On Pseudospheres That Are Quasispheres

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ABSTRACT: We construct bounded domains $D \neq$ ball in $n \geq 3$ dimensional Euclidean space, R^n, for which ∂D is homeomorphic to a ball under a quasiconformal mapping of R^n and such that $n-1$ dimensional Hausdorff measure $='$s harmonic measure on ∂D.

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1. Introduction. Denote points in Euclidean space, R^n, by $x = (x_1, ..., x_n)$ and let $\bar{E}$, ∂E, denote the closure and boundary of $E \subseteq R^n$, respectively. Put $B(x, r) = \{y : \ |y - x| < r\}$ and $S(x, r) = \{y : \ |y - x| = r\}$ when $r > 0$. Define k dimensional Hausdorff measure, $1 \leq k \leq n$, in R^n as follows: For fixed $\delta > 0$ and $E \subseteq R^n$, let $L(\delta) = \{B(x_i, r_i)\}$ be such that $E \subseteq \cup B(x_i, r_i)$ and $0 < r_i < \delta, \quad i = 1, 2, ...$ Set

$$\phi_\delta^k(E) = \inf_{L(\delta)} \left(\sum \alpha(k) r_i^k \right),$$

where $\alpha(k)$ denotes the volume of the unit ball in R^k. Then

$$H^k(E) = \lim_{\delta \to 0} \phi_\delta^k(E), \quad 1 \leq k \leq n.$$

Let D be a bounded domain in R^n with $0 \in D$ and $H^{n-1}(\partial D) < +\infty$. Then ∂D is said to be a pseudo sphere (see [S]) if

$$\begin{aligned} &(a) \quad D \neq \text{ ball and there is a homeomorphism } f : R^n \to R^n \text{ with } f(S(0,1)) = \partial D, \\ &(b) \quad h(0) = a \int_{\partial D} h \, dH^{n-1}, \text{ whenever } h \text{ is harmonic in } D \text{ and continuous on } \bar{D}\,. \end{aligned} \tag{1.1}$$

In (b), a denotes a constant. The construction of pseudo spheres in R^2, which are not circles, was first done by Keldysh and Lavrentiev to show the existence of domains not of Smirnov type (see [KL], [P, Ch. 3]). Also a completely different proof of existence in R^2 has been given by Duren, Shapiro, and Shields in [DSS] (see also [Du, Ch. 10]). In higher dimensions we proved in [LV]

Theorem A. *There exists a pseudo sphere D in R^n, $n \geq 3$.*

Recall that a function $g : R^n \to R^n$ is said to be $K \geq 1$ quasiconformal on R^n (see [R], [Re]) if

- (i) g is a homeomorphism of R^n onto R^n,
- (ii) g has distributional partial derivatives that are locally n th power integrable,
- (iii) $\|Dg(x)\|^n \leq K \, J_g(x)$, almost everywhere.

In (iii), $Dg(x) = (\frac{\partial g_i(x)}{\partial x_j})$, is the Jacobian matrix of g and $\|Dg(x)\|$ is the norm of $Dg(x)$ as a linear operator on R^n. Also $J_g(x)$ (the Jacobian of g at x) is the determinant of $Dg(x)$. In [LV] we asked whether f in the definition of a pseudosphere can also be chosen $K > 1$ quasiconformal from R^n to R^n when $n \geq 3$. If so, then ∂D is said to be a K quasisphere. In R^2 it follows easily from the geometric construction of Keldysh and Lavrentiev and the Ahlfors three point condition [A] that there exists pseudospheres which are quasispheres. The construction in [DSS] (see also [D, ch 10]) is also easily seen to produce pseudospheres that are quasispheres. In this note we answer our own conjecture by proving

Theorem 1. *Given $K > 1$ there exists a pseudo sphere D in R^n, $n \geq 3$, which is a K quasisphere.*

We note that the only 1 quasiconformal maps of R^n are Möbius transformations. Also, it was shown by [FL] that convex domains satisfying some mild smoothness conditions cannot be pseudospheres. More generally, in [LV1] we showed that Lipschitz domains cannot be pseudospheres. Thus pseudospheres can be nice (quasispheres) but not too nice (Lipschitz). To point out some of the differences between Theorem 1 and Theorem A we need to recall some details from [LV]. Suppose $a = 1$ in the definition of a pseudosphere. To construct D, let $D_0 = B(0, \rho_0)$ and let

$$G_0(x) = (n(n-2)\alpha(n))^{-1} \left(|x|^{2-n} - \rho_0^{2-n}\right), \quad x \in B(0, \rho_0),$$

be Green's function for $B(0, \rho_0)$, where ρ_0 is chosen so that if $x \in \partial B(0, \rho_0)$, then

$$|\nabla G_0(x)| = (n\alpha(n))^{-1} \rho_0^{1-n} = 2. \tag{1.2}$$

By induction, if D_m has been defined for $m =$ a nonnegative integer, we added certain smooth bumps to ∂D_m to get D_{m+1} with $D_m \subset D_{m+1}$. Then $D = \bigcup D_m$. To obtain f we modified the identity mapping slightly in a neighborhood of each bump, to get h_{m+1} a homeomorphism from R^n into R^n, with $h_{m+1}(\partial D_m) = \partial D_{m+1}$, $h_{m+1}(D_m) = D_{m+1}$, for $m = 0, 1, \ldots$. Put $h_0(x) = \rho_0 x$ and set $f_k(x) = h_k \circ h_{k-1} \circ \cdots \circ h_0(x)$. Then $f = \lim_{k \to \infty} f_k$ uniformly in R^n. The problem with our construction in [LV] was that the distortion (i.e K) could build up under successive iterations. In the present paper we overcome this difficulty by using the so called ' mickey mouse ' construction which is apparently due to Thurston (oral communication to the first author by Seppo Rickman). Under this construction h_{m+1} is defined in such a way that it is 1 quasi-conformal (i.e the restriction of a Möbius transformation) in a neighborhood of H^{n-1} almost every point of ∂D_m. To get D_{m+2} we then only allow bumps to be added that lie in the image of the above neighborhoods. It turns out for H^{n-1} almost every point $x \in R^n$ that we can arrange it so that all functions in the composition defining $f_k(x)$, with one exception, are 1 quasiconformal, while the remaining function can be chosen K quasiconformal for fixed $K > 1$. We note that the construction of a pseudosphere in R^2 given in [P] also uses circles, but for a different reason. To carry out the above program we have had to overcome certain problems not encountered in [LV]. For example in this paper we added C^∞ bumps to ∂D_m and consequently were able to use Schauder type theorems to make the desired estimates on the Green's function of D_{m+1}. However, to get h_{m+1}, as above, we are forced to add non smooth spherical bumps to ∂D_m. Hence we have to argue that our earlier program can still be used. Also in [LV] we used an important lemma of Wolff [W, Lemma 2.7] for the Green's function of a domain obtained by adding a C^∞ bump to a half space. Again we have to verify that Wolff's lemma remains valid for spherical bumps (whose radius is large). As for the proof of Theorem 1 we follow closely the proof of Theorem A in [LV] so the reader is advised to have this paper at hand. In section 2 we discuss adding spherical bumps to a domain and show inequality (1.1) in [LV] (see (2.3)) is still valid. In section 3 we use the ' mickey mouse construction ' to get D and f. In section 4 we add a spherical bump to a half space and show that the conclusion of Wolff's lemma remains true. We then use this lemma in section 5 to show that (1.3) in [LV] (see (3.14)) still holds. (1.2), (1.3), and (1.9) of [LV] imply that (1.1)(b) is valid (see the discussion in section 3 following (3.11)).

Finally the second author would like to thank Seppo Rickman for showing him the 'mickey mouse construction ' and Juha Heinonen for some helpful conversations.

2. Spherical Bumps. We assume throughout this section that Ω is a bounded domain with $0 \in \Omega$. Moreover we assume Ω is locally Lipschitz. That is given $y \in \partial\Omega$ there exists $s > 0$ such that $B(y, s) \cap \partial\Omega$ is a part of the graph of a Lipschitz function defined on a hyperplane in R^n and $B(y, s) \cap \Omega$ lies above the graph. We also assume that $\partial\Omega$ is connected and the union of a finite number of closed spherical caps with centers in Ω and the property that each point of $\partial\Omega$ lies in at most two spherical caps. Thus either two caps are disjoint or their intersection is an $n-2$ dimensional ' circle ' (intuitively cut out by the smaller sphere from the larger sphere). Let T denote the set of points in the union of these ' circles '. Finally we assume that $F \subset R^n$ is a compact set with $F \cap \partial\Omega \subset T$. We remark that in our construction F will be the set of points where a certain iterate is not 1 quasiconformal. Intuitively we want to avoid this set in modifying Ω to get Ω' so that successive iterations will not increase K. Let G be Green's function for Ω with pole at 0. By definition,

$$G(x) - (n(n-2)\alpha(n))^{-1} |x|^{2-n}, \; x \in R^n,$$

is harmonic in Ω and G has boundary value 0 in the sense of Perron-Wiener-Brelot. Using the Kelvin transformation (see [H]) we see that each component of

$$\nabla G(x) = \left(\frac{\partial G}{\partial x_1}, ..., \frac{\partial G}{\partial x_n} \right)$$

extends to a C^∞ function on $\bar{\Omega} \setminus [T \cup \{0\}]$. Under this assumption suppose that

$$|\nabla G| > 1 \text{ on } \partial\Omega \setminus T. \tag{2.1}$$

Given $\delta, 0 < \delta \le 10^{-20}$, we shall add smooth spherical bumps to $\partial\Omega$ by "pushing out" $\partial\Omega$ along certain small surface elements in $\{x \in \partial\Omega \setminus T : \ |\nabla G|(x) > 1 + \delta\}$ of approximate side length r'. Let Ω', G' be the domain and Green's function with pole at 0, obtained from this process. Then $\partial\Omega'$ will have the same properties as $\partial\Omega$, i.e it is locally Lipschitz, connected, and the union of a finite number of closed spherical caps with centers in Ω' and the property that each point of $\partial\Omega'$ lies in at most two spherical caps. Define T' relative to Ω' in the same way that T was defined relative to Ω. Then $\Omega \subset \Omega'$, $T \subset T'$ and we shall choose the spherical bumps so that

$$|\nabla G'| > 1 \text{ on } \partial\Omega' \setminus T'. \tag{2.2}$$

Also for $t \ge \delta$, we shall have

$$H^{n-1}(\partial\Omega') \ge H^{n-1}(\partial\Omega) + \eta(t) H^{n-1}\left(\{x : \ |\nabla G(x)| > 1 + t\}\right), \tag{2.3}$$

where η is a nondecreasing positive function on $(0, \infty)$ which is independent of Ω, Ω'.

Let $0 < \sigma_0 < 10^{-3}$ be a small positive number to be chosen in section 5 and let l be the largest nonnegative integer such that $2^{-l}\sigma_0 > \delta > 0$. Put $\sigma_k = 2^{-k}\sigma_0$, for $k = 0, 1, ...$ and set

$$\begin{aligned} E_k &= \{x \in \partial\Omega : \ 1 + \sigma_k < |\nabla G(x)| \le 1 + \sigma_{k-1}\}, \quad 1 \le k \le l + 1, \\ E_0 &= \{x \in \partial\Omega : \ |\nabla G(x)| > 1 + \sigma_0\}. \end{aligned}$$

Let $d(E_1, E_2)$ denote the Euclidean distance between the sets E_1, E_2 and put $U = \{y \in \partial\Omega : d(\{y\}, T) < 10^8\, \hat{r}_0\}$ where $\hat{r}_0 > 0$ is so small that

$$H^{n-1}\left(E_k \cap U\right) \le (1/4)\, H^{n-1}(E_k) \text{ for } 0 \le k \le l + 1 \tag{2.4}$$

which is possible since $H^{n-1}(T) = 0$. Next if $0 < \hat{r}_1 < 1$ is the smallest radius of the spheres whose caps form $\partial\Omega$ we also choose

$$\hat{r}_0 \le (\delta\, \hat{r}_1 / 10)^{20}. \tag{2.5}$$

Let $V = \{y \in \bar{\Omega} : d(\{y\}, T) \ge 10^4\, \hat{r}_0\} \cap \{y \in \bar{\Omega} : d(\{y\}, \{0\}) \ge \rho_0/2\}$ where ρ_0 is as in (1.2) and set

$$M_1 = \max_{x \in V} \sum |\partial_\beta G(x)|,$$

where $\beta = (\beta_1, \beta_2, ..., \beta_n)$, $\ 0 \le |\beta| \le 2$, is a multiindex and ∂_β denotes the corresponding partial derivative with respect to x^β, $x \in V$. We first choose r_0', $0 < r_0' \leq \hat{r}_0$, so that

$$r_0' \leq \frac{1}{(10nM_1)^{10}} \,. \tag{2.6}$$

Given $y \in \partial\Omega \setminus U$ let $B(\bar{y}, \bar{r}_0)$ denote the reflection of $B(y, 10^4 r_0')$ with respect to the sphere whose spherical cap $\subset \partial\Omega$ contains y. From our assumptions on $\partial\Omega$ we can choose $r_0' > 0$ so small that for any $y \in \partial\Omega \setminus U$,

$$B(y, 10^4 r_0') \text{ intersects exactly one spherical cap } \subset \partial\Omega, \text{ and } [B(y, 10^4 r_0') \cup B(\bar{y}, \bar{r}_0)] \cap F = \emptyset. \tag{2.7}$$

From compactness and a standard covering argument it follows for each r', $0 < r' \leq r_0'$, that there exists, $y^1, y^2, ..., y^N \in \partial\Omega \setminus U$, such that

$$\partial\Omega \setminus U \subseteq \bigcup_{i=1}^{N} B(y^i, 100\, r') \cap \bar{\Omega} \subseteq V \text{ and } B(y^i, 10\, r') \cap B(y^j, 10\, r') = \emptyset, \quad i \neq j. \tag{2.8}$$

We now construct Ω'. Let L be the set of all $y \in \{y^i\}_1^N$ for which

$$B(y, 100\, r') \cap \left(\bigcup_{k=0}^{l+1} E_k \right) \neq \emptyset.$$

Let $\lambda_k \geq 2$, $k = 0...$, be an increasing sequence of positive numbers to be specified later and set $r_k = r'/\lambda_k$ for $k = 0, ..., l+1$. For fixed $y \in L$, let $j = j(y)$ be the smallest nonnegative integer with

$$B(y, 100\, r') \cap E_j \neq \emptyset. \tag{2.9}$$

We draw a sphere $S(\tilde{y}, \tilde{r})$ of radius $\tilde{r}$, center $\tilde{y} \in \Omega$ with the following properties:

$$\begin{aligned} &(a) \quad S(\tilde{y}, \tilde{r}) \cap \partial\Omega = S(y, \sigma_j^2\, r_j) \cap \partial\Omega, \\ &(b) \quad \text{The angle between the normals to } S(\tilde{y}, \tilde{r}) \text{ and } \partial\Omega \text{ at points of intersection is } \sigma_j^2, \\ &(c) \quad B(\tilde{y}, \tilde{r}) \subset \Omega \cup B(y, \sigma_j^2\, r_j). \end{aligned} \tag{2.10}$$

Existence of $S(\tilde{y}, \tilde{r})$ as in (2.10) follows from (2.5) and elementary geometry. Define Ω' by

$$\begin{aligned} &(i) \qquad \Omega \setminus \left(\bigcup_{z \in L} B(z, \sigma_j^2\, r_j) \right) = \Omega' \setminus \left(\bigcup_{z \in L} B(z, \sigma_j^2\, r_j) \right), \\ &(ii) \qquad \partial\Omega' \cap \bar{B}(y, \sigma_j^2\, r_j) = S(\tilde{y}, \tilde{r}) \setminus \Omega \text{ whenever } y \in \{y^i\}_1^N, \\ &(iii) \qquad \Omega' \cap B(y, \sigma_j^2\, r_j) = B(\tilde{y}, \tilde{r}) \cap B(y, \sigma_j^2 r_j). \end{aligned} \tag{2.11}$$

From (2.10), (2.11), and (2.7) it is clear that $\partial\Omega'$ is locally Lipschitz, connected, and the union of a finite number of closed spherical caps with centers in Ω' and the property that each point of $\partial\Omega'$ lies in at most two spherical caps.

We now prove (2.2). If $x \in \partial\Omega' \cap \partial\Omega$, then it follows from (2.1) and the Hopf boundary maximum principle that (2.2) is true. Otherwise, $x \in S(\tilde{y}, \tilde{r}) \cap (\partial\Omega' \setminus T')$ for some $y \in \{y^i\}_1^N$, $S(\tilde{y}, \tilde{r})$, satisfying (2.7)-(2.11). Using (2.5), (2.10) $(a), (b)$ and high school geometry it is easily seen for σ_0 small enough that

$$\tilde{r}/2 \le r_j \le 2\,\tilde{r} \tag{2.12}$$

From (2.12), (2.5) we deduce that $S(\tilde{y}, \tilde{r}) \cap \bar{\Omega} \subset V$ and thereupon from (2.6), (2.1) as well as Taylor's theorem with remainder that

$$[\,1 - (r_0')^{1/2}\,]\,|\nabla G(y)|\,\langle \nu, z - y\rangle \le G(z) \tag{2.13}$$

whenever $z \in S(\tilde{y}, \tilde{r})$ and $\langle \nu, z - y\rangle \ge 8\sigma_j^4 r_j$. Here ν denotes the inner unit normal to $\partial\Omega$ at y and $\langle \cdot, \cdot \rangle$ denotes the inner product on R^n. Let ϕ be a C^∞ function on R with $\phi = 0$ in $(-\infty, 8\sigma_j^4\, r_j)$, $\phi(x) \le x$, for $x \ge 0$, with equality when $x \ge 16\sigma_j^4\, r_j$ and $(\sigma_j^4\, r_j)^{i-1}\,|\,\frac{d^i}{dx^i}(\phi(x) - x)\,| \le 10^4$ whenever $x \ge 0$ and $0 \le i \le 2$. Let h be the harmonic function in $B(\tilde{y}, \tilde{r})$ which is continuous in $\bar{B}(\tilde{y}, \tilde{r})$, with boundary values $h(x) = \phi(\langle \nu, x - y\rangle)$ whenever $x \in S(\tilde{y}, \tilde{r})$. Let

$$H(z) = h(\tilde{y} + \tilde{r}z) - \langle \nu, \tilde{y} - y + \tilde{r}z\rangle \text{ for } z \in B(0,1).$$

Using (2.12) and Schauder type estimates (see [GT]), or direct estimates by way of the Poisson integral for B(0, 1) we find that

$$|\,\nabla H\,|(z) \le c(n)\,\tilde{r}\,\sigma_j^{3/2}$$

whenever $\tilde{y} + \tilde{r}\,z \in \partial\Omega'$. Transferring back we get

$$|\nabla h - \nu| \le c(n)\sigma_j^{3/2}, \tag{2.14}$$

in $S(\tilde{y}, \tilde{r}) \cap \partial\Omega'$, where $c(n) \ge 1$ as in the sequel is a constant which only depends on n, not necessarily the same at each occurence. Since $G \le G'$ in Ω, we see from (2.13) and the boundary values of h that

$$[1 - (r_0')^{1/2}]|\nabla G(y)|\,h \le G' \tag{2.15}$$

on $S(\tilde{y}, \tilde{r})$. Using the Hopf boundary maxium principle and (2.14) , (2.15) it follows that

$$|\nabla G'| \ge [1 - (r_0')^{1/2}][1 - c(n)\sigma_j^{3/2}]|\nabla G(y)| \tag{2.16}$$

on $S(\tilde{y},\tilde{r})\cap\partial\Omega'$. Now from (2.5), (2.6), (2, 9) we deduce that

$$|\nabla G(y)|\geq 1+\sigma_j/2.$$

Putting this inequality in (2.16) we see for $\sigma_0=\sigma_0(n)>0$ small enough that(2.2) is true for $x\in S(\tilde{y},\tilde{r})\cap\Omega'$. Hence (2.2) is true on $\partial\Omega'\setminus T'$.

Next we prove (2.3). To do this observe from (2.5) that since $r'<<\hat{r}_1$ we have

$$H^{n-1}[\partial\Omega\cap B(y,\sigma_j^2\,r_j)]\;\leq\;\alpha(n-1)\,(\sigma_j^2\,r_j\,)^{n-1}\;-\;c(n)\,\hat{r}_1^{-2}\,(\sigma_j^2\,r_j\,)^{n+1}\,. \tag{2.17}$$

Note from (2.5), (2.12), and elementary trigonometry, that the solid angle say θ subtended by $B(\tilde{y},\tilde{r})\cap\partial\Omega$ with respect to $\tilde{y}$ satisfies

$$|\theta-\sigma_j^2|\leq 4\tilde{r}\sigma_j^2/\hat{r}_1 \tag{2.18a}$$

and

$$\tilde{r}\sin\theta\;\geq\;\sigma_j^2 r_j\;\;-\;\;(\sigma_j^2 r_j)^3/(100\,\hat{r}_1^2) \tag{2.18b}$$

for $\sigma_0=\sigma_0(n)>0$ small enough. Now using spherical coordinates and (2.18a) it is easily seen that

$$H^{n-1}(S(\tilde{y},\tilde{r})\setminus\Omega)\;\geq\;\alpha(n-1)\,(1\,+\,\sigma_j^4/c(n))(\tilde{r}\sin\theta)^{n-1}.$$

From this inequality, (2.18b), and once again (2.5) we conclude that

$$H^{n-1}[\partial\Omega'\cap B(y,\sigma_j^2\,r_j)]\;\geq\;(1\,+\,\sigma_j^4/c(n))\,\alpha(n-1)\,(\sigma_j^2\,r_j)^{n-1}\,. \tag{2.19}$$

Combining (2.17), (2.19), and using $\lambda_j\,r_j=r'$, we find for some $c(n)\geq 1$ that

$$H^{n-1}[\partial\Omega'\cap B(y,\sigma_j^2\,r'/\lambda_j)]\;\geq\;(1\,+\,\sigma_j^4/c(n))\,H^{n-1}[\partial\Omega\cap B(y,\sigma_j^2\,r'/\lambda_j)]. \tag{2.20}$$

Let $\eta(t)=\sigma_i^{2n+2}\,\lambda_i^{1-n}\,/c_1(n)$ for $\sigma_{i+1}\leq t<\sigma_i$, $i=0,1,\ldots$ and set $\eta(t)=\sigma_0^{2n+2}\,\lambda_0^{1-n}/c_1(n)$ for $t\geq\sigma_0$. Then from (2.20), (2.4), and (2.8) we conclude for $c_1(n)$ large enough that (2.3) is true for $t\geq\delta$.

3. The Mickey Mouse Construction. We continue with the same notation introduced in sections 1-2. Let $\Omega, \Omega', y \in \{y^i\}_1^N, r', \tilde{r} = \tilde{r}(j), S(\tilde{y}, \tilde{r}), \lambda_j$, and σ_j^2 be as in (2.7) - (2.12). Suppose that $B(y, 100r') \cap \partial\Omega \subset S(w, \rho^*)$ with $B(w, \rho^*) \subset \Omega$. Choose a Möbius transformation L so that

$$\begin{aligned}
&(\alpha) \quad L\,[B(w,\rho^*)] = H = \{x \in R^n : x_2 > 0\},\\
&(\beta) \quad L\,[S(w,\rho^*) \cap S(\tilde{y},\tilde{r})] = \{x \in R^n : x_1 = x_2 = 0\},\\
&(\gamma) \quad L\,[B(\tilde{y},\tilde{r})] = \tilde{H} \text{ and } L(\tilde{H} \setminus H) \subset \{x : x_1, x_2 < 0\},\\
&(\delta) \quad \text{The angle between the normals to } H, \tilde{H} \text{ at points of } \tilde{H} \cap H \text{ is } \sigma_j^2.
\end{aligned} \tag{3.1}$$

(3.1) is easily proven using (2.10) (b), as well as the fact that Möbius transformations preserve angles and map balls into hyperplanes or balls (see [Re, ch 3]). We introduce polar coordinates $x_1 = r\cos\theta,\ x_2 = r\sin\theta,\ r \geq 0,\ 0 \leq \theta < 2\pi$. If $x = (x_1, x_2, \ldots, x_n)$ we put $\hat{x} = (x_3, \ldots, x_n)$ and write $x = (x_1, x_2, \hat{x})$. Next we define a quasiconformal mapping q of R^n as follows:

$$\begin{aligned}
&q(x) = x \text{ when } 0 \leq \theta \leq \sigma_j^2,\\
&q(x) = \left(\, r\cos(\lambda\,[\theta - \sigma_j^2] + \sigma_j^2),\ r\sin(\lambda\,[\theta - \sigma_j^2] + \sigma_j^2),\ \hat{x}\,\right)\\
&\ \text{for } \sigma_j^2 < \theta \leq \pi - \sigma_j^2 \text{ with } \lambda = \tfrac{\pi - \sigma_j^2}{\pi - 2\sigma_j^2},\\
&q(x) = (r\cos(\theta + \sigma_j^2),\ r\sin(\theta + \sigma_j^2), \hat{x}) \text{ for } \pi - \sigma_j^2 < \theta \leq \pi + \sigma_j^2,\\
&q(x) = \left(r\cos(\lambda^*\,[\theta - \pi - \sigma_j^2] + \pi + 2\sigma_j^2),\ r\sin(\lambda^*[\theta - \pi - \sigma_j^2] + \pi + 2\sigma_j^2),\ \hat{x}\,\right)\\
&\ \text{for } \pi + \sigma_j^2 < \theta \leq 2\pi - \sigma_j^2 \text{ with } \lambda^* = \tfrac{\pi - 3\sigma_j^2}{\pi - 2\sigma_j^2},\\
&q(x) = x \text{ for } 2\pi - \sigma_j^2 < \theta < 2\pi.
\end{aligned}$$

From the above definition of q we note that

$$\begin{aligned}
&(i) \quad q \text{ maps } H \text{ onto } H \cup \tilde{H},\\
&(ii) \quad q \text{ is the identity mapping on } \{x : x_1/\sqrt{x_1^2 + x_2^2} \geq \cos(\sigma_j^2)\,\},\\
&(iii) \quad q \text{ is a rotation on } \{x : x_1/\sqrt{x_1^2 + x_2^2} \leq -\cos(\sigma_j^2)\},\\
&(iv) \quad q \text{ is } 1 + 10\,\sigma_j^2 \text{ quasiconformal on } R^n.
\end{aligned} \tag{3.2}$$

Put $g(x) = L^{-1} \circ q \circ L(x)$ when $x \in R^n$. From (2.7), (2.12), (3.1) we note that if

$$\tilde{F} = L^{-1} \circ q \left(\{x : -\cos(\sigma_j^2) \leq x_1/\sqrt{x_1^2 + x_2^2} \leq \cos(\sigma_j^2)\} \right),$$

then

$$[\tilde{F} \cup g^{-1}(\tilde{F})] \subset B(y, 10^4 r_0') \cup B(\bar{y}, \bar{r}_0) \text{ and } [\tilde{F} \cup g^{-1}(\tilde{F})] \cap F = \emptyset \tag{3.3}$$

where the last line follows from (2.7). From (3.2) we also conclude that

$$
\begin{aligned}
&(a) \quad g \text{ is the identity transformation on the unbounded component } I \text{ of } R^n \setminus g^{-1}(\tilde{F}),\\
&(b) \quad g \text{ is a Möbius transformation on the bounded component } J \text{ of } R^n \setminus g^{-1}(\tilde{F}),\\
&(c) \quad g \text{ is } 1+10\,\sigma_j^2 \text{ quasiconformal on } R^n.
\end{aligned}
\tag{3.4}
$$

We do this construction for each $y \in \{y^i\}_1^N$ obtaining functions $g_1, \ldots, g_N$ and sets $\tilde{F}_1, I_1, J_1, \ldots, \tilde{F}_N, I_N, J_N$, corresponding to $y^1, y^2, \ldots, y^N$, in such a way that (3.3), (3.4) hold with $g = g_i$, $1 \le i \le N$. Define $\Phi : R^n \to R^n$ by

$$
\Phi(x) = \begin{cases} x \text{ when } x \in I_1 \cap I_2 \ldots \cap I_N \\ g_i(x) \text{ when } x \in g_i^{-1}(\tilde{F}_i) \cup J_i,\ 1 \le i \le N. \end{cases}
\tag{3.5}
$$

We note that Φ is well defined since from (3.3), (2.8), and (2.12) it follows that the sets $g_i^{-1}(\tilde{F}_i) \cup J_i$, $1 \le i \le N$, are pairwise disjoint. Using this note and (3.5) we conclude that

$$
\begin{aligned}
&(*) \quad \Phi(\Omega) = \Omega' \text{ and } \Phi \text{ is a } 1+10\,\sigma_j^2 \text{ quasiconformal mapping of } R^n \text{ onto } R^n,\\
&(**) \quad \Phi \equiv \text{ a Möbius transformation in each component of } R^n \setminus \Phi^{-1}(\hat{F}) \text{ where } \hat{F} = \textstyle\bigcup_{i=1}^N \tilde{F}_i,\\
&(***) \quad [\hat{F} \cup \Phi^{-1}(\hat{F})] \cap F = \emptyset \text{ and } F' = \hat{F} \cup F \text{ is compact with } F' \cap \partial\Omega' \subset T'.
\end{aligned}
\tag{3.6}
$$

We now construct D, f. Let $D_0 = B(0, \rho_0)$ be as in section 1 where ρ_0 is as in (1.2) and set $F = F_0 = T = T_0 = \emptyset$. Let $\delta = \delta_0 = 10^{-20}\sigma_0$ and put $\Omega = D_0$. We use the results in section 2 to get $\Omega' = D_1$ satisfying (2.2), (2.3) and $\Phi_1 = \Phi$ satisfying (3.6) with $\Phi_1(D_0) = D_1$. Let $F_1 = F'$, $T_1 = T'$ be the sets obtained from this construction. We now proceed by induction. Suppose D_k, Φ_k, T_k, F_k have been constructed using the results in section 2 for $m \ge 1$ with

$$
D_k \subset D_{k+1},\ T_k \subset T_{k+1},\ F_k \subset F_{k+1}, \text{ and } F_{k+1} \cap \partial D_{k+1} \subset T_{k+1}, \text{ for } 0 \le k \le m-1,
\tag{3.7}
$$

in such a way that

$$
|\nabla G_k| > 1 \text{ on } \partial D_k \setminus T_k,
\tag{3.8}
$$

$$
H^{n-1}(\partial D_{k+1}) \ge H^{n-1}(\partial D_k) + \eta(t) H^{n-1}(\{x : \ |\nabla G_k(x)| > 1+t\})
\tag{3.9}
$$

whenever $t \ge \delta_k = 10^{-20}\sigma_k, 0 \le k \le m-1$. Here G_k denotes the Green's function for D_k with pole at 0. We also assume that

$$
\text{(3.6) holds with } \Phi,\ F,\ F' \text{ replaced by } \Phi_{k+1},\ F_k,\ F_{k+1}, \text{ respectively, for } 0 \le k \le m-1.
\tag{3.10}
$$

We put $\Omega = D_m$, $F = F_m$, $T = T_m$, and note from the induction hypothesis, (3.7), that $F \cap \partial\Omega \subset T$. If $\delta = \delta_m = 10^{-20}\sigma_m$, then we can apply the results in section 2 to get $\Omega' = D_{m+1}$, $T' = T_{m+1}$, for which (3.8), (3.9) hold when $k = m+1$. Also using (3.6) we get $F' = F_{m+1}$, $\Phi = \Phi_{m+1}$, satisfying (3.10) with $k = m$. By induction we conclude that (3.7)-(3.10) holds, for each nonnegative integer k.

Put $D = \bigcup_0^\infty D_i$. We note that $f_m = \Phi_m \circ \cdots \circ \Phi_1$ maps D_0 onto D_m. From (3.6) (***), (3.7), and (3.10) it is clear for given $x \in R^n \setminus T_{m-1}$ that each function in the composition defining f_m, with at most one exception, is a Möbius transformation in a neighborhood of x. Moreover such an exception is $1 + 10\sigma_0^2$ quasiconformal in a neighborhood of x. Thus f_m is $1 + 10\sigma_0^2$ quasiconformal on R^n and $f_m(D_0) = D_m$ for $m \geq 1$. Now $(f_m)_1^\infty$ is a locally bounded sequence of $1 + 10\sigma_0^2$ quasiconformal mappings on R^n, so a subsequence (see [Re ch 9] or [R, ch 6]) of this sequence either converges uniformly to a $1 + 10\sigma_0^2$ quasiconformal $\hat{f}$ from R^n to R^n or to a constant. Clearly a constant is ruled out. Put $f(x) = \hat{f}(\rho_0 x)$. Then from our construction we conclude that $f(B(0,1)) = D$, $f(S(0,1)) = \partial D$. Thus if $10\,\sigma_0^2 < K$, then (1.1) (a) in Theorem 1 is true.

For the reader's convenience we outline the proof of (b) given in [LV]. Using (2.5)-(2.12) it is intuitively clear for σ_0 small enough that D is NTA in the sense of Jerison and Kenig [JK] with constant 1000 (see [LV, section 4] for details). Also from Green's theorem and (3.8) we see that

$$H^{n-1}(\partial D_k) \leq \int_{\partial D_k} |\nabla G_k| \, dH^{n-1} = 1. \tag{3.11}$$

From (3.11) we see that D is of finite perimeter in the sense of Federer (see [GE]). Thus as $k \to \infty$,

$$H^{n-1}|_{\partial D_k} \to H^{n-1}|_{\partial^* D} = H^{n-1}|_{\partial D} \, . \tag{3.12}$$

Here the convergence is weak convergence as measures. Also $\partial^* D$ is the reduced boundary of D. To get the last inequality, we note that $\partial^* D$ agrees H^{n-1} almost everywhere with the so called measure theoretic boundary of D, defined as the set of points where the Lebesgue lower n densities of $D, R^n \setminus D$ are positive. Using the fact that D is NTA, it is easily seen that ∂D ='s the measure theoretic boundary of D. Hence (3.12) is true (for a more direct proof see [LV, section 4]). Also observe from (3.9) that

$$\lim_{k \to \infty} H^{n-1}\left(\{x \in \partial D_k : |\nabla G_k(x)| > 1 + \delta\}\right) = 0 \tag{3.13}$$

for each $\delta > 0$, since otherwise we could use (3.9) and iteration to get a contradiction. Finally we shall show in section 5 that

$$\int_{D_k} |\nabla G_k| \log |\nabla G_k| \, dH^{n-1} \leq c < \infty \text{ for } k = 0, 1, \ldots, . \tag{3.14}$$

From (3.14) we deduce for $\alpha > 1, \quad k = 0, 1, ...,$

$$\log \alpha \int_{\{|\nabla G_k| > \alpha\}} |\nabla G_k| dH^{n-1} \leq c < +\infty. \tag{3.15}$$

Let $g \geq 0$ be a harmonic function in D which is continuous on $\bar{D}$. Then from (3.8), (3.12), and Green's theorem we get

$$g(0) = \int_{\partial D_k} g|\nabla G_k| dH^{n-1} \geq \int_{\partial D_k} g dH^{n-1} \to \int_{\partial D} g dH^{n-1}, \tag{3.16}$$

as $k \to \infty$. To obtain the reverse inequality for fixed $\delta < 10^{-3}$ and $\alpha > 10^3$, put

$$\begin{aligned} P_k &= \{x \in \partial D_k : \ 1 \leq |\nabla G_k(x)| \leq 1 + \delta\} \\ Q_k &= \{x \in \partial D_k : \ 1 + \delta < |\nabla G_k(x)| \leq \alpha\} \\ L_k &= \{x \in \partial D_k : \ |\nabla G_k(x)| > \alpha\}, \end{aligned}$$

for $k = 0, 1, 2...$ Then

$$g(0) = \int_{\partial D_k} g|\nabla G_k| dH^{n-1} = \int_{P_k} \cdots + \int_{Q_k} \cdots + \int_{L_k} \cdots = I_1 + I_2 + I_3.$$

Clearly,

$$|I_1| \leq (1 + \delta) \int_{\partial D_k} g dH^{n-1}.$$

Also from (3.13) we find that

$$|I_2| \leq \alpha ||g||_\infty H^{n-1} (\{x \in \partial D_k : \ 1 + \delta \leq |\nabla G_k|\}) \to 0,$$

as $k \to \infty$. Here, $||g||_\infty$ denotes the maximum of g in $\bar{D}$. Using (3.15) we get

$$|I_3| \leq ||g||_\infty \int_{\{|\nabla G_k| > \alpha\}} |\nabla G_k| dH^{n-1} \leq c(\log \alpha)^{-1} ||g||_\infty.$$

Letting $k \to \infty$ we obtain from the above estimates and (3.12) that

$$g(0) \leq (1 + \delta) \int_{\partial D} g dH^{n-1} + c(\log \alpha)^{-1} ||g||_\infty.$$

Finally letting $\delta \to 0, \quad \alpha \to \infty$, we have

$$g(0) \le \int_{\partial D} g dH^{n-1}.$$

In view of (3.16) we conclude that

$$g(0) = \int_{\partial D} g dH^{n-1} \tag{3.17}$$

when $g \ge 0$ is continuous on $\bar{D}$ and harmonic in D. From (3.17) with $g \equiv 1$ we note that, $H^{n-1}(\partial D) = 1$. If g_1 is continuous on $\bar{D}$, harmonic in D, and $g_1 - m \ge 0$ in $\bar{D}$, then from (3.17) and the above note we deduce

$$g_1(0) = (g_1 - m)(0) + m = \int_{\partial D} (g_1 - m) dH^{n-1} + m = \int_{\partial D} g_1 dH^{n-1}.$$

Finally from a simple barrier estimate it is easily seen that for each $y \in T_1$,

$$\infty = \limsup_{x \to y} |\nabla G_1|(x) \le \limsup_{x \to y} |\nabla G|(x).$$

From this inequality we conclude that $D \ne$ ball.

Thus D is a pseudosphere and Theorem 1 is true once we have proved (3.14).

4. Lemma of Wolff. If $x = (x_1, \dots, x_n) \in R^n$ we write $x = (x', x_n)$, where $x' = (x_1, \dots, x_{n-1})$. For given $\epsilon > 0, 0 < \epsilon < 1/10$, define $\phi(\cdot, \epsilon)$ on R^{n-1} by

$$\phi(x', \epsilon) = \begin{cases} \epsilon^{-1} [1 - \sqrt{1 + \epsilon^2 (1 - |x'|^2)}] \text{ when } x' \in R^{n-1} \text{ and } |x'| \le 1 \\ = 0 \text{ when } x' \in R^{n-1} \text{ and } |x'| > 1. \end{cases}$$

Put $K = \{x \in R^n : x_n > 0\}$ and set $K(\epsilon) = \{x = (x', x_n) \in R^n : x_n > \phi(x', \epsilon)\}$. We note that $\partial K(\epsilon) \setminus \partial K$ consists of the part of the sphere with center $(0, \dots, 0, \epsilon^{-1})$ and radius, $\sqrt{\epsilon^{-2} + 1}$ which lies outside K. Thus $K(\epsilon)$ is obtained by adding a spherical bump to K. Let $g(\cdot, \epsilon)$ be the Green's function for $K(\epsilon)$ with pole at ∞. That is, $g(x, \epsilon) - x_n$ is a bounded harmonic function in $K(\epsilon)$ and $g(\cdot, \epsilon)$ is continuous on $\bar{K}(\epsilon)$ with $g(\cdot, \epsilon) \equiv 0$ on $\partial K(\epsilon)$. Set

$$I(\epsilon) = \int_{\partial K(\epsilon)} |\nabla g(\cdot, \epsilon)| \ln |\nabla g|(\cdot, \epsilon) \, dH^{n-1}.$$

Next let $\hat{\theta}(x') = (1 - |x'|^2)^+, x' \in R^{n-1}$, where $a^+ = \max\{a, 0\}$. Let θ denote the bounded harmonic function on K which is continuous on $\bar{K}$ with $\theta = \hat{\theta}$ on $\partial K = R^{n-1}$. Put

$$\Lambda(\theta) = \int_{\mathbb{R}^{n-1}} [\, (\theta_{x_n})^3 - 3|\nabla' \theta|^2 \, \theta_{x_n}] \, dH^{n-1}$$

where ∇' denotes the gradient in x' only. We prove

Lemma 4.1 *If $\Lambda(\theta) > 0$, then there exists $c^* = c^*(n) \geq 1$, such that $I(\epsilon) \leq -\epsilon^3 \Lambda(\theta)/100$, for $0 < \epsilon \leq c^*(n)^{-1} \min(\Lambda(\theta), 1)$.*

Proof: The proof is essentially the same as Lemma 2.12 in [W]. However this lemma was proved under the assumption that $\partial K(\epsilon)$ is smooth (C^∞) where in our case $\partial K(\epsilon)$ is just Lipschitz. Therefore we include some details. We shall show that I has continuous fourth derivatives and $|I''''| \leq c(n)$ on $(0, \epsilon_0)$ for $\epsilon_0 = \epsilon_0(n) > 0$, sufficiently small. Also it will turn out that the derivatives of I can be found by differentiating under the integral sign as in [W] and $I(0) = I'(0) = I''(0) = 0$, while $I'''(0) = -(1/8)\Lambda(\theta)$. Using Taylor's theorem with remainder we then get Lemma 4.1.

To begin, let $y \in \partial K(\epsilon)$ and suppose for some $r > 0$ that w is harmonic in $K(\epsilon) \cap B(y, 2r)$ with continuous boundary values zero on $\partial K(\epsilon) \cap B(y, 2r)$ and $|w| \leq M < \infty$ in $B(y, 2r) \cap K(\epsilon)$. From a barrier type argument we find for $0 < \epsilon \leq \epsilon_0 \leq 1/100$, sufficiently small, that

$$|w|(x) \leq c(n)\, M\, (|x - y|/r)^{9/10} \tag{4.2}$$

for $x \in K(\epsilon) \cap B(y, r)$. With ϵ_0 now fixed let $g(\cdot, z, \epsilon)$ denote Green's function for $K(\epsilon)$ with pole at $z \in K(\epsilon)$ for $0 < \epsilon \leq \epsilon_0$. We note that $g(x, z, \epsilon) \leq c(n)|x - z|^{2-n}$ since the righthand side is a constant multiple of the Green's function for R^n. Let $S = \{(x', 0) \in R^n : |x'| = 1\}$ and let $\hat{x}, \hat{z} \in \partial K(\epsilon) \cap [B(0, 2) \setminus S]$. Let $x \in B(\hat{x}, |1 - |\hat{x}||/2)$, $z \in B(\hat{z}, |1 - |\hat{z}||/2)$, with $|1 - |\hat{x}|| \leq |1 - |\hat{z}||/16$. Then from (4.2) with $r = |x - z|/2$, $y = \hat{x}$, $w = g(\cdot, z, \epsilon)$, and the above note it follows that

$$g(x, z, \epsilon) \leq c\, |1 - |x||^{9/10}\, |x - z|^{1-n+1/10}\,. \tag{4.3}$$

Next suppose that v is harmonic in $K(\epsilon)$ with

$$v(x) = \int_{\partial K(\epsilon) \cap B(0,1)} (1 - |z|)^{4/5}\, |\nabla g(x, z, \epsilon)|\, dH^{n-1}z,\ x \in K(\epsilon)$$

where derivatives of $g(\cdot, \cdot, \epsilon)$ are with respect to z. Under these assumptions we prove for $x \in B(\hat{x}, |1 - |\hat{x}||/2) \cap K(\epsilon)$, and $\hat{x} \in \partial K(\epsilon) \cap B(0, 2)$ that there exists $\hat{c}(n) \geq 1$ with

$$|v(x)| \leq \hat{c}(n)\, |1 - |x||^{4/5}. \tag{4.4}$$

Now from Green's formula,

$$|v(x)| \leq c(n)\,|1-|x||^{4/5} + \int_J (1-|z|)^{4/5}\,|\nabla g(x,z,\epsilon)|dH^{n-1}z \tag{4.5}$$

where $J = \{z \in \partial K(\epsilon) \cap B(0,1) : |1-|x|| < (1/100)(1-|z|)\,\}$. From the Kelvin transformation (see [H]), it is easily seen that $g(x,\cdot,\epsilon)$ extends to a harmonic function in $B(z, |1-|z||)$ whenever x is not in this ball and $z \in \partial K(\epsilon) \setminus S$. We shall also denote this extension by $g(x,\cdot,\epsilon)$. Using this fact, (4.3), and interior estimates for harmonic functions we see that

$$|\nabla g(x,z,\epsilon)| \,\leq\, c(n)\,|1-|x||^{9/10}\,|x-z|^{-n+1/10} \text{ whenever } z \in J.$$

Putting this estimate in (4.5), using $2|x-z| \geq |1-|z||$ when $z \in J$ and integrating we get (4.4).

Again from the Kelvin transformation, (4.2) with $w = g$, and interior estimates for harmonic functions we observe that

$$|\partial^k g/\partial x^\alpha|(x,\epsilon) \,\leq\, c(k,n)\,|1-|x||^{-k+9/10}, x \in B(\hat{x}, \tfrac{1}{2}|1-|\hat{x}||), \tag{4.6}$$

whenever $\hat{x} \in \partial K(\epsilon) \cap [B(0,2) \setminus S], 0 < \epsilon \leq \epsilon_0$, and $k = 0,1,\ldots$. Here $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi index with $|\alpha| = k$ and $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$. Also we have

$$g(x', \phi(x',\epsilon), \epsilon) \equiv 0,\; x' \in R^{n-1},\; 0 < \epsilon \leq \epsilon_0. \tag{4.7}$$

Next observe for $k = 0,1,\ldots,$ that $\partial^k\phi(\cdot,\epsilon)/\partial\epsilon^k$ is uniformly Lipschitz for $x' \in R^{n-1}, 0 < \epsilon \leq \epsilon_0$, with

$$\begin{aligned} &(i) && |\nabla \tfrac{\partial^k \phi(x',\epsilon)}{\partial \epsilon^k}| \,\leq\, c(n,k), \\ &(ii) && |\tfrac{\partial^k \phi(x',\epsilon)}{\partial \epsilon^k}| \,\leq\, c(n,k)\,(1-|x'|)^+ \\ &(iii) && \tfrac{\partial \phi(x',0)}{\partial \epsilon} \,=\, -(1-|x'|^2)^+/2,\; \tfrac{\partial^2\phi(x',0)}{\partial\epsilon^2} \equiv 0. \end{aligned} \tag{4.8}$$

We claim that (4.6) - (4.8) imply $g(\cdot,\epsilon)$ has continuous mixed partials in x, ϵ of all orders whenever, $x \in B(\hat{x}, |1-|\hat{x}||/2), 0 < \epsilon \leq \epsilon_0$ and $\hat{x} \in \partial K(\epsilon) \setminus S$. Moreover if also $|\hat{x}| < 2$,

$$\left|\frac{\partial^{k+l} g(x,\epsilon)}{\partial x^\alpha\,\partial\epsilon^l}\right| \,\leq\, c(k,l,n)\,|1-|x||^{-k+4/5} \tag{4.9}$$

for $k, l = 0,1,\ldots$ while

$$\left|\frac{\partial^{k+l}\,[g(x,\epsilon) - x_n]}{\partial x^\alpha\,\partial\epsilon^l}\right| \,\leq\, c(k,l,n)\,|1+|x||^{-k-n+1},\; x \in \partial K(\epsilon),\; |x| \geq 2. \tag{4.10}$$

(4.9) for $l = 0, k = 0,1,\ldots$ is implied by (4.6). (4.10) follows from the fact that $g(\cdot,\epsilon) - x_n$ extends to a bounded harmonic function in $R^n \setminus B(0,3/2)$ which is zero on $\{(x',0) : x' \in R^{n-1},\, |x'| > 3/2\}$ and the Poisson integral formula for such functions. Thus (4.9), (4.10) are true for $l = 0$, $k = 0, 1, \dots,$. We can now proceed by induction to get (4.9), (4.10). We do only the case $l = 1, k = 0, \dots,$ in detail. From (4.7), (4.8)(ii), the mean value theorem from elementary calculus, and (4.6) for $k = 1$ we see that

$$
\begin{aligned}
&|g(x', \phi(x', \epsilon_2), \epsilon_2) - g(x', \phi(x', \epsilon_2), \epsilon_1)| = |g(x', \phi(x', \epsilon_2), \epsilon_1) - g(x', \phi(x', \epsilon_1), \epsilon_1)| \\
&\quad \leq c(n) \max\{ |\partial g/\partial x_n|(x', \phi(x', \epsilon), \epsilon) \,:\, 0 < \epsilon \leq \epsilon_0\} \; |\phi(x', \epsilon_2) - \phi(x', \epsilon_1)| \\
&\quad \leq c(n)|\epsilon_2 - \epsilon_1| \, [(1 - |x'|)^+]^{4/5}
\end{aligned}
\tag{4.11}
$$

for $x' \in \partial K(\epsilon_2) \setminus S$, $0 < \epsilon_1, \epsilon_2 \leq \epsilon_0$. From (4.11) we deduce that $\{(\epsilon_2 - \epsilon_1)^{-1} \, (g(\cdot, \epsilon_2) - g(\cdot, \epsilon_1))\}$ is uniformly bounded and has a continuous extension to $\partial K(\epsilon_2)$ whenever $0 < \epsilon_1, \epsilon_2 \leq \epsilon_0$ and $\epsilon_1 \neq \epsilon_2$. From the maximum principle for harmonic functions and the Kelvin transformation, it follows that this sequence is harmonic and uniformly bounded in $L(\epsilon_2) = K(\epsilon_2) \bigcup \{B(\hat{x}, |1 - |\hat{x}||) : \hat{x} \in \partial K(\epsilon_2) \setminus S\}$. Letting $\epsilon_1 \to \epsilon_2$ it follows that $\partial g/\partial \epsilon$ is uniformly continous and bounded in $L(\epsilon_2)$ whenever $0 \leq \epsilon_2 \leq \epsilon_0$. Moreover,

$$
\partial g/\partial \epsilon \, (x, \epsilon) = -\partial g/\partial x_n \, (x, \epsilon) \; \partial \phi/\partial \epsilon \, (x', \epsilon), \;\; x = (x', \phi(x', \epsilon)) \in \partial K(\epsilon) \setminus S. \tag{4.12}
$$

Using (4.8) (ii) and (4.6) with $k = 1$ we get $|\partial g/\partial \epsilon \, (\cdot, \epsilon)| \leq c(n)[(1 - |x|)^+]^{4/5}$ on $\partial K(\epsilon)$. Using this inequality and the maximum principle for bounded harmonic functions in $K(\epsilon)$ we conclude first that $|\partial g/\partial \epsilon| \leq c(n)v$, and thereupon from (4.4), the Kelvin transformation, and interior estimates for harmonic functions that (4.9) is true when $l = 1, k = 0, 1, \dots$. (4.10) follows for $l = 1$ by the same reasoning as when $l = 0$. Finally since a uniformly convergence sequence of harmonic functions has derivatives which also converge uniformly, it follows that the mixed partial derivatives consisting of one partial derivative in ϵ and k partial derivatives in the space variable x are independent of the order of differentiation.

Next we use (4.12) and argue as in (4.11) to obtain that

$$
\begin{aligned}
\frac{\partial^2 g}{\partial \epsilon^2}(x, \epsilon) &= -2 \frac{\partial^2 g}{\partial \epsilon \partial x_n}(x, \epsilon) \, \frac{\partial \phi}{\partial \epsilon}(x', \epsilon) \\
&\quad - \frac{\partial g}{\partial x_n}(x, \epsilon) \, \frac{\partial^2 \phi}{\partial \epsilon^2}(x', \epsilon) \, - \, \frac{\partial^2 g}{\partial x_n^2}(x, \epsilon) \, (\frac{\partial \phi}{\partial \epsilon})^2(x', \epsilon)
\end{aligned}
\tag{4.13}
$$

whenever $x \in \partial K(\epsilon)$. Using (4.8) (ii) and (4.9) with $l = 0, 1$, $k = 1, 2$, we conclude first that $|\partial^2 g/\partial \epsilon^2(x, \epsilon)| \leq c(n)[(1 - |x|)^+]^{4/5}$ when $x \in \partial K(\epsilon)$ and thereupon from (4.4), the Kelvin transformation, and interior estimates for harmonic functions, that (4.9) is true when $l = 0, 1, 2$. (4.10) follows by the same reasoning as when $l = 0, 1$. As above we see that the mixed partial derivatives consisting of two partial derivatives in ϵ and k partial derivatives in the space variable x are independent of the order of differentiation. Continuing by induction we get (4.9), (4.10).

Finally observe from a barrier argument that

$$c(n)|\nabla g(\cdot,\epsilon)| \geq 1 \text{ on } \partial K(\epsilon) \setminus S \tag{4.14}$$

for $0 < \epsilon \leq \epsilon_0$. Using (4.9), (4.10), (4.14) we deduce that derivatives of I with respect to ϵ of all orders can be found by differentiating under the integral sign defining I. Doing this and letting $\epsilon \to 0$ we find that the argument of Wolff [W, pp 360-362] can be used essentially verbatim. One only needs to check that the second and third partial derivatives of ϕ with respect to ϵ do not add additional terms in the calculations when $\epsilon = 0$. In fact from (4.8) (iii) we see that the second partial of ϕ with respect to ϵ vanishes identically. Moreover all terms involving the third partial of ϕ with respect to ϵ vanish at $\epsilon = 0$ (since all second partials of $g(x,0) = x_n$ are identically zero and $|\nabla g(x,0)| \equiv 1$). Lemma 4.1 now follows from Wolff's argument in the way mentioned at the beginning of the proof. $\square$

In order to apply Wolff's lemma we need to show that $\Lambda(\theta) > 0$. In fact we shall show in [LVV] that if $\hat\psi \geq 0$, is a radial, nonincreasing, Lipschitz function on R^{n-1} with compact support and $\hat\psi \not\equiv 0$, then $\Lambda(\psi) > 0$. As usual, ψ denotes the bounded harmonic extension of $\hat\psi$ to K which is continuous on $\bar K$ with $\psi = \hat\psi$ on ∂K. Clearly this result implies

$$\Lambda(\theta) > 0. \tag{4.15}$$

Here we outline a direct method for establishing (4.15) which gives a numerical lower bound for the integral when $n \geq 5$. Using separation of variables or the Poisson integral formula for harmonic functions in a half space one can show for $r = |x'|$ that $\theta_{x_n}(x',0) = -c_n F(n/2, -1/2, (n-1)/2, r^2)$, $0 < r < 1$, where $F(a,b,c,z)$ is the usual hypergeometric function,

$$c_n = 2\Gamma(n/2)/[\Gamma((n-1)/2)\,\Gamma(3/2)],$$

and Γ is the Euler gamma function. Writing the hypergeometric function in a series it is easily seen that

$$-F(n/2, -1/2, (n-1)/2, r^2) \geq -1 + \frac{n}{n-1}[1 - (1-r^2)^{1/2}].$$

Using this estimate and doing some arithmetic we find that

$$\begin{aligned}\int_0^1 \theta_{x_n}^3 \, r^{n-2} dr &\geq c_n^3\,(n-1)^{-3} \int_0^1 [-1 + n(1-r^2)^{1/2}]^3\, r^{n-2}\, dr \\ &= c_n^3 (n-1)^{-3}\left(\frac{1}{n-1} + \frac{6n^2}{n^2-1}\right) - 6\, c_n^2 (n-1)^{-3}\left(1 + n^2/(n+2)\right) = b_n.\end{aligned} \tag{4.16}$$

Also we note that

$$3\int_0^1 \theta_{x_n}\,|\nabla'\theta|^2\, r^{n-2}dr \;=\; -12c_n \int_0^1 F(n/2,-1/2,(n-1)/2)\, r^n\, dr \;=\; 48(n-2)/[(n+2)\pi n].$$

The last equality is obtained by writing out the series for the integrand and integrating term by term. The series one gets after evaluating at 1 can be written as the sum of several hypergeometric functions evaluated at 1. Using tables one then gets the last equality. Finally using Stirling's formula and making some more estimates one can show that $b_n + 48(n-2)/[(n+2)\pi n] \geq 0$ for $n \geq 5$, which in view of (4.16), the above equality, and the fact that $\theta_{x_n}(r,0) \geq 0$ for $r \in (1,\infty)$ (by positivity of θ) implies (4.15) for $n \geq 5$. The cases $n = 3,4$ can be done separately. A more involved argument using estimates also for $\int_1^\infty \theta_{x_n}^3\, r^{n-2}\, dr$ can be used to show that for some absolute constant c one has $\Lambda(\theta) \geq c/n$, $n = 3,4,\dots$ (more details will be supplied upon request).

Next we introduce some notation in order to state some consequences of Lemma 4.1 and (4.15). let Ω_1 be a bounded domain with diameter ≈ 1 and NTA constant 1000. Then by definition,

(i)(corkscrew condition) For each $x \in \partial\Omega_1$, $0 < r < 1$, there are points $P_r(x) \in \Omega_1$, $Q_r(x) \in R^n \setminus \Omega_1$, with $|P_r(x) - x| \leq 1000\, r$, $|Q_r(x) - x| \leq 1000\, r$, and dist. $(P_r(x), \partial\Omega_1) \geq 1000^{-1} r$, dist. $(Q_r(x), \partial\Omega_1) \geq 1000^{-1} r$,

(ii)(Harnack chain condition) For each $x, y \in \Omega_1$ there is a path $\gamma : [0,1] \to \Omega_1$ with $\gamma(0) = x, \gamma(1) = y$, and with length $\leq 1000|x-y|$. Also dist. $(\gamma(t), \partial\Omega_1) \geq 1000^{-1} \min(|\gamma(t) - x|, \;\; |\gamma(t) - y|)$ for $t \in [0,1]$.

Next suppose that Ω_1 is Lipschitz on scale t with constant 1000. That is assume for each $z \in \partial\Omega_1$, there exists a coordinate system such that $\partial\Omega_1 \cap B(z,t)$ is the graph of a Lipschitz function defined on R^{n-1} with Lipschitz norm ≤ 1000. Moreover, $\Omega_1 \cap B(z,t)$ lies above the graph of this function. Finally assume for some $w \in \partial\Omega_1$ and $t > 0$ that after a possible rotation of coordinates,

$$\begin{aligned} \partial\Omega_1 \cap B(w,t) &= \{x :\; x_n = w_n\} \cap B(w,t) \\ \Omega_1 \cap B(w,t) &= \{x :\; x_n > w_n\} \cap B(w,t). \end{aligned} \tag{4.17}$$

Let $\phi(\cdot,\epsilon)$ be as defined at the beginning of section 4, $\lambda \geq 2$, and define $\Omega_2(\epsilon) \supset \Omega_1$ for $0 < \epsilon \leq \epsilon_0$, as follows:

(a) $\quad \Omega_1 \setminus B(w,t) = \Omega_2(\epsilon) \setminus B(w,t)$,

(b) $\quad \partial\Omega_2(\epsilon) \cap B(w,t) = \{(x' + w', w_n + t\lambda^{-1}\phi(t^{-1}\lambda x', \epsilon)) :\;\; x' \in R^{n-1}\} \cap B(w,t)$,

(c) $\quad \Omega_2(\epsilon) \cap B(w,t) = \{(x' + w', x_n) :\; x_n > w_n + t\lambda^{-1}\phi(t^{-1}\lambda x', \epsilon)\} \cap B(w,t)$.

We assume

$$B(0,\rho_0) \subseteq \Omega_1 \subseteq B(0,1), \tag{4.18}$$

where ρ_0 is as in (1.2). Denote Green's functions for $\Omega_1, \Omega_2(\epsilon)$, with pole at 0, by $G_1, G_2(\cdot,\epsilon)$ respectively, and let ω_1 be harmonic measure on Ω_1 with respect to 0. With this notation we state

Lemma 4.19. *Let Ω_1 be NTA and Lipschitz on scale t with constant 1000. Suppose Ω_1 satisfies (4.17), (4.18), and Ω_2 is obtained by adding a spherical bump to Ω_1 as in (a)-(c). Let $\epsilon_0 = (2c^*(n))^{-1}\min(\Lambda(\theta),1)$, where c^* is as in Lemma 4.1. If $0 < \hat{\epsilon} \le \epsilon_0$, then there exists $\lambda^* = \lambda^*(\hat{\epsilon}, n)$, $\bar{c} = \bar{c}(\hat{\epsilon}, n) \ge 2$, such that for $\lambda \ge \lambda^*$,*

$$\int_{\partial\Omega_2(\epsilon)} |\nabla G_2(\cdot,\epsilon)| \log |\nabla G_2|(\cdot,\epsilon)\, dH^{n-1} \le \int_{\partial\Omega_1} |\nabla G_1| \log |\nabla G_1|\, dH^{n-1} - \frac{1}{\bar{c}\lambda^{n-1}}\omega_1(B(w,t))$$

whenever $\hat{\epsilon} \le \epsilon \le \epsilon_0$.

Proof: In view of Lemma 4.1 and (4.15) we can essentially apply [W, Lemma 2.7] to get Lemma 4.19 in R^3. The proof in $R^n, n > 3$, is unchanged. □

5. Proof of Theorem 1. Armed with Lemma 4.19 we can use the argument in [LV, section 3] to prove (3.14) and hence complete the proof of Theorem 1. Unfortunately, in [LV, section 3] Schauder estimates for smooth domains were again used, whereas our boundaries are only locally Lipschitz. Thus for the reader's convenience we sketch the argument in [LV, section 3] indicating the necessary changes. We wish to apply Lemma 4.19 to D_m, D_{m+1} constructed in section 3, but in order to do so we need to introduce intermediary domains with flat bumps as in Lemma 4.19 and make some estimates. We shall use the same notation as in section 2. Note that in section 3 we constructed D_{m+1} from D_m by adding spherical bumps as in (2.9)-(2.11). Thus we work with Ω, Ω' as in section 2. We assume, as we may, that $\sigma_0^2 \le \epsilon_0/100$ where σ_0 is yet to be fixed and ϵ_0 is as in Lemma 4.19. We now define (λ_k) introduced above (2.9). Let $\hat{r}_0, \hat{r}_1, \delta, M_1, r_0', r', (E_k), L, l, r_j$ be as in section 2. For fixed $y \in L$ recall from (2.9) that j was the least postive integer such that $B(y, 100r') \cap E_j \neq \emptyset, 1 \le j \le l+1$. Let T be the tangent plane to $\partial\Omega$ at $y \in \partial\Omega$. From the above restriction on σ_0, (2.5), we see as in (2.18a) that the central angle, say $\eta_j = \eta_j(y)$, subtended by $B(\tilde{y}, \tilde{r}) \cap T$ (relative to $\tilde{y}$) satisfies $2^{-1}\sigma_j^2 \le \eta_j \le 2\sigma_j^2 \le \epsilon_0/4$, regardless of the choice of $\lambda_j \ge 2$, $y \in L$

or r'. Put $\epsilon_j = \tan(\sigma_j^2/2)$, $\lambda_j^* = \lambda^*(\epsilon_j, n)$ and set $\lambda_j^{'} = \max(\sigma_j^{-2}, b_j, \lambda_j^*)$, $j = 0, 1, ...,$ where $b_j = \bar{c}(\epsilon_j, n)$. Let $\lambda_k = \max_{0 \le j \le k} \lambda_j^{'}$, $k = 0, 1, ...,$ and observe that $(\lambda_k)_0^\infty$ depends only on n once σ_0 is fixed.

We add flat bumps to Ω, Ω' as follows. Let y, j be as above and as in (3.1) let $S(w, \rho^*)$ be such that $B(y, 100r') \cap \partial\Omega \subset S(w, \rho^*)$ and $B(w, \rho^*) \subset \Omega$. After a rotation if necessary we may assume that $y = (w', w_n - \rho^*)$, where $w = (w', w_n)$. Let $A = \rho^* - \sqrt{(\rho^*)^2 - (r' + (r')^{3/2})^2}$ and define $\psi : R^{n-1} \to R$ by

$$\psi(x') = \begin{cases} w_n - \rho^* \text{ for } |x' - w'| \le r', \\ (r')^{-3/2} A \, (|x' - w'| - r') + w_n - \rho^* \text{ for } r' \le |x' - w'| \le r' + (r')^{3/2}, \\ w_n - \sqrt{(\rho^*)^2 - |x' - w'|^2} \text{ for } r' + (r')^{3/2} \le |x' - w'| \le \rho^*. \end{cases}$$

Note that the graph of ψ coincides with the tangent plane T to $S(w, \rho^*)$ at y when $|x' - w'| \le r'$ and 'linearly' connects this tangent plane with $S(w, \rho^*)$ when $r' \le |x' - w'| \le r' + (r')^{3/2}$. Suppose that $L = \{z_1, z_2, ..., z_p\}$ and put $L_k = \{z_1, ..., z_k\}$, $1 \le k \le p$.

Define $\hat{\Omega}_k$, $1 \le k \le p$, by

$$(\hat{\text{I}}) \qquad \hat{\Omega}_k \setminus \left(\bigcup_{z \in L_k} B(z, 10\, r') \right) = \Omega \setminus \left(\bigcup_{z \in L_k} B(z, 10\, r') \right),$$

$$(\hat{\text{II}}) \qquad \partial\hat{\Omega}_k \cap B(y, 10\, r') = \{(x', \psi(x'))\} \cap B(y, 10\, r'),$$

$$(\hat{\text{III}}) \qquad \hat{\Omega}_k \cap B(y, 10\, r') = \{(x', x_n) : \quad x_n > \psi(x')\} \cap B(y, 10\, r'),$$

whenever $y \in L_k$. Next we define $\tilde{\Omega}_k \supseteq \hat{\Omega}_p$, $1 \le k \le p$, relative to $\hat{\Omega}_k$ in the same way that Ω' was defined by adding spherical bumps to Ω. That is,

$$(\tilde{I}) \qquad \hat{\Omega}_k \setminus \left(\bigcup_{z \in L_k} B(z, \sigma_j^2 r_j) \right) = \tilde{\Omega}_k \setminus \left(\bigcup_{z \in L_k} B(z, \sigma_j^2\, r_j) \right),$$

$$(\tilde{II}) \qquad \partial\tilde{\Omega}_k \cap \bar{B}(y, \sigma_j^2\, r_j) = B(y, \sigma_j^2\, r_j) \cap [\partial\hat{\Omega}_k \setminus B(\tilde{y}, \tilde{r}) \cup S(\tilde{y}, \tilde{r}) \setminus \hat{\Omega}_k] \text{ whenever } y \in L_k,$$

$$(\tilde{III}) \qquad \tilde{\Omega}_k \cap B(y, \sigma_j^2\, r_j) = (B(\tilde{y}, \tilde{r}) \cup \hat{\Omega}_k) \cap B(y, \sigma_j^2 r_j) \text{ whenever } y \in L_k.$$

Here $\tilde{y}, \tilde{r}$ are defined as in (2.10) relative to y. From the definition of Ω, Ω' we see that $\hat{\Omega}_k \supseteq \Omega$, $\tilde{\Omega}_p \supseteq \Omega'$ for $1 \le k \le p$. Also from the definition of ψ and (2.5) it can be shown as in [LV, section 4] that $\hat{\Omega}_k, \tilde{\Omega}_k$, $1 \le k \le p$, are NTA and Lipschitz on scale r' with constant 1000. Let $\hat{\Omega}_0 = \Omega$, $\tilde{\Omega}_0 = \hat{\Omega}_p$. From the defininion of (λ_k) and our restriction on σ_0 we deduce after a possible rotation and translation that Lemma 4.19 can be applied with $\Omega_1 = \tilde{\Omega}_0$, $\Omega_2 = \tilde{\Omega}_1$. Next by the same reasoning we can apply Lemma 4.19 with $\Omega_1 = \tilde{\Omega}_1$ and $\Omega_2 = \tilde{\Omega}_2, ...,$ etc. Let $\hat{G}_k, \tilde{G}_k, \hat{\omega}_k, \tilde{\omega}_k$, be the Green's functions and harmonic measures relative to 0 for $\hat{\Omega}_k, \tilde{\Omega}_k$. Applying the above argument p times we obtain an inequality for $\hat{G}_p = \tilde{G}_0$ and $\tilde{G}_p$. Using the definition of $(\lambda_k)_0^\infty$, we conclude

$$\int_{\partial\Omega_p} |\nabla\tilde{G}_p| \log |\nabla\tilde{G}_p| dH^{n-1} \leq \int_{\partial\Omega_p} |\nabla\hat{G}_p| \log |\nabla\hat{G}_p| dH^{n-1} - c(n)\,(\lambda_{l+1})^{-(n-1)} \sum_{k=0}^{p-1} \tilde{\omega}_k(B(z_{k+1}, 2r')). \tag{5.1}$$

To prove (3.14) we must show that $\hat{G}_p, \tilde{G}_p$, in (5.1) can be replaced by G, G', with a manageable error term. To do so we introduce $\Omega_k^{'}, 0 \leq k \leq p$, defined by $\Omega_0^{'} = \Omega'$, and for $1 \leq k \leq p$,

$$\begin{array}{ll} (\mathrm{I}') & \Omega_k^{'} \setminus \left(\bigcup\limits_{z \in L_k} B(z, 10\,r') \right) = \Omega' \setminus \left(\bigcup\limits_{z \in L_k} B(z, 10\,r') \right), \\ (\mathrm{II}') & \partial\Omega_k^{'} \cap B(y, 10\,r') = \partial\tilde{\Omega}_k \cap B(y, 10\,r'), \\ (\mathrm{III}') & \Omega_k^{'} \cap B(y, 10\,r') = \tilde{\Omega}_k \cap B(y, 10\,r'), \end{array}$$

for each $y \in L_k$. Denote the corresponding Green's functions and harmonic measures relative to 0, by $G_k^{'}, \omega_k^{'}$, $1 \leq k \leq p$. We shall also need the following facts about the NTA domain Ω_1 with constant 1000 satisfying (4.18). If $z \in \partial\Omega_1$ and $0 < \rho \leq 10$, then

$$\begin{aligned} \omega_1(B(z,\rho)) &\leq c(n)\rho^{n-2} \max_{B(z,\rho)\cap\Omega_1} G_1 \\ &\leq c(n)\rho^{n-2} G_1(P_\rho) \leq c(n)\,\omega_1(B(z,\rho)), \end{aligned} \tag{5.2}$$

where $P_\rho = P_\rho(z)$. Moreover,

$$\omega_1(B(z, 2\rho)) \leq c(n)\,\omega_1(B(z,\rho)). \tag{5.3}$$

(5.3) is called the doubling inequality for harmonic measure. Also, there exists $\mu = \mu(A) > 0$ so that for z, P_ρ, as above, and $x \in B(z,\rho) \cap \Omega_1$,

$$G_1(x) \leq c(n)\,(|x - z|/\rho)^\mu\, G_1(P_\rho). \tag{5.4}$$

From Harnack's inequality, it follows that there exists $\nu = \nu(n)$, $1 < \nu < \infty$, with

$$c(n)^{-1}\rho^\nu \leq \omega_1(B(z,\rho)) \leq 1, \quad 0 < \rho \leq 1. \tag{5.5}$$

Next we note that if $z \in \partial\Omega_1$ and u, v, are two positive harmonic functions in Ω_1 which vanish continuously on $\partial\Omega_1 \setminus B(z,\rho)$, and $P_\rho = P_\rho(z)$, then for $x \in \Omega_1 \setminus B(z, 2\rho)$

$$c(n)^{-1} u(P_\rho)/v(P_\rho) \leq u(x)/v(x) \leq c(n) u(P_\rho)/v(P_\rho). \tag{5.6}$$

Moreover, (5.6) is valid when u, v, vanish on $\partial\Omega_1 \cap B(z, 2\rho)$, and $x \in B(z,\rho) \cap \Omega_1$. (5.6) is called the rate inequality. Next since Ω_1 is Lipschitz on scale t, we have for $0 < t_1 \leq t$,

$$t_1^{1-n} \int_{B(z,t_1)} |\nabla G_1|^2\, dH^{n-1} \leq c(n)\, \left({t_1}^{1-n} \omega_1[B(z, t_1)] \right)^2 \tag{5.7}$$

which is called an L^2 reverse Hölder inequality. Using (5.7) and Hölder's inequality one easily deduces the following A_∞ type condition. If $E \subset B(z,t_1)$ is a Borel set, then

$$\frac{\omega_1(E)}{\omega_1(B(z,t_1))} \le c(n) \left(\frac{H^{n-1}(E)}{H^{n-1}(B(z,t_1))} \right)^{1/2} . \tag{5.8}$$

Also using (5.5), (5.7) and Jensen's inequality one deduces,

$$\int_{B(z,t_1)} |\nabla G_1| \, |\log |\nabla G_1| \, |\, dH^{n-1} \le -c(n) \log t_1 \; \omega_1[B(z,t_1)]. \tag{5.9}$$

For the proof of (5.2)-(5.6) see [KJ, sections 4 and 5]. (5.7) follows from (5.6) and a result of Dahlberg (see [D]). Using (5.2)-(5.5) it follows as in [LV, (3.10)] that

$$\sum_{k=0}^{p-1} \omega_k^*(B(z_{k+1}, 6r')) \le c(n), \tag{5.10}$$

whenever $*$ is an element of $\{\wedge, \sim, '\}$. We show for $0 \le k \le p-1$ that

$$\int_{\partial\Omega_k'} |\nabla G_k'| \log |\nabla G_k'| dH^{n-1} \le \int_{\partial\Omega_{k+1}'} |\nabla G_{k+1}'| \log |\nabla G_{k+1}'| dH^{n-1} + c(n)(r')^{1/8} \omega_k'(B(z_{k+1}, 6r')), \tag{5.11}$$

$$\int_{\partial\hat\Omega_{k+1}} |\nabla \hat G_{k+1}| \log |\nabla \hat G_{k+1}| dH^{n-1} \le \int_{\partial\hat\Omega_k} |\nabla \hat G_k| \log |\nabla \hat G_k| dH^{n-1} + c(n)(r')^{1/8} \hat\omega_k(B(z_{k+1}, 6r')). \tag{5.12}$$

Summing (5.11) and using (5.10), it then follows that

$$\int_{\partial\Omega'} |\nabla G'| \log |\nabla G'| dH^{n-1} \le \int_{\partial\tilde\Omega_p} |\nabla \tilde G_p| \log |\nabla \tilde G_p| dH^{n-1} + c(n)(r')^{1/8} \tag{5.13}$$

where we have used the fact that $\Omega_0' = \Omega'$, $\Omega_p' = \tilde\Omega_p$. Summing (5.12) and using (5.10), we find

$$\int_{\partial\hat\Omega_p} |\nabla \hat G_p| \log |\nabla \hat G_p| dH^{n-1} \le \int_{\partial\Omega} |\nabla G| \log |\nabla G| dH^{n-1} + c(n)\,(r')^{1/8} \tag{5.14}$$

since $\hat\Omega_0 = \Omega$. Putting (5.13), (5.14) into (5.1) we get

$$\int_{\partial\Omega'} |\nabla G'| \log |\nabla G'| dH^{n-1} \le \int_{\partial\Omega} |\nabla G| \log |\nabla G| dH^{n-1} + c(n)\,(r')^{1/8}. \tag{5.15}$$

Using this inequality in the definition of D_{m+1} we obtain

$$\int_{\partial D_{m+1}} |\nabla G_{m+1}| \log |\nabla G_{m+1}| dH^{n-1} \le \int_{\partial D_m} |\nabla G_m| \log |\nabla G_m| dH^{n-1} + c(n)\,(r')^{1/8} \tag{5.16}$$

where $r' = r'(m)$. From (2.5) and the definition of δ_k following (3.9) we see that $\sum_{m=0}^{\infty} (r'(m))^{1/8} < \infty$. Hence (3.14) is true and the proof of Theorem 1 is complete after we prove (5.11), (5.12).

We prove only (5.11) for $k = 0$, since the proof of all the other inequalities is the same. To prove (5.11) for $k = 0$ let $y = z_1$ in the definition of Ω'_1 and let ψ be as defined earlier relative to y. If $\tilde{y}, \tilde{r}$ are as in (2.10), put

$$\tau(x') = \begin{cases} \min(\,\psi(x'),\ \tilde{y}_n - \sqrt{(\tilde{r})^2 - |w' - x'|^2}\,) \text{ for } |x' - w'| \le \tilde{r} \\ = \psi(x') \text{ for } \tilde{r} \le |x' - w'| \le \rho^*. \end{cases}$$

Then

$$\partial\Omega'_1 \cap B(y, 10r') \;=\; \{(x', \tau(x'))\} \cap B(y, 10r')$$

$$\Omega'_1 \cap B(y, 10r') \;=\; \{(x', x_n) : x_n > \tau(x')\} \cap B(y, 10r').$$

Also if

$$\sigma(x') = \begin{cases} \min(\,w_n - \sqrt{(\rho^*)^2 - |x' - w'|^2},\ \tilde{y}_n - \sqrt{(\tilde{r})^2 - |x' - w'|^2}) \text{ for } |x' - w'| \le \tilde{r} \\ w_n - \sqrt{(\rho^*)^2 - |x' - w'|^2} \text{ for } \tilde{r} \le |x' - w'| \le \rho^*, \end{cases}$$

then

$$\partial\Omega' \cap B(y, 10r') \;=\; \{(x', \sigma(x'))\} \cap B(y, 10r')$$

$$\Omega' \cap B(y, 10r') \;=\; \{(x', x_n) : x_n > \sigma(x')\} \cap B(y, 10r').$$

Next let

$$K_1 = \{x' : (x', x_n) \in S(\tilde{y}, \tilde{r}) \cap S(w, \rho^*)\}$$

$$K_2 = \{x' : (x', w_n - \rho^*) \in S(\tilde{y}, \tilde{r})\}$$

$$K_3 = \{x' : r' \le |x' - w'| \le r' + (r')^{3/2}\}.$$

Let K be the set of all $x' \in R^{n-1}$ whose distance from $\cup_{i=1}^{3} K_i$ is at most $1000(r')^{3/2}$ and set

$$H = \{x' : |x' - w'| < 3r'\} \setminus K$$

$$K' = \{(x', x_n) \in \partial\Omega' \cap B(z_1, 3r') : x' \in K\}$$

$$K'_1 = \{(x', x_n) \in \partial\Omega'_1 \cap B(z_1, 3r') : x' \in K\}.$$

We have

$$\begin{aligned} &\left| \int_{\partial\Omega' \cap B(z_1,3r')} |\nabla G'| \log |\nabla G'| dH^{n-1} - \int_{\partial\Omega'_1 \cap B(z_1,3r')} |\nabla G'_1| \log |\nabla G'_1| dH^{n-1} \right| \\ &\le \left| \int_{K'} |\nabla G'| \log |\nabla G'| dH^{n-1} - \int_{K'_1} |\nabla G'_1| \log |\nabla G'_1| dH^{n-1} \right| \\ &+ \left| \int_{(\partial\Omega' \setminus K') \cap B(z_1,3r')} |\nabla G'| \log |\nabla G'| dH^{n-1} - \int_{(\partial\Omega'_1 \setminus K'_1) \cap B(z_1,3r')} |\nabla G'_1| \log |\nabla G'_1| dH^{n-1} \right| \\ &= T_1 + T_2. \end{aligned} \tag{5.17}$$

To estimate T_1 we cover K' by balls of radius $10(r')^{3/2}$ with centers in K' and the property that the balls with the same centers and radius $(r')^{3/2}$ are disjoint. Using (5.9) in each ball and (5.3), (5.8) with $E = K', t_1 = 3r'$ we deduce after summing that

$$\left| \int_{K' \cap B(z_1,3r')} |\nabla G'| \log |\nabla G'| dH^{n-1} \right| \le -c(n) \log r' \omega'(K') \le c(n)\, (r')^{1/8}\, \omega'(B(z_1, 3r')).$$

This inequality also holds with K', G', ω' replaced by K_1', G_1', ω_1'. Next we observe that it follows in the same way as (5.10) that $\omega_1'(B(z_1, 3r')) \approx \omega'(B(z_1, 3r'))$ where $\approx$ means the two quantities are constant multiples of each other (depending only on n). From the above inequalities we conclude

$$T_1 \, \leq \, c(n) \, (r')^{1/8} \, \omega'(B(z_1, 3r')). \tag{5.18}$$

To begin the estimate of T_2 we write x for $(x', \sigma(x'))$ and $\hat{x}$ for $(x', \tau(x'))$ in the following integrals.

$$\begin{aligned} T_3 = & \left| \int_{(\partial\Omega' \setminus K') \cap B(z_1, 3r')} |\nabla G_1'| \log \, |\nabla G_1'| dH^{n-1} - \int_{(\partial\Omega_1' \setminus K_1') \cap B(z_1, 3r')} |\nabla G_1'| \log \, |\nabla G_1'| dH^{n-1} \right| \\ \leq & \int_H | \, |\nabla G_1'|(x) \log \, |\nabla G_1'| \, |(x) \, | \sqrt{1 + |\nabla \sigma(x')|^2} - \sqrt{1 + |\nabla \tau(x')|^2} | \, dx' \\ + & \int_H ||\nabla G_1'|(\hat{x}) - |\nabla G_1'|(x)| \, | \log \, |\nabla G_1'||(x) \, \sqrt{1 + |\nabla \tau(x')|^2} dx' \\ + & \int_H |\nabla \hat{G}_1|(\hat{x}) \, | \log \, |\nabla G_1'(x)| - \log \, |\nabla G_1'(\hat{x})| \, | \, \sqrt{1 + |\nabla \tau(x')|^2} dx' \\ & = U_1 + U_2 + U_3 \end{aligned} \tag{5.19}$$

From the definition of τ, σ, and (5.9) we find that

$$U_1 \, \leq \, c(n) \, (r')^{1/8} \, \omega_1'(B(z_1, 3r')) \, \leq \, c(n) \, (r')^{1/8} \, \omega'(B(z_1, 3r')). \tag{5.20}$$

To estimate U_2, U_3 let $\hat{x} = (x', \tau(x')), x = (x', \sigma(x')), x' \in H$, be as in (5.19). Then $\hat{x} \in \partial\Omega_1' \setminus K_1'$ and using the Kelvin transformation it is easily seen that G_1' extends to a harmonic function in $B(\hat{x}, 2(r')^{3/2})$. If $\rho = (r')^{3/2}$, then from standard estimates for harmonic functions in balls, (5.2), and the fact that $|x - \hat{x}| \, \leq \, c(n)(r')^2/\rho^*$, we obtain

$$\begin{aligned} |\nabla G_1'(x) - \nabla G_1'(\hat{x})| \, & \leq \, c(n)|x - \hat{x}| \, \rho^{-2} \max_{B(\hat{x}, \rho)} G_1' \\ & \leq c(n)(r'\rho^*)^{-1} \, G_1'(P_\rho(\hat{x})) \, \leq \, c(n)(r'\rho^*)^{-1} \, \rho^{2-n} \, \omega_1'(B(\hat{x}, \rho)). \end{aligned} \tag{5.21}$$

Using positivity of G_1' and (5.3) we also find that

$$c(n)^{-1} \, \rho^{1-n} \, \omega_1'(B(\hat{x}, \rho)) \leq \, |\nabla G_1'(\hat{x})| \leq c(n) \, \rho^{1-n} \, \omega_1'(B(\hat{x}, \rho)). \tag{5.22}$$

Putting (5.22) in (5.21) we find in view of (2.5) that

$$|\nabla G_1'(x) - \nabla G_1'(\hat{x})| \, \leq \, c(n) \, (r')^{1/4} \, |\nabla G_1'|(\bar{x}) \tag{5.23}$$

where $\bar{x} = x$ or $\hat{x}$. From (5.23) and (5.9) we see that

$$U_2 + U_3 \, \leq \, c(n) \, (r')^{1/8} \, \omega'(B(z_1, 3r')). \tag{5.24}$$

Using (5.24), (5.20) in (5.19) we deduce

$$T_3 \leq c(n)\,(r')^{1/8}\,\omega'(B(z_1, 3r')). \tag{5.25}$$

If $x \in \partial\Omega' \setminus K'$, $\hat{x}$, and $\rho = (r')^{3/2}$, are as above, then again using the Kelvin transformation we deduce first that $G' - G_1'$ has a harmonic extension to $B(x, 2\rho)$ and second that

$$|\nabla G' - \nabla G_1'|(x) \leq c(n)\,\rho^{-1} \max_{B(x,\rho)} |G' - G_1'|. \tag{5.26}$$

We claim that

$$\begin{aligned} \max_{B(x,\rho)} |G' - G_1'| &\leq c(n) \max_{B(x,\rho)\cap\Omega'} |G' - G_1'| + c(n)\,[(r')^{1/2}/\rho^*]\,G'(P_\rho(x)) \\ &\leq c(n) \max_{B(x,\rho)\cap\Omega'} |G' - G_1'| + c(n)\,[(r')^{1/2}\rho/\rho^*]\,|\nabla G'|(x) \end{aligned} \tag{5.27}$$

The second line of (5.27) follows from the first line, (5.2), (5.3), and the same argument as in (5.22). To prove the first line of (5.27) observe that if $x \in S(\tilde{y}, \tilde{r}) \cap K'$, then this inequality is obvious since both functions are extended by essentially reflecting across $S(\tilde{y}, \tilde{r})$. Otherwise suppose $\bar{z} \in B(x, \rho)$ and $\tilde{z}, \hat{z}$ denote the reflection of $\bar{z}$ with respect to the plane $\{u \in R^n : u_n = w_n - \rho^*\}$ and the sphere $S(w, \rho^*)$, respectively. Then $|\tilde{z} - \hat{z}| \leq c(n)(r')^{1/2}\rho/\rho^*$. Using this fact, the definition of the Kelvin transformation, and standard estimates for functions vanishing on $B(x, \rho) \cap S(w, \rho^*)$ we obtain (5.27). As noted earlier we have $\omega'(B(z_1, 3r')) \approx \omega_1'(B(z_1, 3r'))$ so from (5.2), we have $G'(P_{r'}(x)) \approx G_1'(P_{r'}(x))$. Now if $\sigma_0 = \sigma_0(n) > 0$ is small enough, then from this note and a barrier type estimate using interior and exterior cones, we deduce for $r' \geq t \geq 2|x - \hat{x}|$ that

$$c(n)^{-1}\,(t/r')^{11/10} \max\{G'(P_{r'}(x)),\, G_1'(P_{r'}(x))\} \leq \min\{G'(P_t(x)),\, G_1'(P_t(x))\} \tag{5.28 a}$$

and

$$\max\{G'(P_t(x)),\, G_1'(P_t(x))\} \leq c(n)\,(t/r')^{9/10} \min\{G'(P_{r'}(x)),\, G_1'(P_{r'}(x))\}. \tag{5.28 b}$$

We observe that every point of $\partial\Omega' \cap B(z_1, 3r')$ lies within $c(n)(r')^2/\rho^*$ of a point of $\partial\Omega_1'$. From this observation, the maximum principle for harmonic functions, (5.28) and (2.5) we see that

$$\begin{aligned} \max_{B(x,\rho)\cap\Omega'} |G' - G_1'| &\leq c(n) \max_{\partial\Omega'\cap B(z_1,3r')} |G_1'| \leq c(n)(r'/\rho^*)^{9/10}\,G'(P_{r'}(x)) \\ &\leq c(n)(r')^{1/4}\,G'(P_\rho(x)) \leq c(n)\,(r')^{1/4}\,\rho\,|\nabla G'(x)|. \end{aligned} \tag{5.29}$$

Using (5.27), (5.29) in (5.26) we get

$$|\nabla G' - \nabla G_1'|(x) \leq c(n)\,(r')^{1/4}\,|\nabla G'(x)|. \tag{5.30}$$

Finally from (5.30), (5.25) we conclude that

$$
\begin{aligned}
T_2 &\le \left| \int_{(\partial\Omega'\setminus K')\cap B(z_1,3r')} |\nabla G_1'| \log\, |\nabla G_1'| dH^{n-1} - \int_{(\partial\Omega'\setminus K')\cap B(z_1,3r')} |\nabla G'| \log\, |\nabla G'| dH^{n-1} \right| + T_3 \\
&\le \int_H \big| |\nabla G_1'|(x) - |\nabla G'|(x) \big| \, |\log\, |\nabla G_1'||(x) \sqrt{1+|\nabla\sigma(x')|^2} dx' + c(n)\,(r')^{1/8}\,\omega'(B(z_1,3r')) \\
&+ \int_H |\nabla G'|(x)\, |\log\, |\nabla G_1'(x)| - \log\, |\nabla G'(x)|\,| \sqrt{1+|\nabla\sigma(x')|^2} dx' \\
&\qquad\qquad \le c(n)\,(r')^{1/8}\,\omega'(B(z_1,3r')).
\end{aligned} \tag{5.31}
$$

From (5.18), (5.31) and (5.17) we conclude that

$$
\left| \int_{\partial\Omega_1'\cap B(z_1,3r')} |\nabla G_1'| \log\, |\nabla G_1'| dH^{n-1} - \int_{\partial\Omega'\cap B(z_1,3r')} |\nabla G'| \log\, |\nabla G'| dH^{n-1} \right| \le c(n)\,(r')^{1/8}\,\omega'(B(z_1,3r')). \tag{5.32}
$$

Next we note that the argument in [LV] from (3.26) to (3.28) uses only NTA estimates (primarily (5.4) and (5.6)) so is also valid for our current domains. Thus

$$
\left| \int_{\partial\Omega_1'\setminus B(z_1,3r')} |\nabla G_1'| \log\, |\nabla G_1'| dH^{n-1} - \int_{\partial\Omega'\setminus B(z_1,3r')} |\nabla G'| \log\, |\nabla G'| dH^{n-1} \right| \le c(n)\,(r')^{1/8}\,\omega'(B(z_1,3r')). \tag{5.33}
$$

From (5.32), (5.33) we find that (5.11) is valid for $k = 0$. Fix $\sigma_0 = \sigma_0(n) > 0$ subject to the stipulations in sections 2-5. From our earlier remarks we conclude first (3.14) and thereupon that Theorem 1 is valid. □

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