$,

$$b_{ik}(x) = |\nabla u|^{p-4} [(p-2) u_{x_i} u_{x_k} + \delta_{ik} |\nabla u|^2](x), \; 1 \leq i, k \leq 2, \tag{1.4}$$

and δ_{ij} is the Kronecker δ. To prove the above statement regarding ω, we first use basic estimates from [BL] to show that if u is the p capacitary function for a certain ring domain (whose boundary contains $\partial\Omega$), then under the hypotheses of Theorems 1 - 3, $|\nabla u|^{p-2}$ can be extended to an A_2 weight on $\mathbb{R}^2$. In view of this result, we can then use results from [FJK], [FJK1], [FKS] and argue as in [DJK] to obtain that $\omega(\partial G \cap \partial \Omega) > 0$. Finally we get (a) of Theorems 1 - 3 as in [M]. As for (b) of Theorems 1 - 3, we note that in [BL] we proved an inequality which would enable us to prove (b) of Theorems 1 - 3 with λ replaced by λ^* (see [BL, display 6.11]) where

$$\lambda^*(r) = r \, \exp[A^* \sqrt{\log 1/r \, \log\log 1/r}], 0 < r < 10^{-6},$$

and $A^* = A^*(k), 0 < A^* < \infty$. Observe that (b) of Theorems 1 - 3 does not follow from the work in [BL] since $\lambda(r) \leq \lambda^*(r)$ for suitably small r. To obtain the stronger result in (b) of Theorems 1 - 3, we show that if u is the p capacitary function of a certain ring domain, then $\{x : u(x) = t\}$ is $k' = k'(k)$ quasiconformal for $0 < t \leq 1/2$. Using this fact and the fact that $\log |\nabla u|$ is a supersolution to the partial differential equation in (1.3), (1.4) for $1 < p < 2$, we can once again use results from [FJK], [FJK1], [FKS] to obtain estimates for the nontangential maximal function of $\log |\nabla u|$ in $\{x : u(x) > t\}$. Next following [M] we show that for each $\zeta \in \partial\Omega$ there exists a simple smooth open arc, $\gamma_\zeta \subset \Omega$, connecting ζ to a point on $\{x : u(x) = 1/2\}$. Moreover,

$$\limsup_{x \longrightarrow \zeta, x \in \gamma_\zeta} \frac{\log |\nabla u(x)|}{\sqrt{\log 1/u(x) \, \log\log\log 1/u(x)}} \leq \; A' = A'(k) < \infty \tag{1.5}$$

for μ almost every $\zeta \in \partial\Omega$. Finally we show that (1.5) implies (b) of Theorems 1 -3.

As for the organization of this paper, in section 2 we statle some facts about quasiconformal mappings of $\mathbb{R}^2$ amd list some basic estimates from [BL]. We then use these estimates to deduce the A_2 claims for $|\nabla u|^{p-2}$ mentioned above. In section 3 we first state results for elliptic measure defined with respect to a degenerate elliptic equation, whose degeneracy is given in terms of an A_2 weight. Second we prove (a) of Theorems 1 - 3. In section 4 we use our basic estimates from section 2 to show that the levels of u are quasicircles (Lemma 4.1) and then obtain estimates for the nontangential maximal function of $\log |\nabla u|$ in $\{x : u(x) > t\}$ (Lemma 4.21). In section 5 we first prove (1.5) (see (5.26)) and then use this inequality to prove (b) of Theorems 1 - 3. Finally we make some closing remarks.

2. Basic Estimates. In the sequel, unless otherwise stated, c denotes a constant ≥ 1 which may depend on k, p, not necessarily the same at each occurrence. In general, $c(a_1, a_2, \ldots)$ denotes a constant ≥ 1 which may depend only on $a_1, a_2, \ldots,$ not necessarily the same at each occurrence. We shall often use results from [BL], so the reader is advised to have this paper nearby. We begin with a definition. A Jordan curve J is said to be a k quasicircle, $0 < k < 1$, if $J = f(\partial B(0,1))$ where $f \in W^{1,2}(\mathbb{R}^2)$ is a homeomorphism of $\mathbb{R}^2$ and

$$|f_{\bar{z}}| \leq k\,|f_z|, \; H^2 \text{ almost everywhere in } \mathbb{R}^2. \tag{2.1}$$

Here we are using complex notation, $i = \sqrt{-1}$, $z = x_1 + ix_2$, $2f_{\bar{z}} = f_{x_1} + if_{x_2}$, $2f_z = f_{x_1} - if_{x_2}$. If $K = \frac{1+k}{1-k}$ and $z, \zeta, w \in \mathbb{R}^2$ with $|z - w| = |z - \zeta|$, then

$$c_1(k)^{-1} \leq \frac{|f(z) - f(w)|}{|f(z) - f(\zeta)|} \leq c_1(k) \tag{2.2}$$

where $c_1(k)\to 1$ as $k\to 0$. The fact that $c_1(k)\to 1$ as $k\to 0$ can be deduced from compactness arguments and a knowledge of solutions to the Beltrami equation (see [A, chapter 5]). For the optimal value of $c_1(k)$ see the remark following Theorem 2.4 in chapter 1 of [L]. (2.2), the fact that f^{-1} is also k quasiconformal, and Mori's theorem (see [A]) imply the following quasisymmetry property of f :

$$[16c_1(k)]^{-K} \left(\frac{|z - w|}{|z - \zeta|}\right)^K \leq \frac{|f(z) - f(w)|}{|f(z) - f(\zeta)|} \leq 16\, c_1(k) \left(\frac{|z - w|}{|z - \zeta|}\right)^{1/K}, \tag{2.3}$$

(2.2) also implies (see [L, chapter 1, Theorem 6.7]) that if w lies on the open line segment connecting z to ζ, then

$$|f(z) - f(w)| + |f(w) - f(\zeta)| \leq c_1(k)\,|f(z) - f(\zeta)|. \tag{2.4}$$

Next, if $1 \leq q < 1 + 1/k$, then

$$\int_{B(0,4)} |f_z|^q \, dx \leq c_2(k, q)\,|f(1) - f(0)|^q. \tag{2.5}$$

We note that (2.5) was proved in [As] and settled a longstanding conjecture of Gehring and Reich (see also [EH]). For our purposes it would suffice to know that (2.5) held for some $q = q(k) > 2$ and this less sharp result can also be deduced from a knowledge of solutions to the Beltrami equation. We shall need the change of variables formula,

$$\int_{\mathbb{R}^2} \theta(y)\, dy = \int_{\mathbb{R}^2} (|f_z|^2 - |f_{\bar{z}}|^2)(x)\, \theta(f(x))\, dx \tag{2.6}$$

whenever θ is a nonnegative H^2 measurable function on $\mathbb{R}^2$ (see [A, chapter 5]).

To continue our definitions, let w_1, w_2 be distinct points on the Jordan curve J and let J_1, J_2 be the arcs on J with endpoints w_1, w_2. Then J is said to satisfy the Ahlfors three point condition provided there exists $1 \leq M < \infty$ such that

$$\min\{\text{diam } J_1, \text{ diam } J_2\} \leq M\,|w_1 - w_2|. \tag{2.7}$$

J is said to be asymptotically conformal (see [P, chaper 11]) provided for each $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that whenever diam $J_1 < \delta$, we have

$$|w_1 - w| + |w - w_2| < (1+\epsilon)|w_1 - w_2| \text{ for all } w \in J_1. \tag{2.8}$$

Ω is said to be a uniform domain provided there exists $\hat{M}, 1 \leq \hat{M} < \infty$, such that if $w_1, w_2 \in \Omega$, then there is a rectifiable curve $\gamma : [0,1] \to \Omega$ with $\gamma(0) = w_1, \gamma(1) = w_2$, and

$$\begin{aligned} &(a) \quad H^1(\gamma) \, \leq \, \hat{M} \, |w_1 - w_2| \\ &(b) \quad \min\{H^1(\gamma([0,t])), \, H^1(\gamma([t,1]))\} \, \leq \, \hat{M} \, d(\gamma(t), \partial\Omega), \, t \in [0,1]. \end{aligned} \tag{2.9}$$

Here as in the sequel, $d(E, F)$ denotes the distance between the nonempty sets E and F. If $1 \leq \tilde{M} < \infty$ and Ω is a domain, then a ball $B(w, r) \subset \Omega$ is said to be $\tilde{M}$ nontangential provided

$$\tilde{M} r \, > \, d(B(w,r), \partial\Omega) \, > \, \tilde{M}^{-1} r.$$

If $w_1, w_2 \in \Omega$, then a Harnack-chain from w_1 to w_2 in Ω is a sequence of $\tilde{M}$-non-tangential balls such that the first ball contains w_1, the last ball contains w_2, and consecutive balls intersect. A domain Ω is called non-tangentially accessible (NTA) if there exist $\tilde{M}$ (as above) such that:

(α) *Corkscrew condition.* For any $w \in \partial\Omega$, $0 < r \leq$ diam Ω, there exists $a = a_r(w) \in \Omega$ such that $\tilde{M}^{-1} r < |a - w| < r$ and $d(a, \partial\Omega) > \tilde{M}^{-1} r$,

(β) $\mathbb{R}^2 \setminus \Omega$ satisfies the corkscrew condition,

(γ) *Harnack chain condition.* Given $\epsilon > 0$, $w_1, w_2 \in \Omega$, $d(w_j, \partial\Omega) > \epsilon$, and $|w_1 - w_2| < C\epsilon$, there is a Harnack chain from w_1 to w_2 whose length depends on C but not on ϵ.

(2.10)

The following equivalences will be used throughout the proofs of Theorems 1-3.

Lemma 2.11. *If $\partial\Omega = J$ is a Jordan curve, then the conditions,*

J is a quasicircle
J satisfies the Ahlfors three point condition
Ω is a uniform domain
Ω is nontangentially accessible

all imply each other and constants in one definition can be determined from the constants in another definition.

For a proof of Lemma 2.11, see [G]. Note that Ω in Lemma 2.11 can be either of the two components of $\mathbb{R}^2 \setminus J$.

Next let $\partial\Omega$ be a k quasicircle, $r > 0, 1 < p < \infty$, and $\tilde{u}$ a positive weak solution to the p Laplacian in $B(w, 4r) \cap \Omega$ with $\tilde{u} \equiv 0$ in the Sobolev sense on $\partial\Omega \cap B(w, 4r)$ when this set is nonempty. More specifically, $\tilde{u} \in W^{1,p}(B(w, 4r) \cap \Omega)$ and (1.1) holds whenever $\theta \in W_0^{1,p}(B(w, 4r) \cap \Omega)$. Also $\zeta \tilde{u} \in W_0^{1,p}(B(w, 4r) \cap \Omega)$ whenever $\zeta \in C_0^\infty(B(w, 4r))$. Extend $\tilde{u}$ to $B(w, 4r)$ by putting $\tilde{u} \equiv 0$ on

$B(w,4r)\setminus\Omega$. Then there exists a locally finite positive Borel measure $\tilde\mu$ with support $\subset B(w,4r)\cap\partial\Omega$ and for which (1.2) holds with u replaced by $\tilde u$ and $\phi\in C_0^\infty(B(w,4r))$. Let $\max\limits_{B(z,s)}\tilde u$, $\min\limits_{B(z,s)}\tilde u$ be the essential supremum and infimum of $\tilde u$ on $B(z,s)$ whenever $B(z,s)\subset B(w,4r)$. With this notation, we first state Caccioppoli and Harnack inequalities for $\tilde u$.

Lemma 2.12. *Let $\tilde u$ be as above. Then*

$$c^{-1}r^{p-2}\int_{B(w,r/2)}|\nabla\tilde u|^p\,dx\ \le\ \max_{B(w,r)}\tilde u^p\ \le c\,r^{-2}\int_{B(w,2r)}\tilde u^p\,dx.$$

If $B(w,2r)\subset\Omega$, then

$$\max_{B(w,r)}\tilde u\ \le c\min_{B(w,r)}\tilde u.$$

Second we have the following interior regularity for $\tilde u$:

Lemma 2.13. *Let $\tilde u$ be as in Lemma 2.12. Then $\tilde u$ has a representative in $W^{1,p}(B(w,4r)\cap\Omega)$ (also denoted $\tilde u$) with Hölder continuous partial derivatives in $B(w,4r)\cap\Omega$. That is, for some $\lambda=\lambda(p)\in(0,1)$ and $\hat c=\hat c(p), 1\le\hat c<\infty$, we have*

$$\hat c^{-1}\,|\nabla\tilde u(w_1)-\nabla\tilde u(w_2)|\ \le\ (|w_1-w_2|/s)^\lambda\max_{B(z,s)}|\nabla\tilde u|\ \le\ \hat c\,s^{-1}\,(|w_1-w_2|/s)^\lambda\max_{B(z,2s)}\tilde u$$

whenever $w_1,w_2\in B(z,s)$ and $B(z,4s)\subset B(w,4r)\cap\Omega$.

Third we have the following boundary regularity for $\tilde u$.

Lemma 2.14. *Let $\tilde u$ be as in Lemma 2.12, $w\in\partial\Omega$, and $a_r(w)$ as in (2.10). Then*

$$(+)\ \max_{B(w,r)}\tilde u\ \le\ c\,\tilde u(a_r(w))$$

and there exists $\alpha=\alpha(p)\in(0,1)$ such that $\tilde u$ has a Hölder α continuous representative in $B(w,r)$ (also denoted $\tilde u$). Moreover if $x,y\in B(w,r)$ then

$$(++)\ |\tilde u(x)-\tilde u(y)|\le\ c\,(|x-y|/r)^\alpha\,\tilde u(a_r(w)).$$

Next we have

Lemma 2.15. *Let $\tilde u$ be as in Lemma 2.12, $w\in\partial\Omega$, and $\tilde\mu$ as in (1.2) (relative to $\tilde u$). Then*

$$r^{p-2}\,\tilde\mu(B(w,r))\approx\tilde u(a_r(w))^{p-1}\approx\left(\max_{B(w,r)}\tilde u\right)^{p-1}.$$

In Lemma 2.15, as in the sequel, $e\approx f$ means e/f is bounded above and below by positive constants depending only on p,k.

References for the proofs of Lemmas 2.12 - 2.15 can be found in [BL]. Next we state a boundary Harnack theorem proved in [BL, Lemma 2.16].

Lemma 2.16. *Let $w \in \partial\Omega$ and $\tilde u_1, \tilde u_2$ as in Lemma 2.12 with $\tilde u$ replaced by $\tilde u_i, i = 1, 2$. If $\tilde u_1(a_r(w)) = \tilde u_2(a_r(w))$, then*

$$c^{-1} < \frac{\tilde u_1(x)}{\tilde u_2(x)} < c \text{ for all } x \in B(w, r/c) \cap \Omega.$$

Let Ω_1, Ω_2 denote the bounded and unbounded components, respectively, of $\mathbb{R}^2 \setminus \partial\Omega$, so that either $\Omega = \Omega_1$ or $\Omega = \Omega_2$. In the proof of Theorems 1 - 3, we assume, as we may, that

$$0 \in \Omega_1 \text{ and diam } \partial\Omega \leq c\, d(0, \partial\Omega) \tag{2.17}$$

since otherwise we can get (2.17) from a translation thanks to (2.10) (α). This translation is permissible since the p Laplacian is invariant under translations. Thus we assume the existence of r_0, R_0 with $0 < r_0 < R_0$ and

$$B(0, 4r_0) \subset \Omega_1 \subset B(0, R_0/4) \text{ with } r_0 \approx R_0 \approx \text{ diam } \partial\Omega. \tag{2.18}$$

Let u_1 be the weak solution to the p Laplacian in $D_1 = \Omega_1 \setminus \bar B(0, r_0)$ with $u_1 = 0$ on $\partial\Omega_1$ and $u_1 = 1$ on $\partial B(0, r_0)$ in the Sobolev sense. Also let u_2 be a weak solution to the p Laplacian in $D_2 = B(0, R_0) \setminus \bar\Omega_1$ with $u_2 = 0$ on $\partial\Omega$ and $u_2 = 1$ on $\partial B(0, R_0)$. $u_1, 1 - u_2$, are called the p capacitary functions for the rings D_1, D_2. We have (see [BL, Lemma 2.26]) :

Lemma 2.19. *u_1 is real analytic in an open set containing $D_1 \cup \partial B(0, r_0)$ while u_2 is real analytic in an open set containing $D_2 \cup \partial B(0, R_0)$. Also if $x \in D_i, i = 1, 2$, then*

$$c^{-1}\, d(x, \partial\Omega)^{-1}\, u_i(x) \;\leq\; |\nabla u_i(x)| \;\leq\; c\, d(x, \partial\Omega)^{-1}\, u_i(x)$$

$$\text{and } \sum_{l,m=1}^{2} |(u_i)_{x_l x_m}|(x) \;\leq\; c\, d(x, \partial\Omega)^{-2}\, u_i(x).$$

It is easily shown using the weak maximum principle for p harmonic functions and Lemma 2.15 that it suffices to prove Theorems 1 - 3 for u_1, u_2 (see the inequality above (2.24) in [BL] and the discussion following this inequality). Thus in addition to (2.18) we assume in the proofs of Theorems 1 -3 that

$$u = u_1 \text{ when } \Omega = \Omega_1 \text{ and } u = u_2 \text{ when } \Omega = \Omega_2. \tag{2.20}$$

For ease of notation we now drop the subscripts and write u for u_1 when $\Omega = \Omega_1$ and u for u_2 when $\Omega = \Omega_2$. μ, as usual, denotes the measure corresponding to u. We prove,

Lemma 2.21. *There exists $\epsilon_0 = \epsilon_0(k) > 0$ such that if $0 < |p - 2| < \epsilon_0$, then $|\nabla u|^{p-2}$ extends to an A_2 weight on $\mathbb{R}^2$ with constant depending only on p, k.*

Proof: Recall that a non negative H^2 measurable function h on $\mathbb{R}^2$ is said to be an A_2 weight with constant γ if

$$\int_{B(x,r)} h\,dx \cdot \int_{B(x,r)} h^{-1}\,dx \,\leq\, \gamma\,\pi^2\,r^4 \tag{2.22}$$

whenever $x \in \mathbb{R}^2$ and $r > 0$. In the proof of Lemma 2.21, we first assume that $\Omega = \Omega_1$. Set $h(x) = d(0, \partial\Omega)^{-|p-2|}$ when $x \in \bar{B}(0, r_0)$ and $h(x) = |\nabla u|^{|p-2|}$ when $x \in \Omega \setminus \bar{B}(0, r_0)$. To define h on $\mathbb{R}^2 \setminus \Omega$, let $S_j = S(y_j, r_j), j = 1, 2, \dots,$ be a Whitney decomposition of $\mathbb{R}^2 \setminus \bar{\Omega}$ into open squares with center at y_j and sidelength r_j. That is,

$$\begin{aligned}
&(a) \quad \textstyle\bigcup_j \bar{S}(y_j, r_j) = \mathbb{R}^2 \setminus \bar{\Omega},\\
&(b) \quad S(y_j, r_j) \cap S(y_i, r_i) = \emptyset \text{ when } i \neq j,\\
&(c) \quad 10^{-4}\, d(S_j, \partial\Omega) \leq r_j \leq 10^{-2}\, d(S_j, \partial\Omega).
\end{aligned} \tag{2.23}$$

If $r_j <$ diam $\partial\Omega$, choose $w_j \in \Omega$ with

$$d(w_j, \partial\Omega) \approx d(w_j, y_j) \approx d(y_j, \partial\Omega). \tag{2.24}$$

Put $h(x) = h(w_j)$ when $x \in S(y_j, r_j)$. Observe from Lemma 2.19 that

$$h(x) = h(w_j) \approx h(w) \text{ whenever } x \in S_j \text{ and } w \in B(w_j, d(w_j, \partial\Omega)/2). \tag{2.25}$$

If $r_j \geq$ diam $\partial\Omega$ we put $h(x) = d(0, \partial\Omega)^{-|p-2|}$. Then h is defined H^2 almost everywhere on $\mathbb{R}^2$, as we see from (2.6) with θ equal to the characteristic function of $\partial\Omega$.

To prove (2.22) we consider several cases. First if $r < \frac{1}{2}\, d(x, \partial\Omega)$, then from Lemma 2.19 and (2.25) we see that h is essentially constant on $B(x, r)$, so (2.22) is true in this case. Otherwise, choose $\hat{x} \in \partial\Omega$ with $d(x, \partial\Omega) = d(x, \hat{x})$. If also $r \leq c^{-1}$ diam Ω, where c is so large that $B(\hat{x}, 4r) \cap B(0, 2r_0) = \emptyset$, then we can use Lemmas 2.12, 2.14, (2.23) - (2.25), and Hölder's inequality to find that

$$\begin{aligned}
\int_{B(x,r)} h\,dx \,&\leq\, \int_{B(\hat{x},4r)} h\,dx \,\leq c \int_{\Omega\cap B(\hat{x},cr)} |\nabla u|^{|p-2|}\,dx\\
&\leq c \left(\int_{B(\hat{x},cr)\cap\Omega} |\nabla u|^p\,dx \right)^{|1-2/p|} r^{2(1-|1-2/p|)} \,\leq\, c\,u(a_r(\hat{x}))^{|p-2|}\, r^{2-|p-2|}\,.
\end{aligned} \tag{2.26}$$

If $r > c^{-1}$ diam $\partial\Omega$, then $\partial\Omega$ can be covered by at most $\hat{c} = \hat{c}(k, p)$ balls of radius, c^{-1} diam $\partial\Omega$. Using this fact, (2.17), the definition of h and arguing as in (2.26), we see that

$$\int_{B(x,r)} h\,dx \leq cr^2\, d(0, \partial\Omega)^{-|p-2|}. \tag{2.27}$$

To estimate $\int_{B(x,r)} h^{-1} dx$ when $\frac{1}{2} d(x, \partial\Omega) \leq r \leq c^{-1}$ diam $\partial\Omega,$ we note from Harnack's inequality (see Lemma 2.12) and (2.9) that there exists $\beta = \beta(k, p) \geq 1$, for which

$$c\,u(y) \,\geq\, u(a_r(\hat{x})) \left(\frac{d(y, \partial\Omega)}{r} \right)^{\beta} \text{ whenever } y \in B(\hat{x}, 4r) \cap \Omega. \tag{2.28}$$

Using (2.28) and Lemma 2.19, we obtain

$$\int_{B(x,r)} h^{-1}dx \;\le\; c r^{\beta|p-2|} u(a_r(\hat{x}))^{-|p-2|} \int_{B(\hat{x},4r)\cap\Omega} d(y,\partial\Omega)^{(1-\beta)|p-2|}\, dy. \tag{2.29}$$

To estimate the integral on the righthand side of (2.29) let

$$\begin{gathered}\rho = \text{ diam } f^{-1}[B(\hat{x},4r)\cap\Omega] \\ \zeta = f^{-1}(\hat{x}) \\ \tilde{f}(x) = f(\zeta+\rho x), x\in\mathbb{R}^2.\end{gathered}$$

Then $B(\hat{x},4r)\cap\Omega \subset \tilde{f}(B(0,1))$ and from the quasisymmetry property of $\tilde{f}$ (see (2.3)) we deduce that if $y=\tilde{f}(w)\in B(\hat{x},4r)\cap\Omega$ then

$$\frac{r}{d(y,\partial\Omega)} \le c\,(1-|w|)^{-K}. \tag{2.30}$$

Using (2.30) in (2.29) and changing variables (see (2.6)), we find

$$\int_{B(\hat{x},4r)\cap\Omega} d(y,\partial\Omega)^{(1-\beta)|p-2|}\, dy \le c\, r^{(1-\beta)|p-2|} \int_{B(0,1)} (1-|w|)^{K(1-\beta)|p-2|}\, |\tilde{f}_z(w)|^2\, dw. \tag{2.31}$$

Again from quasisymmetry of $\tilde{f}$ we see that $|\tilde{f}(1)-\tilde{f}(0|\le c\,r$. Using this inequality, Hölder's inequality, and (2.5) with f replaced by $\tilde{f}$, in (2.31), we obtain for $2<q<1+\frac{1}{k}$ that

$$\begin{aligned}&\int_{B(0,1)} (1-|w|)^{K(1-\beta)|p-2|}\, |\tilde{f}_z(w)|^2\, dw \\ &\le \left(\int_{B(0,1)} (1-|w|)^{Kq(1-\beta)|p-2|/(q-2)}\, dw\right)^{1-2/q} \cdot \left(\int_{B(0,1)} |f_z(w)|^q\, dw\right)^{2/q} \\ &\le\; c(k,p,q) r^2\end{aligned} \tag{2.32}$$

provided

$$|p-2| < \frac{(q-2)}{Kq(\beta-1)}. \tag{2.33}$$

We note that β can be chosen independent of p when $3/2<p<5/2$. Using this note and combining (2.29), (2.31), (2.32), we deduce the existence of $\epsilon_0(k)>0$ such that if $|p-2|<\epsilon_0$, then

$$\int_{B(x,r)} h^{-1}dx \;\le\; c\, r^{2+|p-2|} u(a_r(\hat{x}))^{-|p-2|}\,. \tag{2.34}$$

If $r > c^{-1}$ diam $\partial\Omega$, then $\partial\Omega$ can be covered by at most $\hat{c}=\hat{c}(k,p)$ balls of radius, c^{-1} diam $\partial\Omega$. Using this fact, (2.17), the definition of h and arguing as in (2.34) we find for $|p-2|<\epsilon_0$,

$$\int_{B(x,r)} h^{-1}\, dx \le c\, r^2\, d(0,\partial\Omega)^{|p-2|}\,. \tag{2.35}$$

It now follows from (2.26), (2.27), (2.34) and (2.35) that Lemma 2.21 is true when $\Omega = \Omega_1$.

The proof of Lemma 2.21 when $\Omega = \Omega_2$ is quite similar. In this case we put $h \equiv d(0, \partial\Omega)^{-|p-2|}$ when $|x| \geq R_0$, while $h(x) = |\nabla u|^{p-2}$ on the rest of Ω. Define h on $\mathbb{R}^2 \setminus \bar{\Omega}$ as in (2.23) - (2.25). Proceeding as in (2.26) - (2.35), we get Lemma 2.21 for $\Omega = \Omega_2$. The proof of Lemma 2.21 is now complete. □

Next in section 2 we prove,

Lemma 2.36. *Given* $p, 1 < p < \infty$, *there exists* $k_0 = k_0(p)$, $0 < k_0(p) \leq 1/2$, *such that if* $\partial\Omega$ *is a* k *quasicircle and* $0 < k \leq k_0(p)$, *then* $|\nabla u|^{p-2}$ *extends to an* A_2 *weight on* $\mathbb{R}^2$ *with constant depending only on* p, k.

Proof : We prove Lemma 2.36 only when $\Omega = \Omega_1$. The proof for $\Omega = \Omega_2$ is essentially unchanged. Defining h as in Lemma 2.21 (see (2.23)-(2.25)) and arguing as in this lemma we get (2.26), (2.27). To estimate $\int_{B(x,r)} h^{-1} dx$ when $\frac{1}{2} d(x, \partial\Omega) \leq r \leq c^{-1}$ diam $\partial\Omega$, we note that the constants in (2.9), (2.10) can be chosen independent of k when $0 < k \leq 1/2$. With this understanding let $\eta = 20^{-1} \min\{1, |p-2|^{-1}\}$. We shall show that if $0 < k_0 \leq 1/2$ is small enough, then

$$c(p)\, u(y) \geq \left(\frac{d(y, \partial\Omega)}{r}\right)^{1+\eta} u(a_r(\hat{x})),\ y \in B(\hat{x}, 4r) \cap \Omega. \tag{2.37}$$

Once (2.37) is proved we can repeat the argument following (2.28) with β replaced by $1+\eta$ to get (2.34) provided (2.33) holds with $\beta - 1$ replaced by η. Using $k_0 \leq 1/2, q = 5/2$, and the above value of η, we deduce from (2.5) that (2.33) holds. If $r > c^{-1}$ diam $\partial\Omega$ we can argue as previously to get (2.35). Thus to prove Lemma 2.36 we need only prove (2.37).

To continue, we show that (2.37) is implied by a certain flatness condition on $\partial\Omega$. In order to state this condition, recall that the Hausdorff distance between the sets E, F (denoted $\rho(E, F)$) is

$$\rho(E, F) = \sup\{d(y, F) : y \in E\} + \sup\{d(y, E) : y \in F\}.$$

We say that $\partial\Omega$ is θ Reifenberg flat (see [R]) on a scale δ provided the following is valid. Given $w \in \partial\Omega, 0 < r \leq \delta$ diam $\partial\Omega$, there exists a line $l = l_{w,r}$ such that

$$w \in l \text{ and } \rho\left(\partial\Omega \cap B(w, r/\theta), l \cap B(w, r/\theta)\right) \leq \theta\, r. \tag{2.38}$$

We shall show that if (2.38) holds for a certain $\theta = \theta(p), \delta = \delta(p) > 0$, then (2.37) is valid. To begin the proof, suppose $\hat{x} \in \partial\Omega$ and $r < \delta$ diam $\partial\Omega$. Let $l = l_{\hat{x},r}$ be the line guaranteed by (2.38). For ease of notation we assume that l is parallel to the x_1 axis and contains $\hat{x} = (0, s)$. Let

$$T_1 = \{(x_1, x_2) : |x_1| < \tfrac{1}{4} r/\theta,\ s + 2\theta r < x_2 < \tfrac{1}{4} r/\theta\}$$

$$T_2 = \{(x_1, x_2) : |x_1| < \tfrac{1}{4} r/\theta,\ s - 2\theta r > x_2 > -\tfrac{1}{4} r/\theta\}.$$

From (2.38) we see that either $T_1 \subset \Omega, T_2 \subset \mathbb{R}^2 \setminus \Omega$ or vice versa. Assume the first possibility holds. Let g be the p harmonic function in $G = T_1 \setminus \bar{B}((0, s+8r), r)$ with boundary values 0 on ∂T_1 and 1 on $\partial B((0, s+2r), r)$ in the Sobolev sense. From Harnack's inequality we find that $c(p)\, u \geq u(a_r)\, g$ in G. Also we can use Lemma 2.16 to compare $u(a_r)\, g$ with $u(a_r)\, (x_2 - s - 2r\theta)$

in $T_1 \cap \{(x_1, x_2) : |x_1| < 2r, |x_2 - s| < 2r\}$, since both functions vanish in the Sobolev sense on $\partial T_1 \cap \{(x_1, x_2) : x_2 = s + 2r\theta\}$. Moreover the constants in the comparison depend only on p. We conclude for some $c_+(p)$ that

$$u(y) \geq c_+^{-1}\, u(a_r) d(y, \partial\Omega)/r \tag{2.39}$$

whenever $y \in B(\hat{x}, r) \cap \Omega$ and $d(y, \partial\Omega) \geq 4\theta r$. We now proceed by induction. Suppose we have shown for $1 \leq m \leq n$ that

$$u(y) \geq c_+^{-m}\, u(a_r)\, d(y, \partial\Omega)/r \tag{2.40}$$

whenever $y \in B(\hat{x}, r) \cap \Omega$ and $(4\theta)^m r \leq d(y, \partial\Omega) < (4\theta)^{m-1} r$. Note that (2.40) holds for $n = 1$ thanks to (2.39). If now $y \in B(\hat{x}, r) \cap \Omega$ and $(4\theta)^{(n+1)} r \leq d(y, \partial\Omega) < (4\theta)^n r$, we can choose $\hat{y} \in \partial\Omega$ with $d(y, \hat{y}) = d(y, \partial\Omega)$. Putting $\rho = 10^{-2}\, d(y, \partial\Omega)/\theta$, we can repeat the above argument with $\hat{y}, \rho$ playing the role of $\hat{x}, r$ to find as in (2.39) that

$$u(y) \geq c_+^{-1}\, u(y^*)\, \frac{d(y, \partial\Omega)}{d(y^*, \partial\Omega)} \tag{2.41}$$

for some $y^* \in B(\hat{x}, r) \cap \Omega$ with $d(y^*, \partial\Omega) \geq (4\theta)^n\, r$. Using (2.41) and (2.40) with y replaced by y^* we obtain that (2.40) is true for $m = n + 1$. Thus by induction (2.40) is valid for $n = 1, 2, \ldots$ From (2.40) we see that (2.38) holds provided $\theta = \theta(p)$ is chosen so small that $c_+^{-1} \geq \theta^\eta$. With θ now fixed we note that if δ diam $\partial\Omega \leq r \leq c^{-1}$ diam $\partial\Omega$ for some $\delta = \delta(p)$, then we can cover $\partial\Omega$ by at most $c = c(p)$ disks of radius δ. Using this fact, Harnack's inequality, and arguing as above we get (2.37) in this case also. The same argument works if $r > c^{-1}$ diam $\partial\Omega$. Thus to prove (2.37) and hence Lemma 2.36, it suffices to show that (2.38) is true.

To prove (2.38) let $z_1, z_2 \in \partial B(0, 1)$ with $|z_1 - z_2| \leq 1/2$ and suppose that z lies on the short arc of $\partial B(0, 1)$ connecting z_1 to z_2. Let z^* be the intersection point of the line segment from z_1 to z_2 with the ray from the origin to z. From (2.4) we see that

$$|f(z_1) - f(z^*)| + |f(z^*) - f(z_2)| \leq c_1(k)\, |f(z_1) - f(z_2)|. \tag{2.42}$$

Also from (2.3) and basic geometry we deduce that

$$|f(z) - f(z^*)| \leq 16\, c_1(k)\, |z_1 - z_2|^{1/K}\, [|f(z_1) - f(z)| + |f(z) - f(z_2)|]. \tag{2.43}$$

Using the triangle inequality and (2.43) in (2.42) we find that

$$(1 - 32c_1(k)|z_1 - z_2|^{1/K})[\,|f(z_1) - f(z)| + |f(z) - f(z_2)|\,] \leq c_1(k)\, |f(z_1) - f(z_2)|. \tag{2.44}$$

Once again from (2.3) we have

$$[16c_1(k)]^{-K}\, (|z_1 - z_2|/2)^K \leq \frac{|f(z_1) - f(z_2)|}{\text{diam } \partial\Omega} \leq 16c_1(k)\, (|z_1 - z_2|/2)^{1/K}. \tag{2.45}$$

Given $\theta, 0 < \theta \leq 10^{-6}$, put $\delta = \theta^{50}$ and choose $k_1 = k_1(\theta), 0 < k_1 \leq 1/4$ so that $c_1(k) \leq 1 + \theta^{10}$, whenever $0 < k \leq k_1$. This choice is possible since $c_1(k) \to 1$ as $k \to 0$. Let $\hat{x} \in \partial\Omega$, $0 < r \leq \delta$ diam $\partial\Omega$, and suppose that w_1, w_2 are the first and last times $\partial\Omega$ enters $\bar{B}(\hat{x}, r/\theta^2)$ in a certain parametrization of $\partial\Omega$. Choose $z_1, z_2, z_3 \in \partial B(0, 1)$ with $f(z_1) = w_1, f(z_2) = w_2, f(z_3) = \hat{x}$. Observe that $1 < K \leq 2$ and $|w_1 - w_2| \leq 2r/\theta^2 \leq 2\theta^{48}$ diam $\partial\Omega$. This observation, (2.45), and a ball park estimate yield

$$c_1(k)\, \left(1 - 32c_1(k)\, |z_1 - z_2|^{1/K}\right)^{-1} \leq 1 + 2\theta^{10}. \tag{2.46}$$

From (2.44) and (2.46) we see that

$$|f(z_1) - f(z)| + |f(z) - f(z_2)| \le (1 + 2\theta^{10})|f(z_1) - f(z_2)| \tag{2.47}$$

whenever z lies on the short arc of $\partial B(0,1)$ connecting z_1 to z_2. Taking $z = z_3$ in (2.47) we find

$$2\,(1 + 2\theta^{10})^{-1}\, r/\theta^2 \le |w_2 - w_1| \le 2r/\theta^2. \tag{2.48}$$

Draw the line l' connecting w_1 to w_2 and put $S = \{x \in \mathbb{R}^2 : d(x, l') \le r\theta^2\}$. Then from (2.47), (2.48) and the Pythagorean theorem we find that

$$\text{If } z \text{ lies on the short arc connecting } z_1 \text{ to } z_2, \text{ then } f(z) \in S. \tag{2.49}$$

From (2.49) and quasisymmetry of f we can deduce that

$$\partial\Omega \cap B(\hat{x}, r/\theta) \subset S. \tag{2.50}$$

Indeed, otherwise from (2.49) we see there exists z_4 on the long arc of $\partial B(0,1)$, connecting z_1 to z_2 with $f(z_4) \in \bar{B}(\hat{x}, r/\theta)$. Then either $|z_1 - z_3| \le |z_4 - z_3|$ or $|z_2 - z_3| \le |z_4 - z_3|$. In either case since $r/\theta^2 = |w_i - w_3|, i = 1, 2$, we obtain from (2.3) that $\theta^{-1} \le 16c_1(k)$ which is a contradiction. Thus (2.50) is true. Let l be the line through $\hat{x}$ that is parallel to l'. From (2.49), (2.50) we see that (2.38) is true. In view of our earlier remarks, the proof of Lemma 2.36 is now complete. □

Finally in section 2 we prove,

Lemma 2.51. *If $\partial\Omega$ is asymptotically conformal and $1 < p < \infty$, then $|\nabla u|^{p-2}$ extends to an A_2 weight on $\mathbb{R}^2$ with constant depending only on $p, \partial\Omega$.*

Proof: We first show that the Ahlfors three point condition (see (2.7)) holds. If diam $J_1 < \delta(1)$ where $\delta(1)$ is chosen so that (2.8) holds with $\epsilon = 1$, then (2.7) is trivial. Otherwise (2.7) follows for some $M = M(\partial\Omega) < \infty$ from a compactness argument and the fact that $\partial\Omega$ is a Jordan curve. From (2.7) and Lemma 2.11 we conclude for some $k = k(\partial\Omega), 0 < k < 1$, that $\partial\Omega$ is a k quasicircle. Using this fact, the definition of asymtotically conformal in (2.7) and proceeding as in the proof of (2.49), (2.50) we find that (2.38) is valid. Armed with (2.38) we can now argue as in Lemma 2.36, to first get (2.37) and then Lemma 2.51. □

3. Proof of Theorems 1 - 3 (*a*). In this section we first state some theorems from [FKS], [FKJ], [FKJ1], [DJK], tailored to our situation. As mentioned in section 1, these theorems will play a major role in our proof of Theorems 1-3. Second we prove Theorems 1 - 3 (*a*). We continue with the same assumptions and notation as in section 2.

Let D be the doubly connected defined relative to u, Ω (see Lemma 2.19) and as in (1.4) put

$$b_{ik}(x) = |\nabla u|^{p-4}[(p-2)u_{x_i}u_{x_k} + \delta_{ik}|\nabla u|^2](x)\, 1 \le i, k \le 2, \tag{3.1}$$

whenever $x \in D$. Again, δ_{ij} is the Kronecker δ. Observe from Lemma 2.19 that $b_{ij}, 1 \le i, j \le 2$, is real analytic in $D \cup (\partial D \setminus \partial\Omega)$. Also, if $x \in D$, and $\xi \in \mathbb{R}^2 \setminus \{0\}$, then

$$\min\{2-p, 1\}\, |\nabla u(x)|^{p-2}\, |\xi|^2 \le \sum_{i,j=1}^{2} b_{ij}(x)\, \xi_i\, \xi_j \le p\, |\nabla u|^{p-2}\, |\xi|^2. \tag{3.2}$$

Put

$$L\,\zeta(x)\;=\;\sum_{i,k=1}^{2}\frac{\partial}{\partial x_i}\,[\,b_{ik}(x)\zeta_{x_k}(x)\,],\;x\in D,\tag{3.3}$$

when $\zeta\in C_0^\infty(D)$. Let $h(x)$ be the extension of $|\nabla u|^{p-2}$ defined in section 2. Throughout section 3 we assume that one of the scenarios in Lemmas 2.21, 2.36, 2.51 is valid, Under the hypotheses of Lemmas 2.21 or 2.36, we continue to allow $c\geq 1$ to depend only on k,p. However, under the assumptions in Lemma 2.51, c may depend only on $p,\partial\Omega$. Thus h is an A_2 weight on $\mathbb{R}^2$ with constant c. From (3.2) we see at $x\in D$ that the divergence form operator in (3.3) has ellipticity constants $\approx h(x)$ when $x\in D$. Set $b_{ij}(x)=h(x)\,\delta_{ij}$ when $x\in\mathbb{R}^2\setminus\bar D$ and $1\leq i,j\leq 2$. To state the results of the above authors we shall need some more notation. If O is open let $\tilde W^{1,2}(O)$ be the weighted Sobolev space of equivalence classes of functions w with distributional gradient ∇w and norm

$$\|w\tilde\|_{1,2}=\|w\,h^{1/2}\|_2+\|(\nabla w)\,h^{1/2}\|_2<\infty.$$

Again, $\|\cdot\|_2$ is the Lebesgue 2 norm in O. Let $\tilde W_0^{1,2}(O)$ be the closure of $C_0^\infty(O)$ in the norm of $\tilde W^{1,2}(O)$. If $O\subset D$ we say that v is a weak solution to $Lv=0$ in O provided $v\in\tilde W^{1,2}(O)$ and

$$\int\sum_{i,j=1}^{2}b_{ij}\,v_{x_i}\phi_{x_j}\,dx\;=\;0\text{ whenever }\phi\in C_0^\infty(O).\tag{3.4}$$

Next let $\hat D\subset D$ be an NTA domain with NTA constants (see (2.10)) having the same dependence as c (discussed above). Let $r>0$, and suppose that v is a positive weak solution to $Lv=0$ in $\hat D\cap B(w,4r)$ and $v\equiv 0$ in the weighted Sobolev sense on $\partial\hat D\cap B(w,4r)$ when this set is nonempty. Extend v to $B(w,4r)$ by putting $v\equiv 0$ on $B(w,4r)\setminus\hat D$. Under these assumptions we state some basic estimates for v similar to Lemmas 2.12, 2.14.

Lemma 3.5. *Let v be as above. Then*

$$c^{-1}r^2\int_{B(w,r/2)}|\nabla v|^2\,h\,dx\;\leq\;\left(\int_{B(w,r)}h\,dx\right)\cdot\max_{B(w,r)}v^2\;\leq c\int_{B(w,2r)}v^2\,h\,dx.$$

If $B(w,2r)\subset\hat D$, then

$$\max_{B(w,r)}v\;\leq c\min_{B(w,r)}v.$$

Lemma 3.6. *Let v be as in Lemma 3.5, $w\in\partial\hat D,0<r\leq$ diam $\hat D$, and $\hat a_r(w)$ as in (2.10) with Ω replaced by $\hat D$. Then*

$$(+)\;\max_{B(w,r)}v\;\leq\;c\,v(\hat a_r(w))$$

and there exists $\sigma\in(0,1)$ such that v has a Hölder σ continuous representative in $B(w,r)$ (also denoted v). Moreover if $x,y\in B(w,r)$ then

$$(++)\;|v(x)-v(y)|\leq\;c\,(|x-y|/r)^\sigma\,v(\hat a_r(w)).$$

For a proof of Lemmas 3.5, 3.6, see [FKS]. σ in Lemma 3.6 has the same dependence as c. In general the constants in [FKS], [FKJ], [FKJ1], [DJK], depend only on the NTA constants, the A_2

constant of the weight, and the elliptictity constants (as in (3.2)). Hence in our case corresponding constants will have the same dependence as c.

Next we list some results for the Green's function, $\hat{g}(\cdot,\cdot)$, mapping $\hat{D}\times\hat{D}$ into the extended positive real numbers (see [FKJ], [FKJ1] for proofs). $\hat{g}$ is uniquely determined by

$$\begin{aligned}&(\alpha)\quad \hat{g}(\cdot,y)\in \tilde{W}_0^{1,2}(\hat{D}\setminus B(y,s)) \text{ whenever } \bar{B}(y,s)\subset\hat{D},\\ &(\beta)\quad \int\sum_{i,j=1}^{2} b_{ij}\,\hat{g}_{x_i}(\cdot,y)\,\phi_{x_j}\,dx=\phi(y) \text{ whenever } \phi\in C_0^\infty(\hat{D}),\ y\in\hat{D}.\end{aligned}\tag{3.7}$$

Moreover,

$$\begin{aligned}&(i)\quad \hat{g}(x,y)=\hat{g}(y,x) \text{ whenever } x\neq y\in\hat{D},\\ &(ii)\quad \hat{g}(\cdot,y) \text{ is real analytic in } \hat{D}\cup(\partial\hat{D}\setminus\partial\Omega),\\ &(iii)\quad h(y)\,\hat{g}(x,y)\approx\log\tfrac{d(y,\partial\hat{D})}{|x-y|} \text{ whenever } x\in B(y,\tfrac12 d(y,\partial\hat{D})).\end{aligned}\tag{3.8}$$

(ii) follows from the fact that b_{ij} is real analytic in $\hat{D}\cup(\partial\hat{D}\setminus\partial\Omega)$ and a theorem of Hopf (see [BL] for references). Put $\hat{g}(\cdot,y)\equiv 0$ in $\mathbb{R}^2\setminus\hat{D}$. Observe from (3.7) (β) and (3.8) (i) that $\hat{g}(\cdot,y)$ and $\hat{g}(y,\cdot)$ are weak solutions to L in $\hat{D}$. Using this fact and Lemmas 3.5, 3.6 with $v=\hat{g}(\cdot,x)$, one can deduce for given $x\in\hat{D}$, the existence of a positive measure $\hat{\omega}(x,\cdot)$ with support contained in $\partial\hat{D}$, $\hat{\omega}(x,\partial\hat{D})=1$, and

$$\int\sum_{i,j=1}^{2} b_{ij}\,\hat{g}_{w_i}(\cdot,x)\,\psi_{w_j}\,dw \;=\; \psi(x)-\int\psi\,d\hat{\omega}(x,\cdot)\tag{3.9}$$

whenever $\psi\in C_0^\infty(\mathbb{R}^2)$. If E is a Borel subset of $\partial\hat{D}, x\in\hat{D}$, we say that $\hat{\omega}(x,\cdot)$ is elliptic measure for $\hat{D}$ corresponding to x, L. Next we state Lemmas 3.10, 3.11 which are analogous to Lemmas 2.15, 2.16.

Lemma 3.10. *Let $w\in\partial\hat{D}, r>0,$ and $x\in\hat{D}\setminus B(w,4r)$. Then*

$$\hat{\omega}(x,B(w,r))\approx\hat{g}(x,\,\hat{a}_r(w))\,\hat{h}(a_r(w)).$$

Lemma 3.11. *Let $w,r,x,\hat{D}$ be as in Lemma 3.10 and suppose that v_1,v_2 are as in Lemma 3.5 with v replaced by $v_i, i=1,2$. If $v_1(\hat{a}_r)=v_2(\hat{a}_r)$, then $v_1/v_2\leq c$ in $B(w,r/2)$ and there exists $\tilde{\sigma}, 0<\tilde{\sigma}<1$, such that*

$$\left|\frac{v_1(y)}{v_2(y)}-\frac{v_1(z)}{v_2(z)}\right|\leq c\left(\frac{|y-z|}{r}\right)^{\tilde{\sigma}}$$

whenever $y,z\in B(w,r)$.

From 3.8 (iii), Harnack's inequality, and Lemma 3.10 we see that

$$c\,\hat{\omega}(\hat{a}_r(w),B(w,r/2))\geq 1.\tag{3.12}$$

Lemmas 3.10, 3.11 and (3.12) imply that if $x \in \hat{D} \setminus B(w, 4r)$ and $E \subset B(w, r/2) \cap \partial\hat{D}$ is Borel, then

$$\frac{\hat{\omega}(x, E)}{\hat{\omega}(x, B(w, r/2))} \approx \hat{\omega}(\hat{a}_r(w), E). \tag{3.13}$$

Fix $x_0 \in \hat{D}$. From (3.13) or Harnack's inequality we see that $\hat{\omega}(x, \cdot), x \in \hat{D}$, is absolutely continuous with respect to $\hat{\omega}(x_0, \cdot)$ and vice versa. Let $k(x, \cdot)$ be the Radon - Nikodym derivative of $\hat{\omega}(x, \cdot)$ with respect to $\hat{\omega}(x_0, \cdot)$. Using Lemma 3.6 and (3.7) - (3.9), it is not difficult to show that the continuous Dirichlet problem for L, $\hat{D}$ always has a solution. That is, given a continuous function θ on $\partial\hat{D}$, there exists a weak solution Θ to L in $\hat{D}$ with continuous boundary values $= \theta$. In fact for $x, x_0 \in \hat{D}$,

$$\Theta(x) = \int \theta \, d\hat{\omega}(x, \cdot) = \int \theta \, k(x, \cdot) \, d\hat{\omega}(x_0,). \tag{3.14}$$

It is also true for $z \in \partial\hat{D}$, that

$$k(x, z) \; = \; \lim_{y \longrightarrow z} \frac{\hat{g}(x, y)}{\hat{g}(x_0, y)}. \tag{3.15}$$

If $w \in \partial\hat{D}, 0 < 4r \leq x_0$, then from (3.15), Lemma 3.10, and Lemma 3.11 with $v_1 = \hat{g}(x, \cdot), \; v_2 = \hat{g}(x_0, \cdot)$, it follows that

$$|k(\hat{a}_r(w), z) - k(\hat{a}_r(w), z')| \; \leq \; c \left(\frac{|z - z'|}{r} \right)^{\tilde{\sigma}} [\omega(x_0, B(w, r))]^{-1} \tag{3.16}$$

whenever $z, z' \in \partial\hat{D} \cap B(w, r/2)$. Finally we state the Main Lemma in [DJK].

Lemma 3.17. *Let $\hat{D}, \hat{\omega}$ be as above and let ω be elliptic measure for D corresponding to L. Let $0 < r \leq$ diam $D, w \in \partial\hat{D}$ and E a Borel subset of $\partial\hat{D} \cap \partial D \cap B(w, r)$. There exists $\hat{\sigma}, 0 < \hat{\sigma} < 1$, such that*

$$\omega(\hat{a}_r(x), E) \; \leq \; c \, \hat{\omega}(\hat{a}_r(x), E)^{\hat{\sigma}} \, .$$

As usual $\hat{\sigma}$ has the same dependence as c. Observe that $\hat{\omega}(\hat{a}_r(w), E) \leq \omega(a_r(w), E)$ follows from the weak maximum principle for L. We note that Lemma 3.17 is proved in detail only for uniformly elliptic operators and when $D, \hat{D}$ are Lipschitz. However this lemma remains valid in our situation (see the last section in [DJK]). For our purposes, it is enough to know that Lemma 3.17 is true when E is most of $\partial D \cap B(w, r)$. We indicate the proof of this simpler result, during the proof of Theorems 1 - 3 (a).

We now begin the proof of Theorems 1 - 3 (a). Let $\tau = \log |\nabla u|$. Then in [BL, section 1] we show that

$$Lu = 0 \text{ and } \; (p-2)^{-1} \, L\tau \approx |\nabla u|^{p-4} \sum_{i,j=1}^{2} (u_{x_i x_j})^2 \text{ in } D \tag{3.18}$$

where proportionality constants depend only on p and are bounded on any compact subset of $(1, \infty)$. Thus τ is a supersolution to L when $1 < p < 2$ while τ is a subsolution to L when $p > 2$. We claim there exists $\bar\sigma, 0 < \bar\sigma < 1$, such that if $w \in \partial\Omega$, $0 < s < r <$ diam $\partial\Omega$, and $y \in B(w, r)$, then

$$c^{-1}\left(\frac{s}{r}\right)^{1/\bar\sigma} \leq \frac{\mu(B(y,s))}{\mu(B(w,r))} \leq c\left(\frac{s}{r}\right)^{\bar\sigma}. \tag{3.19}$$

Indeed, from Lemma 2.14 and Harnack's inequality applied to $B(y, r/4)$ and a certain ball of radius $r/4$ contained in $B(w, r) \setminus B(y, r/4)$ we deduce that

$$c^{-1}\,\mu(B(w,r)) \leq \mu(B(y,r/4)) \leq (1-c^{-1})\,\mu(B(w,r)).$$

Iterating this inequality we get (3.19). To prove Theorems 1 - 3 for $p > 2$, let $\hat K$ be the set of all $x \in \partial\Omega$ with

$$\limsup_{r\to 0} \frac{\mu(B(x,r))}{r} > 0. \tag{3.20}$$

We claim that

$$\hat K \text{ has } \sigma \text{ finite linear measure.} \tag{3.21}$$

Indeed, given n a positive integer, let $\hat K_n$ be the subset of $\hat K$ for which the above lim sup is $> 1/n$. Using a well known covering lemma, we can choose a covering $\{B(x_i, 5r_i)\}$ of $\hat K_n$ with $x_i \in \hat K$, $r_i \leq \delta/5$, $\{B(x_i, r_i)\}$ pairwise disjoint and

$$\mu(B(x_i, 5r_i)) > 5r_i/n. \tag{3.22}$$

Thus

$$\begin{aligned}\sum_i r_i &< n \sum_i \mu(B(x_i, 5r_i)) \\ &\leq c\,n \sum_i \mu(B(x_i, r_i)) \leq c^2\, n\, \mu(\partial\Omega) < \infty,\end{aligned} \tag{3.23}$$

thanks to (3.22), (3.19). Letting $\delta\to 0$ and using the definition of H^1 measure we conclude that $H^1(\hat K_n) < \infty$. Hence (3.21) is true. To complete the proof of Theorems 1 - 3 (a) we show that

$$\mu(\hat K) = \mu(\partial\Omega). \tag{3.24}$$

The proof is by contradiction. Otherwise, if $w \in \partial\Omega \setminus \hat K$, then from Lemmas 2.15, 2.19, we see for $x \in B[a_r(w), d(a_r(w), \partial\Omega)/2]$ that

$$|\nabla u(x)| \approx r^{-1} u(a_r(w)) \approx \left(r^{-1}\,\mu[B(a_r(w), r)]\right)^{1/(p-1)} \to 0 \text{ as } r\to 0. \tag{3.25}$$

Clearly (3.25) implies that $\tau(x) \to -\infty$. We use this fact to construct $\hat D$ an NTA domain with τ bounded above in $\hat D$ and $\tau(x)\to -\infty$ as $x\to y \in F$ compact $\subset \partial\hat D \cap (\partial\Omega\setminus\hat K)$. Furthermore $\mu(F) > 0$. To keep the geometry simple, let f be the quasiconformal map with $f(\partial B(0,1)) = \partial\Omega$, used in the proof of Lemmas 2.21, 2.36, 2.51. Given $0 < \epsilon < 10^{-1}$, we use a point of density argument (see [Ma, Cor 2.14]) to get F compact $w, r > 0$ (small) with $w \in \partial\Omega \setminus \hat K$, $F \subset B(w, r) \cap (\partial\Omega \setminus \hat K)$, and such that

$$\frac{\mu(B(w,s)\setminus F)}{\mu(B(w,s))} < \epsilon \text{ for } 0 < s \leq r. \tag{3.26}$$

Let $\Lambda = f^{-1}(F), b = f^{-1}(a_r(w))$, and put $\rho =$ diam Λ. Observe from the quasisymmetry of f (see (2.3)) and (3.19) that

$$1 - |b| \approx \rho \approx d(b, \Lambda) \tag{3.27}$$

provided ϵ is small enough ($\epsilon \leq c^{-1}$). Moreover, there exists ρ_1 with $\rho_1 \geq \rho/c$ and

$$f[B(b, \rho_1)] \subset D \cap B(w, 2r). \tag{3.28}$$

If $y \in \Gamma$ draw the two rays from y that are tangent to $B(b, \rho_1)$. let $\Gamma(y)$ be the bounded region whose boundary consists of these rays and part of $\partial B(b, \rho_1)$. Let

$$G = \bigcup_{y \in \Lambda} \Gamma(y) \text{ and } \hat{D} = f(G).$$

We note that G is Lipschitz and starlike with respect to b. Using this note, (3.27), (3.28), and quasi symmetry of f, it is easily checked that $\partial\hat{D}$ satisfies the Ahlfors three point condition (see (2.7)) with constant having the same dependence as c. From our construction, (3.25), and once again, quasisymmetry of f, we see that

$$\tau(x) \rightarrow -\infty \text{ as } x \in \hat{D} \rightarrow z \in F \subset \partial\hat{D}. \tag{3.29}$$

Let $\theta(z) = \limsup_{x \in \hat{D} \rightarrow z} \tau(x)$ whenever $z \in \partial\hat{D}$. From a compactness argument and our construction we see that τ is bounded above in $\hat{D}$. Thus θ is upper semicontinuous on $\partial\hat{D}$ and there exists a decreasing sequence (θ_n) of continuous functions with $\lim_{n \rightarrow \infty} \theta_n = \theta$ pointwise on $\partial\hat{D}$. Let Θ_n be the solution to the Dirichlet problem for $L, \hat{D}$ with boundary function θ_n. Since τ is a subsolution to L (see (3.18)) and $\theta \leq \theta_n$, it follows from the weak maximum principle for L that $\tau \leq \Theta_n$. From this inequality, (3.14), and the monotone convergence theorem we conclude that

$$\begin{aligned} \tau(a_r(w)) &\leq \lim_{n \rightarrow \infty} \Theta_n(a_r(w)) \\ &= \lim_{n \rightarrow \infty} \int \Theta_n \, d\hat{\omega}(a_r(w), \cdot) = \int \theta \, d\hat{\omega}(a_r(w), \cdot). \end{aligned} \tag{3.30}$$

Since $\theta = -\infty$ on F we conclude from (3.30) that

$$\hat{\omega}(a_r(w), F) = 0. \tag{3.31}$$

Let g be Green's function for D. Then from (3.18), (3.8) (iii), and Lemma 3.11 we see that $g(\cdot, x_0) \approx u$ in $B(w, 4r) \cap D$ where $x_0 = (0, 2r_0)$ when Ω is bounded and $x_0 = (0, R_0/2)$ when Ω is unbounded. Using this fact, Lemma 2.15, and Lemma 3.10, we deduce first that

$$\mu(B(x, s)) \approx \omega(x_0, B(x, s)), \, 0 < s \leq \text{ diam } \partial\Omega,$$

and second from a covering argument, as well as (3.13) with $\hat{D}, \hat{\omega}$, replaced by D, ω, we find

$$\frac{\mu(E)}{\mu(B(w, r/2))} \approx \frac{\omega(x_0, E)}{\omega(x_0, B(w, r/2))} \approx \omega(a_r(w), E) \tag{3.32}$$

provided E is a Borel subset of $B(w, r/2)$. At this stage we could invoke Lemma 3.17 to conclude first from (3.31) that $\omega(a_r(w), F) = 0$ and thereupon from (3.32) with $E = F \cap B(w, r/2)$ that $\mu(F \cap B(w, r/2)) = 0$ which contradicts (3.26). To give a self contained proof that (3.31), (3.32) lead to a contradiction, we note that if $x \in (\partial \hat{D} \setminus F) \cap B(w, r/c')$, and c' is large enough, there exists $\hat{x} \in (\partial D \setminus F) \cap B(w, r/4)$, with

$$d(\hat{x}, x) \approx d(x, F) \approx d(\hat{x}, F). \tag{3.33}$$

This note can be verified for example by using quasisymmetry of f^{-1}, the definition of G, and (3.26), (3.27). Let $\tilde{r} = r/c'$. From (3.33) and (3.12) with $\hat{\omega}, r, w$ replaced by $\omega, d(\hat{x}, F), \hat{x}$, we see that $c\,\omega(\cdot, B(w, r/2) \setminus F) \geq 1$ on $(\partial \hat{D} \setminus F) \cap B(w, \tilde{r})$ so from the weak maximum principle for L,

$$\hat{\omega}[a_r(w), B(w, \tilde{r}) \setminus F] \leq c\,\omega(a_r(w), B(w, r/2) \setminus F). \tag{3.34}$$

In view of (3.34), (3.32), (3.31), and (3.26), we have

$$\begin{aligned} &\hat{\omega}(a_r(w), B(w, \tilde{r})) = \hat{\omega}(a_r(w), B(w, \tilde{r}) \setminus F) \\ &\leq c\,\omega(a_r(w), B(w, r/2) \setminus F) \leq c^2 \frac{\mu(B(w, r) \setminus F)}{\mu(B(w, r))} \leq c^2\,\epsilon\,. \end{aligned} \tag{3.35}$$

However from Harnack's inequality and (3.12) we see that $c\,\hat{\omega}(a_r(w), B(w, \tilde{r})) \geq 1$. Since ϵ can be arbitrarily small we have reached a contradiction. Thus (3.24) is true and the proof of Theorems 1 - 3 (a) is complete. $\Box$

4. Preliminary Reductions for Theorems 1 - 3 (b)**.** We continue with the notation in sections 2 - 3. c has the same dependence as in section 3. Recall that $\partial\Omega$ is a k quasicircle where $k = k(p)$ in Lemma 2.36 while $k = k(\partial\Omega)$ in Lemma 2.51. Let $t_0 = \min_{B(0, 3\, r_0)} u$ when Ω is bounded while $t_0 = \min_{B(0, R_0/2)} u$ when Ω is unbounded. In section 4 we first prove,

Lemma 4.1. *If $t \in (0, t_0]$, then $\{x : u(x) = t\}$ is a $k' = k'(k, p)$ quasicircle.*

Proof: The proof is by contradiction using the Ahlfors three point condition (see (2.7) and Lemma 2.11). Let Γ be a parametrization of $\{x : u(x) = t\}$, oriented counterclockwise. Note from Lemma 2.19 and the implicit function theorem, that Γ is C^∞ smooth. Also Γ is a Jordan curve as follows from the maximum principle for p harmonic functions. If Lemma 4.1 is false, then for arbitrarily small $\eta > 0$ there exists $t \in (0, t_0]$, and $z_1, z_2, z_1 \neq z_2$, such that if Γ_1, Γ_2, denote the arcs of Γ with endpoints z_1, z_2, then

$$|z_1 - z_2| \leq \eta \min\{\text{diam } \Gamma_1, \ \text{diam } \Gamma_2\}. \tag{4.2}$$

We shall obtain a contradiction to (4.2) for $\eta \leq c^{-1}$. Let $\gamma : [0, 1] \rightarrow D$ with $\gamma(0) = z_1, \gamma(1) = z_2$, satisfy

$$\begin{aligned} &(*) \quad H^1(\gamma) \leq c\,|z_1 - z_2|, \\ &(**) \quad \min\{H^1(\gamma([0, t])), H^1(\gamma([t, 1]))\} \leq c\,d(\gamma(t), \partial\Omega),\ t \in [0, 1]. \end{aligned} \tag{4.3}$$

Existence of γ follows from (2.9). Clearly we may assume that γ is a Jordan arc. We also assume, as we may, thanks to smoothness of Γ, that γ consists of a finite number of line segments with the interior of each segment contained in either $\{x : t < u(x) < 1\}$ or $\{x : 0 < u(x) < t\}$. Choose $w_i \in \Gamma_i, i = 1, 2$, with $d(w_i, z_i) \approx \text{diam } \Gamma_i, i = 1, 2$. Let $\Theta_i, i = 1, 2$ be the arcs of Γ connecting w_1 to w_2 with $z_i \in \Theta_i, i = 1, 2$. Let $t_1 \in [0, 1)$ be the last time γ touches Θ_1 and let $t_2 \in (0, 1], t_1 < t_2$, be the first time γ touches Θ_2. Let γ' be the restriction of γ to $[t_1, t_2]$ and set $z_1' = \gamma(t_1), z_2' = \gamma(t_2)$. From (4.3) and our other assumptions on γ, we see that

$$\text{Either } \gamma' \setminus \{z_1', z_2'\} \subset \{x : t < u(x) < 1\} \text{ or } \gamma' \setminus \{z_1', z_2'\} \subset \{x : 0 < u(x) < t\}. \tag{4.4}$$

From (4.4) and the Jordan curve theorem we find that in either ease there exists a bounded Jordan domain, $\hat{\Omega} \subset D$ with

$$\begin{aligned} &(\alpha) \quad c \text{ diam } \partial\hat{\Omega} \geq \min\{\text{diam } \Gamma_1, \text{ diam } \Gamma_2\}, \\ &(\beta) \quad \partial\hat{\Omega} \text{ is a Jordan curve consisting of } \gamma' \text{ and an} \\ &\qquad \text{arc of } \{x : u(x) = t\}. \end{aligned} \tag{4.5}$$

Let $\Lambda = \partial\hat{\Omega} \cap \{x : u(x) = t\}$. In view of (4.5), Lemma 2.19, and (3.1) we can apply the divergence theorem to the vector field whose i th component, $i = 1, 2$ at $x \in \hat{\Omega}$ is $\sum_{k=1}^{2} b_{ik} u_{x_k}$. Since $Lu = 0$ in D it follows that

$$\int_{\partial\hat{\Omega}} \sum_{i,k=1}^{2} b_{ik}\, \xi_i\, u_{x_k}\, dH^1 = 0 \tag{4.6}$$

where ξ denotes the outer unit normal to $\partial\hat{\Omega}$. From (4.4), (4.5) we deduce that either $\xi = \nabla u/|\nabla u|$ on Λ or $\xi = -\nabla u/|\nabla u|$ on Λ. From this fact and the definition of b_{ik} in (1.4) or (3.1) we find that

$$\left| \int_{\Lambda} \sum_{i,k=1}^{2} b_{ik}\, \xi_i\, u_{x_k}\, dH^1 \right| = (p-1) \int_{\Lambda} |\nabla u|^{p-1}\, dH^1 \,. \tag{4.7}$$

Using this equality in (4.6) and once again (3.1) we conclude that

$$\begin{aligned} (p-1) \int_{\Lambda} |\nabla u|^{p-1}\, dH^1 &= \left| \int_{\gamma'} \sum_{i,k=1}^{2} b_{ik}\, \xi_i\, u_{x_k}\, dH^1 \right| \\ &\leq c(p) \int_{\gamma'} |\nabla u|^{p-1}\, dH^1. \end{aligned} \tag{4.8}$$

Next we claim for small $\eta > 0$, having the same dependence as c, that

$$d(\gamma', \partial\Omega) \leq \eta^{1/2} \text{ diam } \partial\hat{\Omega}. \tag{4.9}$$

In fact otherwise, we have

$$|\nabla u| \approx |\nabla u(z_1')| \text{ on } B[z_1', \tfrac{1}{2}\eta^{1/2} \text{ diam } \partial\hat{\Omega}]. \tag{4.10}$$

Using (4.8), (4.5) (α), (4.2), (4.3), and Lemma 2.19, it follows that

$$\begin{aligned}\eta^{1/2}\,(\text{diam }\partial\hat\Omega)\,|\nabla u(z_1')|^{p-1} \,&\leq\, c\int_\Lambda |\nabla u|^{p-1}\,dH^1 \,\leq c^2\int_{\gamma'}|\nabla u|^{p-1}dH^1 \\ &\leq\, c^3|z_1-z_2||\nabla u(z_1')|^{p-1} \,\leq\, c^4\,\eta\,(\text{diam }\partial\hat\Omega)\,|\nabla u(z_1')|^{p-1}\,.\end{aligned} \tag{4.11}$$

Clearly (4.11) cannot hold if $\eta^{-1} > c^8$. Thus (4.9) is true.

Next we claim that if $0<\delta\leq \eta^{\frac{\alpha}{4\beta}}$, where α is as in Lemma 2.14 (++) and β as in (2.28), then for sufficiently small η,

$$d(w',\partial\Omega)\leq \delta \text{ diam }\partial\hat\Omega \text{ whenever } w'\in\partial\hat\Omega \text{ and } d(w',\gamma')\geq \tfrac{1}{8}\text{ diam }\partial\hat\Omega. \tag{4.12}$$

Indeed, otherwise choose $w'\in\partial\hat\Omega$ with $d(w',\gamma')\geq\frac18$ diam $\partial\hat\Omega$ and $w\in\partial\Omega$ such that

$$d(w,w')=d(w',\partial\Omega)=r>\delta\text{ diam }\partial\hat\Omega.$$

From Harnack's inequality and (2.9) we see that $u(a_r(w))\approx u(w')$. Let $s=$ diam $\partial\hat\Omega$, and choose $\tilde z\in\partial\Omega$ with $d(z_1',\tilde z)=d(z_1',\partial\Omega)$. Then from Lemma 2.14, (4.9), and (2.28) we get

$$t=u(z_1')\,\leq\, c\,\eta^{\alpha/2}\,u(a_s(\tilde z))\,\leq\, c\eta^{\alpha/2}\,\delta^{-\beta}\,u(a_r(w))<t. \tag{4.13}$$

Again we have reached a contradiction. Hence (4.12) is valid. To continue, let $\hat w\in\partial\hat\Omega$ with $d(\gamma',\hat w)\geq\frac14$ diam $\partial\hat\Omega$. Existence of $\hat w$ follows from (4.2), (4.3), (4.5), if η is small enough. Choose a maximal subarc $\Lambda_1\subset\Lambda$ with $\hat w\in\Lambda_1$ and the property that every point in Λ_1 is at least $\frac18$ diam $\partial\hat\Omega$ distance from γ'. Also, take a covering $\{B(\zeta_j,\frac12 d(\zeta_j,\partial\Omega))\}_1^N$ of Λ_1 with $\zeta_j\in\Lambda_1$ and $\{B(\zeta_j,\frac{1}{20}\,d(\zeta_j,\partial\Omega))\}$ pairwise disjoint. Next choose $\zeta_j'\in\partial\Omega\cap B(\zeta_j,2\,d(\zeta_j,\partial\Omega))$ and let Γ' be an arc of $\partial\Omega$ containing $\{\zeta_j'\}_1^N$ with endpoints in this set. We also choose Γ' so that if ζ_l',ζ_j' are successive points on Γ' and one point is not an endpoint of Γ', then the subarc joining these points has diameter

$$\leq\, c\,|\zeta_l'-\zeta_j'|\,\leq\, c^2\,\delta\,\text{ diam }\partial\hat\Omega.$$

This choice is possible, thanks to (2.9), (4.12). We assert that

$$B(\hat w,\delta^{1/2}\text{ diam }\partial\hat\Omega)\cap\partial\Omega\subset\Gamma'. \tag{4.14}$$

Indeed, there is $\zeta\in\Gamma'$ with $d(\hat w,\zeta)\leq 4\delta$ diam $\partial\hat\Omega$ thanks to (4.12) and our construction. Moreover, from our construction and (4.12), we see that both subarcs of $\Gamma'\setminus\{\zeta\}$ are of length $\geq c^{-1}$ diam $\partial\hat\Omega$. Hence (4.14) must hold for δ small enough (with the same dependence as c), since otherwise the Ahlfors three point condition would be violated.

From Lemma 2.19, (3.19), (4.9), (4.12), we find for $x'\in\partial\hat\Omega$ that

$$\int_{\partial\hat\Omega\cap B(x',\frac{1}{20}d(x',\partial\Omega))}|\nabla u|^{p-1}\,dH^1\,\approx\,\mu[B(x,c\,d(x',\partial\Omega))]\,\frac{H^1[B(x',\frac{1}{20}d(x',\partial\Omega))\cap\partial\hat\Omega]}{d(x',\partial\Omega)} \tag{4.15}$$

whenever $x\in\partial\Omega$ with $d(x,x')\leq cd(x',\partial\Omega)$. Using (4.15), (4.14), we conclude

$$\begin{aligned}\mu[B(\hat w,\delta^{1/2}\text{ diam }\partial\hat\Omega)]\,&\leq\, c\sum_j\int_{B[\zeta_j,\frac{1}{20}d(\zeta_j,\partial\Omega)]\cap\Lambda_1}|\nabla u|^{p-1}\,dH^1 \\ &\leq\, c^2\int_{\Lambda_1}|\nabla u|^{p-1}\,dH^1\,.\end{aligned} \tag{4.16}$$

Similarly, from (4.2), (4.3), (3.19), (4.9), and (4.15) we deduce

$$\int_{\gamma'} |\nabla u|^{p-1} \, dH^1 \, \leq \, c \, \mu[B(z_1', c\, \eta^{1/2} \text{ diam } \partial\hat{\Omega})] \, . \tag{4.17}$$

Also, (4.17), (4.16), and (4.8) imply that

$$\mu[B(\hat{w}, \delta^{1/2} \text{ diam } \partial\hat{\Omega})] \, \leq \, c \, \mu[B(z_1', c\, \eta^{1/2} \text{ diam } \partial\hat{\Omega})] \, . \tag{4.18}$$

To reach a contradiction to (4.18), let $\tilde{z}$ be as in (4.13) and observe from (3.19) that

$$\begin{aligned} c \, \mu[B(z_1', c\, \eta^{1/2} \text{ diam } \partial\hat{\Omega})] \, &\leq \, c \, \eta^{\bar{\sigma}/2} \mu[B(\tilde{z}, \text{ diam } \partial\hat{\Omega})] \\ &\leq \, c \, \eta^{\bar{\sigma}/2} \, \delta^{-\frac{1}{2\bar{\sigma}}} \; \mu[B(\hat{w}, \, \delta^{1/2} \text{ diam } \partial\hat{\Omega})] \, < \mu[B(\hat{w}, \, \delta^{1/2} \text{ diam } \partial\hat{\Omega})] \end{aligned} \tag{4.19}$$

provided δ, η are small enough and $\delta \geq \eta^{\bar{\sigma}^3}$. From this contradiction we conlude that (4.2) is false for some $\eta = c^{-1}$. Thus the Ahlfors three point condition holds and Lemma 4.1 is true. □

Before stating our next lemma, we shall need some more notation. Let $\hat{D} \subset D$ be the NTA domain in section 3 and $z \in \partial\hat{D}$. Put $K(z) = \{y \in \hat{D} : |z - y| < (1 + c)d(y, \partial\Omega)\}$. If c is large enough, then from (2.10), we have $K(z) \cap B(z, r) \neq \emptyset$ for each $r > 0$. Given a function $F : \hat{D} \rightarrow \mathbb{R}$, define the nontangential maximal function, $N(F)$ of F on $\partial\hat{D}$ by

$$N(F)(z) = \sup\{|F(y)| : y \in K(z)\}, \; z \in \partial\hat{D}.$$

The following lemma is stated in [FKJ1] (see also [JK]).

Lemma 4.20. *Fix $x_0 \in \hat{D}$ with $d(x_0, \partial\hat{D}) \approx$ diam $\partial\hat{D}$ and suppose that $\theta : \partial\hat{D} \rightarrow \mathbb{R}$ is integrable with respect to $\hat{\omega}(x_0, \cdot)$. Then*

$$\Theta(x) \, = \, \int \theta \, k(x, \cdot) \, d\hat{\omega}(x_0, \cdot)$$

is a weak solution to $L\Theta = 0$ in $\hat{D}$ and $\lim\limits_{y \in K(z) \longrightarrow z} \Theta(y) = \theta(z)$ for $\omega(\cdot, x_0)$ almost every $z \in \partial\hat{D}$. Moreover, if $E(\lambda) = \{z \in \partial\hat{D} : N(\Theta)(z) > \lambda\}$, then

$$\omega(x_0, E(\lambda)) \, \leq \, c \, \lambda^{-1} \int_{\partial\hat{D}} |\theta| \, d\omega(x_0, \cdot) \, .$$

In addition if θ is q th power integrable with respect to $\omega(x_0, \cdot)$, then

$$\int_{\partial\hat{D}} N(\Theta)^q \, d\omega(x_0, \cdot) \, \leq \, c^* \int_{\partial\hat{D}} |\theta|^q \, d\omega(x_0, \cdot).$$

We note that $k, \hat{\omega}$, are as in (3.14). Also c^* depends on q and the A_2, NTA constants.

Next for given $t \in (0, t_0]$, we put $\hat{D} = \{x : t < u(x) < 1\}$ and $x_0 = (0, 2r_0)$ when Ω is bounded while $x_0 = (0, 3R_0/4)$ when Ω is unbounded. Set

$$\psi(x) = \max\{\log(|\nabla u|) - \log(\text{ diam } D) - \tilde{c}, \, 0\}$$

where $\tilde{c}$ is the least positive number for which $\psi \equiv 0$ in $\hat{D} \cap \bar{B}(0, 3r_0)$ when Ω is bounded while $\psi \equiv 0$ in $\hat{D} \setminus B(0, R_0/2)$ when Ω is unbounded. Existence of $\tilde{c}$ (with the same dependence as c) is a consequence of Lemma 2.19. Given $m \geq 10$, a positive integer, put $\Psi(x) = \psi(x)^{2m}$, $x \in \hat{D}$. We use Lemma 4.20 to prove,

Lemma 4.21. *Let $E(\lambda)$ be as in Lemma 4.20 with Θ replaced by Ψ. Then for $\lambda > 0$,*

$$\omega(x_0, E(\lambda)) \leq c\,\lambda^{-1} I(x_0) \text{ where if } x \in \hat{D},$$

$$I(x) = 2m\,(2m-1) \int_{\hat{D}} \hat{g}(x,y)\, \psi^{2m-2}(y)\, |\nabla u|^{p-2}(y)\, |\nabla \psi|^2(y) dy\,.$$

Proof: Observe from Lemma 2.19 that Ψ has continuous second partials in $\hat{D}$. Using this fact, (3.8), and integrating by parts in (3.9) with ψ replaced by Ψ we get for $x \in \hat{D}$,

$$\Psi(x) = \int_{\partial \hat{D}} k(x,y) d\hat{\omega}(x_0, y) \; - \int_{\hat{D}} L\Psi(y)\, \hat{g}(x,y)\, dy. \tag{4.22}$$

Moreover,

$$\begin{aligned} L(\Psi)(x) = L(\psi^{2m})(x) &= \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i} [b_{ij}(x)\, (\psi^{2m})_{x_j}(x)] \\ &= 2m(2m-1)\, \psi^{2m-2}(x) \sum_{i,j=1}^{2} b_{ij}(x) \psi_{x_i}(x)\, \psi_{x_j}(x) \;+\; 2m\, \psi^{2m-1}(x)\, L\psi(x) \\ &= S_1(x) \qquad + \qquad S_2(x). \end{aligned} \tag{4.23}$$

From (3.2) we have,

$$S_1(x) \approx 2m(2m-1)\, |\nabla u(x)|^{p-2}\, \psi^{2m-2}(x)\, |\nabla \psi|^2(x) = S_3(x)\,. \tag{4.24}$$

From (3.18) we get for fixed $p, 1 < p < 2$,

$$-S_2 \approx \; (2-p)\, m\, \psi^{2m-1}(x)\, |\nabla u|^{p-4}(x) \sum_{i,j=1}^{2} (u_{x_i x_j}(x))^2 = S_4(x). \tag{4.25}$$

From (4.22)-(4.25) and the definition of I in Lemma 4.21, it follows for $x \in \hat{D}$ that

$$\begin{aligned} \Psi(x) + \int_{\hat{D}} \hat{g}(x,y)\, S_3(y) dy &= \Psi(x) + I(x) \\ &\approx \int_{\partial \hat{D}} \Psi(y)\, k(x,y)\, d\hat{\omega}(x_0, y) + \int_{\hat{D}} \hat{g}(x,y)\, S_4(y)\, dy \\ &= \Psi_1(x) \quad + \quad \Psi_2(x). \end{aligned} \tag{4.26}$$

Given $\lambda > 0$, let $E_i(\lambda) = \{x \in \partial\hat{D} : N(\Psi_i)(x) > \lambda\}$ for $i = 1, 2$. From Lemmas 4.1, 4.20, we see that

$$\lambda\,\hat{\omega}(x_0, E_1(\lambda/2)) \,\le\, c\Psi_1(x_0). \tag{4.27}$$

We also assert that

$$\lambda\,\hat{\omega}(x_0, E_2(\lambda/2)) \,\le\, c\,\Psi_2(x_0). \tag{4.28}$$

(4.28) is proved in a similar way to (4.27). That is, if $z \in \partial\hat{D}$, let

$$M(z) = \sup_{0<r<\text{ diam }\partial\hat{D}} \left[\omega[x_0, B(z,r)]^{-1} \int_{\hat{D}\cap B(z,r)} g(x_0, y)\, S_4(y)\, dy\right].$$

Let $x \in K(z)$ and put

$$G_0 = \{y \in \hat{D} : d(x, y) < 2d(x, \partial D)\} \setminus B(x, \tfrac{1}{2}d(x, \partial\hat{D})).$$

$$G_k = \{y \in \hat{D} : 2^k d(x, \partial\hat{D}) \,\le\, d(x, y) < 2^{k+1} d(x, \partial\hat{D})\}, k = 1, 2, \ldots$$

From (3.8) (iii), smoothness of S_4, and Lemma 3.10, we find

$$\begin{aligned}\int_{B(x,\frac{1}{2}d(x,\partial\hat{D}))} \hat{g}(x, y)\, S_4(y)\, dy \,&\le\, ch(x)^{-1} \int_{B(x,\frac{1}{2}d(x,\partial\hat{D}))} S_4(y)\, dy \\ &\le\, c\,\omega[x_0, B(z, d(x, \partial\hat{D}))]^{-1} \int_{B(x,\frac{1}{2}d(x,\partial\hat{D}))} \hat{g}(x_0, y)\, S_4(y)\, dy \le cM(z).\end{aligned} \tag{4.29}$$

Using Lemma 3.11 with v_1, v_2 normalized multiples of $\hat{g}(x, \cdot), \hat{g}(x_0, \cdot)$, and Lemma 3.10, we deduce that

$$\int_{G_0} g(x, y) S_4(y) dy \le c\,\omega[x_0, B(z, d(x, \partial\hat{D}))]^{-1} \int_{G_0} g(x_0, y) S_4(y)\, dy \,\le\, c\, M(z). \tag{4.30}$$

Next if $a_i \in G_i \setminus B(x, d(x, \partial\hat{D}))$ with $d(a_i, \partial\hat{D}) \approx 2^i\, d(x, \partial\hat{D}), i = 1, 2, \ldots,$ then from Lemmas 3.10, 3.11 and Lemma 3.6 with $v = g(x, \cdot)$ we obtain for $y \in G_i$,

$$\frac{g(x, y)}{g(x_0, y)} \approx \frac{g(x, a_i)}{g(x_0, a_i)} \,\le\, c\, 2^{-\alpha i}\, \omega[B(z, 2^i d(x, \partial\hat{D}))]^{-1}. \tag{4.31}$$

From (4.31) and the same argument as in (4.29), (4.30), we get

$$\int_{G_i} g(x, y) S_4(y) dy \,\le\, c\, 2^{-i\alpha}\, M(z). \tag{4.32}$$

Summing (4.32) over i and using (4.29), (4.30), we conclude that

$$\Psi_2(x) \,\le\, M(z) \text{ whenever } x \in K(z) \text{ and } z \in \partial\hat{D}.$$

Hence

$$N(\Psi_2)(z) \le M(z),\ z \in \partial\hat{D}. \tag{4.33}$$

(4.33) and the same argument which shows the Hardy - Littlewood maximal function is of weak type 1 - 1 imply assertion (4.28). Since $0 \leq \Psi \leq \Psi_1 + \Psi_2$, we see from (4.28), (4.27) that

$$\omega(x_0, E(\lambda)) \leq \omega(x_0, E_1(\lambda/2)) + \omega(x_0, E_2(\lambda/2)) \leq c\,\lambda^{-1} \left[\Psi_1(x_0) + \Psi_2(x_0)\right]. \tag{4.34}$$

Finally Lemma 4.21 follows from (4.34), (4.26) and the fact that $\Psi(x_0) = 0$.

5. Proof of Theorems 1 - 3 (b)**.** To continue the proof of Theorems 1 - 3 (b), we argue as in [M] (see [P, ch 8]). We continue with the same notation as in section 4. Thus x_0 is as in Lemma 4.21. Also, given $t \in (0, t_0]$, we write $\hat{D}(t) = \{x : t < u(x) < 1\}, \Gamma(t) = \{x : u(x) = t\}$, for $\hat{D}, \Gamma$, respectively, in order to avoid confusion. Let $\mu_t(\cdot)$ denote the positive Borel measure corresponding to $u-t$ in $\hat{D}(t)$ (as in (1.2)), with support in $\Gamma(t)$. From smoothness of $u, \Gamma(t)$, it is easily seen that

$$\mu_t(E) = \int_{E\cap\Gamma(t)} |\nabla u|^{p-1}\, dH^1 \tag{5.1}$$

when $E \subset \mathbb{R}^2$ is a Borel set. Throughout section 5, we shall freely use results from sections 1-3 with u, μ replaced by $u-t, \mu_t$. This replacement is permissible, thanks to Lemma 4.1. To begin, we note from Lemma 4.1, Lemma 2.19, Harnack's inequality, and several applications of Lemma 2.14 with $\tilde{u}$ replaced by $1-u, u-t$, that

$$c^{-1} \leq u(x_0) - t, \tag{5.2}$$

whenever $t \in (0, t_0]$. From (5.2), (3.18), and Lemma 3.11 we see that

$$\hat{g}(x_0, \cdot) \approx u - t \text{ on } \{x : \psi(x) \neq 0\}. \tag{5.3}$$

Moreover,

$$\hat{\omega}(x_0, E(\lambda)) \approx \mu_t[E(\lambda)],\ 0 < \lambda < \infty, \tag{5.4}$$

where we have also used, Lemmas 2.15 and 3.10. From (5.3) and the definition of I in Lemma 4.21, we conclude that

$$I(x_0) \leq 2m\,(2m-1) \int_{\hat{D}(t)} u(y)\, \psi^{2m-2}(y)\, |\nabla u|^{p-2}(y)\, |\nabla \psi|^2(y) dy = J(t). \tag{5.5}$$

To continue, in [BL, see (6.5)-(6.8)] we show that

$$J(t) \leq c^m\, m!\, [\log(e/t)]^m. \tag{5.6}$$

Let $N_t(\Psi)(z), z \in \partial\hat{D}(t)$, be the nontangential maximal function of Ψ in $\hat{D}(t)$ defined relative to the nontangential approach region, $K_t(z)$, and a constant c independent of t. Now from (5.3) - (5.6), Lemma 4.21, and the same argument which shows the Hardy - Littlewood maximal function is fractionally integrable, we get

$$\begin{aligned}\int_{\Gamma(t)} |\nabla u|^{p-1}\, N_t(\Psi)^{1/2}\, dH^1 &\leq c \int_{\Gamma(t)} N_t(\Psi)^{1/2}\, d\omega(x_0, \cdot) \leq c^2 I^{1/2}(x_0) \\ &\leq c^2\, J^{1/2} \leq c^{m/2}\, (m!)^{1/2}\, [\log(e/t)]^{m/2}.\end{aligned} \tag{5.7}$$

Next we discuss the orthogonal trajectories to the levels of u. Existence and properties of these trajectories can be deduced from standard ordinary differential equation theory. Another way is to construct a local ' conjugate, ' w to u (see [Le]) defined by

$$w_y = -|\nabla u|^{p-2}\, u_x, \;\; w_x = \,|\nabla u|^{p-2}\, u_y. \tag{5.8}$$

Since u is a solution to the p Laplacian, the above differential equation is exact, so w exists locally and is unique up to a constant. Furthermore, It is easily checked that w is locally a solution to the p' Laplacian, where $p' = p/(p-1)$. Using this fact, one can show as in Lemma 2.19 that w is real analytic, locally in D. Clearly the levels of w are orthogonal to the levels of u. Observe that

$$\mu_t(\Gamma(t)) \;=\; \int_{\Gamma(t)} |\nabla u|^{p-1} dH^1 = \xi \neq 0, \tag{5.9}$$

is independent of t. Put

$$F(x) = \exp[(2\pi/\xi)(u(x) + iw(x))],$$

for x in a neighborhood of x_0 (as in section 4), where $i = \sqrt{-1}$ and again we are using complex notation. If $w(x_0) = 0$, then from our previous observation we see that F can be continued uniquely throughout D to get a function mapping $D \rightarrow \{y : 1 < |y| < e^{2\pi/\xi}\}$. Moereover since w increases on $\Gamma(t)$, as this curve is traversed counterclockwise, it follows that F is 1 - 1 and onto the above annulus. Given $\hat{x} \in \Gamma(t_0)$, draw the ray from (0, 0) through $F(\hat{x})$. Let $T(\hat{x}, \cdot)$ be the intersection of this ray with the above annulus. Set $\sigma(\hat{x}, \cdot) = F^{-1}(T(\hat{x}, \cdot))$. Then from our construction it is easily checked that a branch of w is constant on $\sigma(\hat{x}, \cdot)$ and ∇u is tangent to $\sigma(\hat{x}, \cdot)$. Thus $\sigma(\hat{x}, \cdot)$ is orthogonal to $\Gamma(t)$ whenever $t \in (0, 1)$. Since u is strictly decreasing along $\sigma(\hat{x}, \cdot)$, we see that $\sigma(\hat{x}, \cdot)$ is a Jordan arc and can be parametrized by

$$u(\sigma(\hat{x}, s)) = 1 - s, \;\text{ for } 0 < s < 1,\, \hat{x} \in \Gamma(t_0). \tag{5.10}$$

From our construction and the implicit function theorem we see that σ is infinitely differentiable in x, s. Also, it is easily seen that for each $y \in D$, there is a unique $\hat{y} \in \Gamma(t_0)$ with y on $\sigma(\hat{y}, \cdot)$.

If $u(y) = t$ ($y, \hat{y}$ as above), we claim that

$$\sigma(\hat{y}, s) \cap \partial\hat{D} \subset K_t(y) \tag{5.11}$$

whenever $1 - t_0 \leq s < 1 - t$. Indeed, suppose $z \in [\hat{D}(t) \setminus \hat{D}(t_0)] \cap \sigma(\hat{y}, \cdot)$ and

$$d(z, \Gamma(t)) \;\leq\; \delta\, |z - y|\,. \tag{5.12}$$

Let $z' \in \Gamma(t)$ with $|z - z'| = d(z, \Gamma(t))$. Let τ_1 be the line segment from z to z' and let τ_2 be the line segment connecting z to the first point of intersection, say z'', of $\sigma(\hat{y}, \cdot)$ with $\partial B(z, |z - z'|)$, starting from y. Let σ' be the arc of $\sigma(\hat{y}, \cdot)$ connecting y to z''. From the Jordan curve theorem, we find that $\tau_1 \cup \tau_2 \cup \sigma'$ divides $\{x : u(x) > t\}$ into two disjoint Jordan domains. Let G be the domain with smallest diameter. Then from Lemma 4.1, Lemmas 2.15, 2.19 with u replaced by $u - t$, the divergence theorem applied to $|\nabla u|^{p-2}\nabla u$, the fact that ∇u is tangent to $\sigma(\hat{x}, \cdot)$, and the definition of μ_t in (5.1), we find

$$\begin{aligned} \mu_t(\partial G \cap \Gamma(t)) &= \int_{\partial G \cap \Gamma(t)} |\nabla u|^{p-1}\, dH^1 \\ &\leq \int_{\tau_1 \cup \tau_2} |\nabla u|^{p-1}\, dH^1 \;\leq\; c\mu_t[B(z', |z - z'|)]. \end{aligned} \tag{5.13}$$

In view of (5.12), this inequality contradicts (3.19) with μ replaced by μ_t for $\delta \leq c^{-1}$. Thus (5.11) is true.

Next suppose, $u(x) = u(y) = t_1, x \neq y, 0 < t_1 \leq t_0$, and $\sigma(\hat{x}, \cdot), \sigma(\hat{y}, \cdot)$ are the orthogonal trajectories passing through, x, y, respectively. Given, $0 < t_2 < t_1$, let G' be one of the two Jordan domains bounded by these trajectories and $\Gamma(t_1), \Gamma(t_2)$. Let α_1, α_2 be the arcs on $\Gamma(t_1), \Gamma(t_2)$, which are contained in $\partial G'$. Using the divergence theorem and once again the fact that ∇u is tangent to each orthogonal trajectory, we deduce

$$\mu_{t_1}(\alpha_1) = \mu_{t_2}(\alpha_2)\,. \tag{5.14}$$

Again from (3.19) with μ replaced by μ_t, we have for $i = 1, 2$,

$$c^{-1}\,\xi \left(\frac{\text{diam }\alpha_i}{\text{diam }D}\right)^{1/\bar{\sigma}} \leq\, \mu_{t_i}(\alpha_i) \,\leq\, c\,\xi\left(\frac{\text{diam }\alpha_i}{\text{diam }D}\right)^{\bar{\sigma}}. \tag{5.15}$$

Let $z \in \partial\Omega$ and let $z_n \in \Omega, n = 1, 2, \dots,$ with $\lim\limits_{n\to\infty} z_n = z$. Let $\sigma(\hat{z}_n, \cdot), \hat{z}_n \in \Gamma(t_0), n = 1, 2, \dots,$ be the orthogonal trajectories through $z_n, n = 1, 2, \dots$. Let $(\hat{y}_n)$ be a subsequence of $(\hat{z}_n)$ with $\lim\limits_{n\to\infty} \hat{y}_n = \hat{y}$. Then from (5.14), (5.15), we see that $\lim\limits_{n\to\infty} \sigma(\hat{y}_n, \cdot) = \sigma(\hat{y}, \cdot)$ uniformly on $[1-t_0, 1)$. Also, since $z_n \to z$ as $n\to\infty$, it follows from (5.11) that $\sigma(\hat{y}, \cdot) \subset K(z)$ and thereupon that $\lim\limits_{s\to 1} \sigma(\hat{y}, s) = z$. We can now repeat the argument using (5.14), (5.15), and (5.11) to get first that $\hat{z}_n \to \hat{y}$ as $n\to\infty$ and second that $\sigma(\hat{y}, \cdot)$ is independent of the choice of (z_n). Thus, for each $z \in \partial\Omega$, there exists a unique orthogonal trajectory through z. The above argument also shows that $\lim\limits_{s\to 1} \sigma(\hat{x}, s)$ exists and is in $\partial\Omega$ whenever $\hat{x} \in \Gamma(t_0)$.

Given $y \in \partial\Omega$, let $b(y) = \hat{y}$, where $\hat{y}$ is the unique point in $\Gamma(t_0)$ which lies on the orthogonal trajectory with endpoint y. From our previous remarks we see that $\hat{y}$ exists. We note that μ_t converges weakly to μ in the sense of measures, as can be seen from using (1.2) for $u - t, \mu_t$ and taking limits. From this note, the definition of b, (5.14) and the above discussion, we deduce that

$$\int_{\Gamma(t_0)} \eta\, d\mu_{t_0} \,=\, \int_{\partial\Omega} \eta \circ b\, d\mu. \tag{5.16}$$

Let $\hat{x}_0 \in \Gamma(t_0)$ be the intersection of the orthogonal trajectory through x_0 with $\Gamma(t_0)$. Let O be the interior of $D \setminus [\hat{D}(t_0) \cup \sigma(\hat{x}_0, \cdot)]$ and observe that O is simply connected. From the definition of F, (5.9), we deduce that a branch of w can be chosen so that the function $x \to (u, w)$ maps O, one to one and onto the rectangle, $U = \{(u, w) : 0 < u < t_0, 0 < w < \xi\}$. Moreover from the change of variables formula for multiple integrals, we have

$$\int_0^\beta \int_0^{t_0} \Theta(u, w)\, du dw \,=\, \int_O \Theta(u(x), w(x))\, |\nabla u(x)|^p\, dx \tag{5.17}$$

whenever $\Theta : U \to \mathbb{R}$ is positive and Lebesgue measurable. We now follow the proof of Theorem 8.10 in [P]. Observe that $N_t(w), w \in \Gamma(t), 0 < t < t_0$, is lower semicontinuous on O. Let $\theta(t) = \log(e/t), 0 < t \leq t_0$, and put

$$\phi(x) = (u(x))^{-1}\, [\theta \circ u(x)]^{-1}\, [1 -\, \theta \circ \theta \circ u(x)]^{-(1+2/m)},\ x \in O.$$

Note from (5.10) for $\hat{x} \in \Gamma(t_0)$, $1 - t_0 \leq s < 1$, that

$$m \, \tfrac{d}{ds} \, [1 - (\theta \circ \theta)(1-s)]^{-2/m} \, = \, - 2 \, \phi(\sigma(\hat{x}, s)). \tag{5.18}$$

Thus,

$$\int_{1-t_0}^{1} \, \phi(\sigma(\hat{x}, s)) \, ds \, \leq \, c \, m \tag{5.19}$$

whenever $\hat{x} \in \Gamma(t_0)$.

Using (5.19), (5.7), and (5.17), we find

$$\begin{aligned} L \, &= \, \int_O |\nabla u|^p(x) \, N_t(\Psi)^{1/2}(x) \, \theta(u(x))^{-m/2} \, \phi(x) dx \\ &\leq \, c^{m/2} \, (m!)^{1/2} \int_{1-t_0}^{1} \, \phi[\sigma(\hat{x}_0, s)] \, ds \, \leq \, c^m \, (m!)^{1/2}. \end{aligned} \tag{5.20}$$

On the other hand using (5.17), the Tonelli theorem to interchange the order of integration, and (5.16) we see that

$$\begin{aligned} L \, &= \, \int_{\Gamma(t_0)} |\nabla u|^{p-1}(\hat{x}) \, \left(\int_{1-t_0}^{1} \, N_{1-s}(\Psi)^{1/2}(\sigma(\hat{x}, s)) \, \theta(1-s)^{-m/2} \, \phi(\sigma(\hat{x}, s)) \, ds \right) \, dH^1 \hat{x} \\ &= \, \int_{\partial\Omega} q(x) \, d\mu(x), \end{aligned} \tag{5.21}$$

where we have put

$$q(x) = \int_{1-t_0}^{1} \, N_{1-s}(\Psi)^{1/2}[\sigma(b(x), s)] \, \theta(1-s)^{-m/2} \, \phi[\sigma(b(x), s)] \, ds, \ \ x \in \partial\Omega.$$

Let E_m be the set of $x \in \partial\Omega$ with

$$q(x) \, \geq \, c^{2m} \, (m!)^{1/2}.$$

From (5.20), (5.21) and weak type estimates we see that

$$\mu(E_m) \leq c^{-m} \text{ for } m \geq 10. \tag{5.22}$$

Let

$$N^*(\hat{x}, \tau) = \sup \{ \Psi^{1/2}(\sigma(\hat{x}, s)) : 1 - t_0 \leq s \leq \tau \} \ \text{ when } \hat{x} \in \Gamma(t_0) \text{ and } \tau \in [1 - t_0, 1).$$

Clearly, $N^*(\hat{x}, \cdot)$ is nondecreasing on $[1 - t_0, 1)$. Moreover, from (5.11) we have,

$$N^*(\hat{x}, \tau) \, \leq \, N_{1-s}(\Psi)^{1/2}(\sigma(\hat{x}, s)) \text{ for } \tau \in [1 - t_0, 1), \tau \leq s < 1, \hat{x} \in \Gamma(t_0). \tag{5.23}$$

From (5.23) and the fact that

$$- \tfrac{d}{ds} \, \left([\theta(1-s)]^{-m/2} \, \cdot \, [1 - (\theta \circ \theta)(1-s)]^{-(1+2/m)} \right) \, \leq \, cm \, [\theta(1-s)]^{-m/2} \, \phi[\sigma(b(x), s)]$$

for $s \in [1-t_0, 1), x \in \partial\Omega$, we find

$$N^*(b(x),\tau)\left([\theta(1-\tau)]^{-m/2} \cdot [1-(\theta\circ\theta)(1-\tau)]^{-(1+2/m)}\right) \leq c\, m\, q(x). \tag{5.24}$$

If $x \notin E_m$, it follows from (5.24) that

$$\psi^m[\sigma(b(x),\tau)]\ \left([\theta(1-\tau)]^{-m/2} \cdot [1-(\theta\circ\theta)(1-\tau)]^{-(1+2/m)}\right) \leq c^{3m}\,(m!)^{1/2}.$$

Now suppose that

$$m \geq \log\log[-\log(1-\tau)] = \kappa(\tau) > m-1$$

for τ near 1. Taking $1/m$ roots of the above inequality, using Stirling's formula, and doing some arithmetic, it follows that

$$\frac{\psi[\sigma(b(x),\tau)]}{\sqrt{\theta(1-\tau)\,\kappa(\tau)}} \leq \hat{c}, \tag{5.25}$$

where $\hat{c}$ has the same dependence as c in sections 3-5. We conclude from (5.25) for $x \notin \bigcup_{m=l}^{\infty} E_m$ that

$$\limsup_{\tau\longrightarrow 1}\left(\frac{\log|\nabla u[\sigma(b(x),\tau)]|}{\sqrt{-\log(1-\tau)\,\cdot\,\log\log[-\log(1-\tau)]}}\right) \leq \hat{c}. \tag{5.26}$$

Since l is arbitrary we see from (5.22) that (5.26) holds μ almost everywhere.

Theorems 1 - 3 (b) follow easily from (5.26). Indeed let

$$\lambda(r) = r\,\exp\left[c_+\sqrt{\log 1/r\ \log\log\log 1/r}\right],\ 0 < r < 10^{-6},$$

and suppose $F \subset \partial\Omega$ is a Borel set with $H^{\lambda}(F) = 0$. Let F_1 be the subset of all $x \in F$ with

$$\limsup_{r\longrightarrow 0}\frac{\mu(B(x,r))}{\lambda(r)} \leq 2. \tag{5.27}$$

Then from the definition of λ, it is easily shown that $\mu(F_1) = 0$. Thus to prove Theorems 1 - 3 (b) it suffices to show for c_+ sufficiently large, with the same dependence as c, that

$$\mu(\hat{F}) = 0 \tag{5.28}$$

whenever $\hat{F}$ is Borel and equal, μ almost everywhere, to the set of all points in $\partial\Omega$ for which (5.27) is false. To prove (5.28) suppose (5.27) is false for $x \in \partial\Omega$. Then $\mu[B(x,r)] \geq \lambda(r)$ for r in a sequence (r_j) with $\lim\limits_{j\longrightarrow\infty} r_j = 0$. From Lemmas 2.15 and 2.19, we recall that

$$\mu[B(x,r)]/r \approx [u(a_r(x))/r]^{p-1} \approx |\nabla u(a_r(x))|^{p-1}.$$

The above inequality yields for small $r > 0$,

$$c\,\log|\nabla u(a_r(x))| \geq c_+\sqrt{\log(1/r)\ \log\log\log(1/r)}. \tag{5.29}$$

From the same argument as in (2.28) we see that

$$-\ln u(a_r(x)) \leq \beta\,[\log(\text{ diam }\partial\Omega) - \log r]. \tag{5.30}$$

Next from (5.11), Harnack's inequality, and Lemma 2.19, we see that if $\sigma(b(x),\tau)$ is the intersection of $\sigma(b(x),\cdot)$ with $\partial B(x,r)$, then

$$1-\tau = u(\sigma(b(x),\tau)) \approx u(a_r(x)) \text{ and } |\nabla u(a_r(x))| \approx |\nabla u[\sigma(b(x),\tau)]|. \tag{5.31}$$

Using (5.31), (5.30) in (5.29) we conclude that

$$c \log|\nabla u[\sigma(b(x),\tau)]| \geq c_+ \sqrt{-\log(1-\tau)\,\log\log[-\log(1-\tau)]} \tag{5.32}$$

for τ in a sequence tending to zero. Clearly for c_+ large enough, (5.32) can only hold if (5.26) is false. Thus (5.28) is valid and the proof of Theorems 1 - 3 (b) is complete. □

Closing Remarks. We believe it is unlikely that Theorems 1 - 3 remain valid when the assumption, '$\partial\Omega$ is a quasicircle', is replaced by, ' Ω is simply connected.' Clearly our methods depend heavily on the fact that $\partial\Omega$ is a quasicircle. As pointed out in [BL], Theorem C would likely follow from a proof of the first display in Lemma 2.19 for simply connected domains. Numerous related problems are listed in [BL]. One problem not listed there, which illustrates the difficulty in replacing a quasidisk by a simply connected domain, is the p harmonic version of the Hayman-Wu theorem. To be more specific, is it true for any line l that

$$\int_{l\cap D} |\nabla u|^{p-1}\, dH^1 < \infty. \tag{5.33}$$

Here u, D are as in section 2, but we assume only that Ω is simply connected. (5.33) follows easily from Lemmas 2.15, 2.19 if $\partial\Omega$ is a quasicircle.

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