Boundary Behaviour of p-Harmonic Functions in Domains Beyond Lipschitz Domains

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Abstract

In this paper we prove the boundary Harnack inequality and Hölder continuity for ratios of p-harmonic functions vanishing on a portion of certain Reifenberg flat and Ahlfors regular NTA-domains. Applications are given to the p-Martin boundary problem for these domains.

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1 Introduction

In [LN], [LN1], [LN2], see also [LN3], we proved a number of results concerning the boundary behaviour of positive p-harmonic functions, $1 < p < \infty$, in a bounded Lipschitz domain $\Omega \subset \mathbf{R}^n$. In particular, we established the boundary Harnack inequality as well as Hölder continuity for ratios of positive p-harmonic functions, $1 < p < \infty$, vanishing on a portion of $\partial\Omega$.

Recall that in the case of harmonic functions, i.e., in the case $p = 2$, the boundary Harnack inequality for ratios of positive harmonic functions, vanishing on a portion of a Lipschitz domain, was first introduced in [K] and later proved independently in [A], [D], [W]. These results were extended in [JK] for $p = 2$ to non-tangentially accessible domains (NTA-domains), where Hölder continuity of the quotient was also established.

Currently we are unable to extend our results for $p \neq 2$, $1 < p < \infty$, to the general setting of NTA-domains. However in this paper we consider certain Reifenberg flat and Ahlfors regular NTA-domains, beyond Lipschitz domains, for which our methods are applicable. Neither class of domains is contained in the other and our proofs of the p-harmonic boundary Harnack results differ at key points for each class. We also study the Martin boundary in each class of domains.

To state our results we need to introduce some notation. Points in Euclidean n-space $\mathbf{R}^n$ are denoted by $x = (x_1, \ldots, x_n)$ or (x', x_n) where $x' = (x_1, \ldots, x_{n-1}) \in \mathbf{R}^{n-1}$. We let $\bar{E}, \partial E$, diam E, be the closure, boundary, diameter, of the set $E \subset \mathbf{R}^n$ and we define $d(y, E)$ to equal the distance from $y \in \mathbf{R}^n$ to E. $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbf{R}^n$ and we let $|x| = \langle x, x \rangle^{1/2}$ be the Euclidean norm of x. $B(x, r) = \{y \in \mathbf{R}^n : |x - y| < r\}$ is defined whenever $x \in \mathbf{R}^n$, $r > 0$, and dx denotes Lebesgue n-measure on $\mathbf{R}^n$. Let

$$h(E, F) = \max(\sup\{d(y, E) : y \in F\}, \sup\{d(y, F) : y \in E\})$$

be the Hausdorff distance between the sets $E, F \subset \mathbf{R}^n$. If $O \subset \mathbf{R}^n$ is open and $1 \leq q \leq \infty$, then by $W^{1,q}(O)$ we denote the space of equivalence classes of functions f with distributional gradient $\nabla f = (f_{x_1}, \ldots, f_{x_n})$, both of which are q th power integrable on O. Let $\|f\|_{1,q} = \|f\|_q + \| \, |\nabla f| \, \|_q$ be the norm in $W^{1,q}(O)$ where $\| \cdot \|_q$ denotes the usual Lebesgue q norm in O. Next let $C_0^\infty(O)$ be the set of infinitely differentiable functions with compact support in O and let $W_0^{1,q}(O)$ be the closure of $C_0^\infty(O)$ in the norm of $W^{1,q}(O)$.

Given G a bounded domain, i.e., a connected open set, and $1 < p < \infty$, we say that u is p-harmonic in G provided $u \in W^{1,p}(G)$ and

$$\int |\nabla u|^{p-2} \langle \nabla u, \nabla \theta \rangle \, dx = 0 \tag{1.1}$$

whenever $\theta \in W_0^{1,p}(G)$. Observe that if u is smooth and $\nabla u \neq 0$ in G, then

$$\nabla \cdot (|\nabla u|^{p-2} \nabla u) \equiv 0 \text{ in } G \tag{1.2}$$

and u is a classical solution in G to the p-Laplace partial differential equation. Here, as in the sequel, $\nabla \cdot$ is the divergence operator.

We now introduce the domains considered in this paper. We define,

Definition 1.3. *A bounded domain Ω is called non-tangentially accessible (NTA) if there exist*

$M \geq 2$ and r_0 such that the following are fulfilled:

(i) *corkscrew condition: for any $w \in \partial\Omega, 0 < r < r_0$, there exists $a_r(w) \in \Omega$ satisfying $M^{-1}r < |a_r(w) - w| < r, d(a_r(w), \partial\Omega) > M^{-1}r$,*

(ii) *$\mathbf{R}^n \setminus \bar{\Omega}$ satisfies the corkscrew condition,*

(iii) *uniform condition: if $w \in \partial\Omega, 0 < r < r_0$, and $w_1, w_2 \in B(w, r) \cap \Omega$, then there exists a rectifiable curve $\gamma : [0, 1] \to \Omega$ with $\gamma(0) = w_1, \gamma(1) = w_2$, and such that*
(a) $H^1(\gamma) \leq M\,|w_1 - w_2|$,
(b) $\min\{H^1(\gamma([0, t])),\, H^1(\gamma([t, 1]))\} \leq M\, d(\gamma(t), \partial\Omega)$.

In Definition 1.3, H^1 denotes length or the one-dimensional Hausdorff measure. We note that (iii) is different but equivalent to the usual Harnack chain condition given in [JK] (see [BL], Lemma 2.5). We choose this definition in order to emphasize the dependence of Ω on M. M will be called the NTA-constant of Ω. By a Lipschitz graph, Λ, with constant at most N, we mean, after a possible rotation, that $\Lambda = \{(x', \phi(x')) : x' \in \mathbf{R}^{n-1}\}$, where $\phi : \mathbf{R}^{n-1} \to \mathbf{R}$ is Lipschitz with $\||\nabla\phi|\|_\infty \leq N$.

Definition 1.4. *Let $\Omega \subset \mathbf{R}^n$ be a bounded domain, $w \in \partial\Omega$, and $0 < r < r_0$. Then Ω and $\partial\Omega$ are said to be uniformly (δ, N, r_0)-approximable by Lipschitz graph domains provided that there exists, whenever $w \in \partial\Omega$ and $0 < r < r_0$, a Lipschitz graph Λ containing w with constant at most N such that*

$$h(\partial\Omega \cap B(w, r), \Lambda \cap B(w, r)) \leq \delta r.$$

We let $\mathcal{F}(\delta, N, r_0)$ denote the class of all domains Ω which satisfy Definition 1.4. Moreover, note that $\mathcal{F}(\delta, 0, r_0)$ consists of all domains which are uniformly δ-approximable by planes in the Hausdorff distance sense, i.e., in this case Λ in Definition 1.4 must be a plane. Finally, for short we say that Ω is uniformly δ-approximable by Lipschitz graph domains, provided $\Omega \in \mathcal{F}(\delta, N, r_0)$ for some (δ, N, r_0).

We now discuss the rationale for considering NTA-domains which are uniformly approximable by Lipschitz graph domains. Let $\Omega \subset \mathbf{R}^n$ be a bounded domain and let p, $1 < p < \infty$, be given. Let $w \in \partial\Omega$, $0 < r < r_0$, and put $\Delta(w, r) = \partial\Omega \cap B(w, r)$. Suppose that u is a positive p-harmonic function in $\Omega \cap B(w, 4r)$, that u is continuous in $\bar{\Omega} \cap B(w, 4r)$, and that $u = 0$ on $\Delta(w, 4r)$. Then for our methods to work we need the 'fundamental inequality'

$$\hat{\lambda}^{-1}\frac{u(x)}{d(x, \partial\Omega)} \leq |\nabla u(x)| \leq \hat{\lambda}\frac{u(x)}{d(x, \partial\Omega)} \tag{1.5}$$

to hold whenever $x \in \Omega \cap B(w, r/\hat{c})$. In (1.5) the constants $\hat{\lambda}, \hat{c} > 1$ are independent of $w \in \partial\Omega$ and r. In section 3 we show, see Lemma 3.8, that there exists $\delta^* = \delta^*(p, n, N, M)$ such that if $0 < \delta < \delta^*$, $\Omega \in \mathcal{F}(\delta, N, r_0)$ and if Ω is an NTA-domain with constants M, r_0, then (1.5) holds.

Next we note that if $\Omega \in \mathcal{F}(\delta, N, r_0)$ and Ω is an NTA-domain with constants M, r_0, then there exists $\tilde{\delta} = \tilde{\delta}(n, N, M) > 0$ such that $\partial\Omega$ separates $B(w, r/2)$ in the sense that

$$\{x \in \Omega \cap B(w, r/2) : d(x, \partial\Omega) \geq 2\delta r\} \subset \text{ one component of } \mathbf{R}^n \setminus \Lambda, \tag{1.6}$$

whenever $w \in \partial\Omega, 0 < r < r_0$, and $0 < \delta < \tilde{\delta}$. Here Λ, corresponding to δ, N, r_0, w, r is defined as in Definition 1.4.

Definition 1.7. *Let $\Omega \subset \mathbf{R}^n$ be a bounded domain. Then Ω and $\partial\Omega$ are said to be (δ, r_0)-Reifenberg flat provided $\Omega \in \mathcal{F}(\delta, 0, r_0)$ and provided (1.6) holds whenever $0 < r < r_0$ and $w \in \partial\Omega$.*

For short we say that Ω is δ-Reifenberg flat whenever Ω is (δ, r_0)-Reifenberg flat for some $r_0 > 0$. We note that an equivalent definition of a Reifenberg flat domain is given in [KT]. As in [KT] one can show that a δ-Reifenberg flat domain is an NTA-domain with constant $M = M(n)$, provided $0 < \delta < \hat{\delta}$, and provided $\hat{\delta}$ is small enough.

We are now ready to state the first main result proved in this paper.

Theorem 1. *Let $\Omega \subset \mathbf{R}^n$ be a (δ, r_0)-Reifenberg flat domain. Given p, $1 < p < \infty$, there exists $\delta_1 = \delta_1(p, n) < \hat{\delta}$, such that the following is true. Let $w \in \partial\Omega$, $0 < r < r_0$, and $0 < \delta < \delta_1$. Suppose that u, v are positive p-harmonic functions in $\Omega \cap B(w, 4r)$, that u, v are continuous in $\bar{\Omega} \cap B(w, 4r)$ and $u = 0 = v$ on $\Delta(w, 4r)$. There exists $c_1 = c_1(p, n) \geq 1$ such that if $\tilde{r} = r/c_1$, then*

$$c_1^{-1} \frac{u(a_{\tilde{r}}(w))}{v(a_{\tilde{r}}(w))} \leq \frac{u(y)}{v(y)} \leq c_1 \frac{u(a_{\tilde{r}}(w))}{v(a_{\tilde{r}}(w))}$$

whenever $y \in \Omega \cap B(w, r/c_1)$. Moreover, there exist $c_2 = c_2(p, n) \geq c_1$ and $\alpha = \alpha(p, n)$, $\alpha \in (0, 1)$, such that

$$\left| \log \frac{u(y_1)}{v(y_1)} - \log \frac{u(y_2)}{v(y_2)} \right| \leq c_2 \left(\frac{|y_1 - y_2|}{r} \right)^{\alpha}$$

whenever $y_1, y_2 \in \Omega \cap B(w, r/c_2)$.

We note that the second display in Theorem 1 implies the first display (take $y_2 = a_{\tilde{r}}(w)$). In our proofs we first obtain the boundary Harnack inequality in the top display and then use it to prove Hölder continuity of the logarithm of the ratio in the bottom display. Thus for proofs, we prefer to separate the statements of the above inequalities. To state a corollary to Theorem 1 we need to introduce the notion of Reifenberg flat domains with vanishing constant.

Definition 1.8. *Let $\Omega \subset \mathbf{R}^n$ be a (δ, r_0)-Reifenberg flat domain for some $0 < \delta < \hat{\delta}, r_0 > 0$, and let $w \in \partial\Omega$, $0 < r < r_0$. We say that $\Delta(w, r)$ is Reifenberg flat with vanishing constant, if for each $\epsilon > 0$, there exists $\tilde{r} = \tilde{r}(\epsilon) > 0$ with the following property. If $x \in \Delta(w, r)$ and $0 < \rho < \tilde{r}$, then there exists a plane $P' = P'(x, \rho)$ containing x such that*

$$h(\partial\Omega \cap B(x, \rho), P' \cap B(x, \rho)) \leq \epsilon\rho.$$

The following corollary follows readily from Theorem 1.

Corollary 1.9. *Let $0 < \delta < \hat{\delta}$ and let $\Omega \subset \mathbf{R}^n$ be a (δ, r_0)-Reifenberg flat domain. Suppose also that $w \in \partial\Omega$, $0 < r < r_0$, and that $\Delta(w, 4r)$ is Reifenberg flat with vanishing constant. Assume for given $p, 1 < p < \infty$, that u, v are positive p-harmonic functions in $\Omega \cap B(w, 4r)$, u, v are continuous in $\bar{\Omega} \cap B(w, 4r)$ and $u = 0 = v$ on $\Delta(w, 4r)$. Then there exist $r^* = r^*(p, n) > 0$,*

$c = c(p,n) \geq 1$, *and* $\alpha = \alpha(p,n)$, $\alpha \in (0,1)$, *such that if* $w' \in \Delta(w,r)$ *and* $0 < r' < r^*$, *then*

$$c^{-1}\frac{u(a_{r'}(w'))}{v(a_{r'}(w'))} \leq \frac{u(y)}{v(y)} \leq c\frac{u(a_{r'}(w'))}{v(a_{r'}(w'))},$$

whenever $y \in \Omega \cap B(w',r')$. *Moreover,*

$$\left|\log\frac{u(y_1)}{v(y_1)} - \log\frac{u(y_2)}{v(y_2)}\right| \leq c\left(\frac{|y_1-y_2|}{r'}\right)^{\alpha}$$

whenever $y_1, y_2 \in \Omega \cap B(w',r')$.

Given a bounded domain Ω we shall in the following always let σ denote the $(n-1)$-dimensional Hausdorff measure on $\partial\Omega$.

Definition 1.10. *Let* $\Omega \subset \mathbf{R}^n$ *be a bounded domain. We say that* Ω *and* $\partial\Omega$ *are Ahlfors regular provided that there exist* $r_0 > 0, C \geq 1$, *such that*

$$C^{-1} \leq \frac{\sigma(\Delta(w,r))}{r^{n-1}} \leq C$$

whenever $w \in \partial\Omega$, $0 < r < r_0$.

We are now ready to state the second main result proved in this paper.

Theorem 2. *Let* $\Omega \subset \mathbf{R}^n$ *be an Ahlfors regular NTA-domain with constants* M, r_0, *which is uniformly* (δ, N, r_0)*-approximable by Lipschitz graph domains. Given* p, $1 < p < \infty$, *there exists* $\delta_2 = \delta_2(p,n,N,M,C)$ *such that if* $0 < \delta < \delta_2$, *then the following is true. Let* $w \in \partial\Omega$, $0 < r < r_0$, *suppose that* u, v *are positive* p*-harmonic functions in* $\Omega \cap B(w,4r)$, *that* u, v *are continuous in* $\bar{\Omega} \cap B(w,4r)$ *and* $u = 0 = v$ *on* $\Delta(w,4r)$. *There exists a constant* $c_1 = c_1(p,n,N,M,C)$, $1 \leq c_1 < \infty$, *such that if* $\tilde{r} = r/c_1$, *then*

$$c_1^{-1}\frac{u(a_{\tilde{r}}(w))}{v(a_{\tilde{r}}(w))} \leq \frac{u(y)}{v(y)} \leq c_1\frac{u(a_{\tilde{r}}(w))}{v(a_{\tilde{r}}(w))}$$

whenever $y \in \Omega \cap B(w, r/c_1)$. *Moreover, there exist* $c_2 = c_2(p,n,N,M,C) \geq c_1$, $1 \leq c_2 < \infty$, *and* $\alpha = \alpha(p,n,N,M,C), \alpha \in (0,1)$, *such that*

$$\left|\log\frac{u(y_1)}{v(y_1)} - \log\frac{u(y_2)}{v(y_2)}\right| \leq c_2\left(\frac{|y_1-y_2|}{r}\right)^{\alpha}$$

whenever $y_1, y_2 \in \Omega \cap B(w, r/c_2)$.

Next we describe the road map for the proof of Theorem 1 and Theorem 2. To begin, we let $\Omega \subset \mathbf{R}^n$, $w \in \partial\Omega$, $0 < r < r_0$, and we assume that Ω is either a δ-Reifenberg flat domain or an Ahlfors regular NTA-domain which is uniformly approximable by Lipschitz graph domains, satisfying the conditions in either Theorem 1 or Theorem 2.

Let $r^* = r/c$, where c is a large positive constant which will be chosen at the end of the proof of Theorem 1 and 2, so that the following statements are true. Suppose that u, v are positive p-harmonic functions in $\Omega \cap B(w,4r)$, u, v are continuous in $\bar\Omega \cap B(w,4r)$ and $u = 0 = v$ on $\Delta(w,4r)$. We extend u and v to $B(w,4r)$ by defining $u \equiv 0 \equiv v$ on $B(w,4r) \setminus \bar\Omega$. Then we assume, mainly for technical reasons, that

$$\begin{aligned}
&(a) \qquad 0 \leq u \leq v/2 \text{ in } \bar\Omega \cap \bar B(w,4r^*),\\
&(b) \qquad c^{-1} \leq u(a_{r^*}(w)), v(a_{r^*}(w)) \leq c,\\
&(c) \qquad c^{-1} h(a_{r^*}(w)) \leq \max_{B(w,4r^*)} h \leq c h(a_{r^*}(w)) \text{ whenever } h = u \text{ or } v.
\end{aligned} \tag{1.11}$$

In (1.11) $(b), (c)$, the constant c depends only on p, n. In the final part of our arguments, see the end of subsection 3.3, we show that the assumptions in (1.11) can be easily removed.

Our arguments are based on certain deformations of p-harmonic functions previously introduced in [LN1] in our study of p-harmonic functions in Lipschitz domains. In particular, we let $u(\cdot,\tau), 0 \leq \tau \leq 1$, be the p-harmonic function in $\Omega \cap B(w,4r^*)$ with continuous boundary values,

$$u(x,\tau) = \tau v(x) + (1-\tau) u(x) \text{ whenever } x \in \partial(\Omega \cap B(w,4r^*)) \text{ and } \tau \in [0,1]. \tag{1.12}$$

Using (1.11), (1.12), we see that if $t, \tau \in [0,1]$, then

$$\frac{u(\cdot,t) - u(\cdot,\tau)}{t-\tau} = v - u \leq c(p,n,M,N,C) \tag{1.13}$$

on $\partial(\Omega \cap B(w,4r^*))$. From the maximum principle for p-harmonic functions it then follows that the inequality in (1.13) also holds in $\Omega \cap B(w,4r^*)$. Therefore, using (1.13) we see that $\tau \to u(x,\tau), \tau \in [0,1]$, for fixed $x \in \Omega \cap B(w,r^*)$, is Lipschitz continuous with Lipschitz norm $\leq c$. Thus $u_\tau(x,\cdot)$ exists almost everywhere in [0,1]. Let $\{x_\nu\}$ be a dense sequence of $\Omega \cap B(w,4r^*)$ and let W be the set of all $\tau \in [0,1]$ for which $u_\tau(x_m,\cdot)$ exists, in the sense of difference quotients, whenever $x_m \in \{x_\nu\}$. We note that $H^1([0,1] \setminus W) = 0$ where H^1 is one-dimensional Hausdorff measure.

To proceed we make use of the 'fundamental inequality' stated in (1.5) and proved in Lemma 3.8 under the assumptions in either Theorem 1 or 2. Thus there exist constants $\hat c$ and $\bar\lambda$, which depend on the constants in Theorems 1 or 2, but are independent of τ, $\tau \in [0,1]$, such that if $x \in \Omega \cap B(w,r')$, $r' = r^*/\hat c$ and $\tau \in [0,1]$ then

$$\bar\lambda^{-1} \frac{u(x,\tau)}{d(x,\partial\Omega)} \leq |\nabla u(x,\tau)| \leq \bar\lambda \frac{u(x,\tau)}{d(x,\partial\Omega)}. \tag{1.14}$$

Define for $t, \tau \in [0,1]$ and $x \in \Omega \cap B(w,4r^*)$,

$$U(x) = U(x,t,\tau) = \frac{u(x,t) - u(x,\tau)}{t-\tau}. \tag{1.15}$$

From (1.11) and (1.13), we see that $U(\cdot) = U(\cdot,t,\tau)$ is a non-negative function in $\Omega \cap B(w,4r^*)$ with $U \equiv 0$ on $\partial\Omega \cap B(w,4r^*)$. Using (1.14) it follows as in [LN1, section 4] that if $x \in \Omega \cap B(w,r'/4)$ then U satisfies

$$\tilde L U(x) = \sum_{i,j=1}^n \frac{\partial}{\partial x_i}(A_{ij}(x)\, U_{x_j}) = 0 \tag{1.16}$$

where

$$A_{ij}(x) \;=\; A_{ij}(x,t,\tau) = \int_0^1 a_{ij}(\lambda \nabla u(x,t) + (1-\lambda)\nabla u(x,\tau))\, d\lambda,\ 1 \leq i,j \leq n, \quad (1.17)$$

whenever $x \in \Omega \cap B(w, r'/4)$. In (1.17)

$$a_{ij}(\eta) \;=\; |\eta|^{p-4}\, [(p-2)\eta_i\, \eta_j \,+\, \delta_{ij}\, |\eta|^2] \text{ for } \eta \in \mathbf{R}^n \setminus \{0\} \quad (1.18)$$

where δ_{ij} is the Kronecker δ. Moreover, as in [LN1, section 4], we note that if $x \in \Omega \cap B(w, r'/4)$ then

$$\begin{aligned} &(i) \qquad c^{-1}|\xi|^2\Big(|\nabla u(x,t)| + |\nabla u(x,\tau)|\Big)^{p-2} \;\leq\; \sum_{i,j=1}^{n} A_{ij}(x)\xi_i\xi_j \text{ for } \xi \in \mathbf{R}^n \setminus \{0\},\\ &(ii) \qquad \sum_{i,j=1}^{n} |A_{ij}(x)| \leq c\Big(|\nabla u(x,t)| + |\nabla u(x,\tau)|\Big)^{p-2}. \end{aligned} \quad (1.19)$$

Extend U continuously to $B(w, 4r^*)$ by putting $U \equiv 0$ on $B(w, 4r^*) \setminus \Omega$. Since $u(x,t) - u(x,\tau) = (t-\tau)(v(x) - u(x))$ on $\partial(\Omega \cap B(w, 4r^*))$ it follows from the maximum principle for p-harmonic functions that $u(x,t) \rightarrow u(x,\tau)$ uniformly in the closure of $\Omega \cap B(x, r'/4)$ as $t \rightarrow \tau$. Moreover, we establish, under the assumptions of Theorem 1 and Theorem 2, the following claim.

Claim 1. *Assume that (1.11) holds. Then U is a bounded weak solution to a locally uniformly elliptic PDE in divergence form in $\Omega \cap B(w, r'/8)$. All constants can be chosen independent of t, τ. Moreover $U \equiv 0$ on $B(w, r'/8) \setminus \Omega$ and if $\tau, t \in (0,1], \tau < t$, then there exists a constant c, independent of t, τ, such that if $r'' = r'/32$, then $U(\cdot, t, \tau)$ is Hölder continuous in $B(w, 4r'')$, with Hölder exponent depending only on the constants in either Theorem 1 or 2 while the Hölder norm depends on these constants as well as τ.*

Assuming Claim 1 we consider limits as $t \rightarrow \tau$. Using Claim 1 and elliptic PDE theory it follows that there exists a subsequence of $\{U(\cdot, t, \tau)\}$, $\{U(\cdot, t_k, \tau)\}$, with $t_k \rightarrow \tau$ as $k \rightarrow \infty$, which converges uniformly, in $B(w, 2r'')$, to a function $f = f(\cdot, \tau)$. Moreover, $f \in C^{0,\alpha}(\bar{B}(w, 2r''))$ for some $\alpha = \alpha(p, n, N, M, C) \in (0,1)$ and f is locally a weak solution, in $\Omega \cap B(w, 2r'')$, to the equation

$$L\zeta \;=\; \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\,(\, b_{ij}(x,\tau)\zeta_{x_j}(x)\,) = 0. \quad (1.20)$$

In (1.20),

$$b_{ij}(x,\tau) = |\nabla u(x,\tau)|^{p-4}[(p-2)u_{x_i}(x,\tau)u_{x_j}(x,\tau) + \delta_{ij}|\nabla u(x,\tau)|^2], \quad (1.21)$$

whenever $x \in \Omega \cap B(w, 2r'')$ and $1 \leq i,j \leq n$. It is also easy to check that $\zeta = u(\cdot, \tau)$ is a weak solution to L in $\Omega \cap B(w, 2r'')$ (see [LN1, display (1.6)]). Moreover, the function f has the following important properties,

$$\begin{aligned} &(i) \qquad f \text{ is continuous in } \bar{B}(w, 2r'') \text{ with } f \equiv 0 \text{ on } \bar{B}(w, 2r'') \setminus \Omega,\\ &(ii) \qquad f(x_m, \tau) = u_\tau(x_m, \tau) \text{ whenever } x_m \in (x_\nu),\ x_m \in \Omega \cap B(w, 2r''), \tau \in W. \end{aligned} \quad (1.22)$$

Let $x_m \in \{x_\nu\}$, $x_m \in \Omega \cap B(w, 2r'')$. Using (1.22), the fundamental theorem of calculus and the fact that $H^1([0,1] \setminus W) = 0$ we see that

$$\log\left(\frac{v(x_m)}{u(x_m)}\right) = \log\left(\frac{u(x_m, 1)}{u(x_m, 0)}\right) = \int_0^1 \frac{f(x_m, \tau)}{u(x_m, \tau)}\, d\tau. \tag{1.23}$$

As $\{x_\nu\}$ is a dense sequence in $\Omega \cap B(w, 2r'')$ we can therefore conclude that the proofs of Theorems 1 and 2 are reduced to proving the corresponding results for the functions $v_1(\cdot) = f(\cdot, \tau)$, $v_2(\cdot) = u(\cdot, \tau)$ with constants independent of τ, whenever $\tau \in (0, 1]$. In particular, assuming Claim 1, we see that v_1 as well as v_2 are non-negative solutions to the operator L, see (1.20)-(1.21), vanishing continuously on $\Delta(w, 2r'')$. We establish the following claim under the assumptions of Theorem 1 and Theorem 2.

Claim 2. *Let $\tau \in (0, 1]$ be fixed and let L be the operator defined in (1.20)-(1.21). Let v_1 and v_2 be non-negative solutions to the operator L in $\Omega \cap B(w, 2r'')$, vanishing continuously on $\Delta(w, 2r'')$. Then there exist a constant $c = c(p, n, N, M, C)$, $1 \leq c < \infty$, and $\alpha = \alpha(p, n, N, M, C)$, $\alpha \in (0, 1)$, such that if $r''' = r''/c$, then*

$$c^{-1} \frac{v_1(a_{r'''}(w))}{v_2(a_{r'''}(w))} \leq \frac{v_1(y)}{v_2(y)} \leq c \frac{v_1(a_{r'''}(w))}{v_2(a_{r'''}(w))},$$

$$\left| \log \frac{v_1(y_1)}{v_2(y_1)} - \log \frac{v_1(y_2)}{v_2(y_2)} \right| \leq c \left(\frac{|y_1 - y_2|}{r''} \right)^\alpha,$$

whenever $y \in \Omega \cap B(w, r''')$ and $y_1, y_2 \in \Omega \cap B(w, r''')$.

We note, as is discussed below, that the proof of Claim 2 in the case of Theorem 1 and Theorem 2 are considerably different. Moreover, assuming (1.11), Claim 1, and Claim 2, we see that Theorem 1 and Theorem 2 now follow from (1.23) as $r'' = r/(32\hat{c}c)$ and as

$$f(a_{r'''}(w), \tau) \leq c, \ u(a_{r'''}(w), \tau) \geq c^{-1}, \text{ whenever } \tau \in (0, 1]. \tag{1.24}$$

Moreover, (1.24) follows from (1.11) and the NTA-character of Ω.

As applications of Theorems 1 and 2 we consider the Martin boundary problem for p-harmonic functions and we refer to section 5 for definitions and discussion of this problem. In particular we prove the following two theorems.

Theorem 3. *Let Ω, δ, p, r_0 be as in Theorem 1. There exists $\delta_+ = \delta_+(p, n) > 0$ such that if $0 < \delta < \delta_+ < \hat{\delta}$, then the p-Martin boundary of Ω can be identified with $\partial\Omega$.*

Theorem 4. *Let $\Omega, \delta, p, r_0, N, M, C$, be as in Theorem 2. If $w \in \partial\Omega$, and $\partial\Omega$ has a tangent plane at w, then a minimal positive p-harmonic function corresponding to w is unique up to constant multiples.*

We note that if $\partial\Omega$ is as in Theorem 4 then $\partial\Omega$ has tangent planes almost everywhere with respect to the $(n-1)$-dimensional Hausdorff measure, σ, on $\partial\Omega$. To see this one first observes that σ almost every point in $\partial\Omega$ lies in the reduced or measure theoretic boundary in the sense of geometric measure theory (see [EG]). Now there must be a tangent plane to Ω at each point in these sets as follows easily from the fact that Ω is an NTA-domain. We also note that given Theorem 1 and 2 we can reuse the arguments in [LN1] to get Theorems 3 and 4.

The rest of this paper is organized as follows. In section 2 we state a number of basic estimates for p-harmonic functions in NTA-domains as well as some of the results from [LN, LN1, LN3, LN2] concerning the boundary behaviour of p-harmonic functions in bounded Lipschitz domains. In section 3 we prove (1.5) and consider p-harmonic functions in Reifenberg flat NTA-domains. In particular, we establish Claim 1, Claim 2 and we remove assumption (1.11) to finally get Theorem 1. In section 4 we then establish Claim 1 and 2 under the assumptions of Theorem 2 and prove Theorem 2. Finally, in section 5 we first introduce the Martin boundary problem and minimal positive p-harmonic functions. We then indicate how Theorem 3 and 4 follow from Theorem 1 and 2, and the arguments in [LN1]. In this section we also make concluding remarks.

As noted above, there are major technological differences between the arguments in section 3 and section 4. On the one hand, in the setting of δ-Reifenberg flat domains, section 3, we are able, once we have established the uniform non-degeneracy of $|\nabla u(\cdot,\tau)|$, to make use of the results of [FKS], [FJK] and [FJK1] on degenerate elliptic equation to prove Claim 1 and Claim 2. In this argument we need to be able to extend $|\nabla u(\cdot,\tau)|^{p-2}$ to the set $B(w,r'/4)\setminus\Omega$ in such a way that the extended function is an A_2-weight in $B(w,r'/c)$ with constant independent of τ. On the other hand if $\Omega\in\mathcal{F}(\delta,N,r_0)$ and Ω is an Ahlfors regular NTA-domain, as in section 4, we then need to make a more involved analysis of the elliptic measure associated to the operator L defined in (1.20). To carry out our program we in this case use the fact, proved in [DJ], that if Ω is an Ahlfors regular NTA-domain then $\partial\Omega$ contains big pieces of Lipschitz graphs contained in $\bar\Omega$, as well as modifications of arguments from [LN1, LN2]. Indeed, one of our main contributions in this paper is that that we are able to carry out the analysis in sections 2 and 3 of [LN] in the above settings.

We emphasize that this paper is not self-contained and that it relies heavily on work in [LN, LN1, LN2]. Thus the reader is advised to have these papers at hand[1].

Finally, we also note that applications of the results proved in this paper to problems concerning boundary and free boundary regularity for the p-Laplacian, $1<p<\infty$, in Reifenberg flat chord arc and Ahlfors regular NTA-domains will appear elsewhere.

2 Estimates of p-harmonic functions in Lipschitz and NTA-domains

By definition, $\Omega\subset\mathbf{R}^n$ is a bounded Lipschitz domain, if there exists a finite set of balls $\{B(x_i,r_i)\}$, with $x_i\in\partial\Omega$ and $r_i>0$, such that $\{B(x_i,r_i)\}$ constitutes a covering of an open neighbourhood of $\partial\Omega$ and such that, for each i,

$$\begin{aligned}\Omega\cap B(x_i,r_i) &= \{x=(x',x_n)\in\mathbf{R}^n : x_n>\phi_i(x')\}\cap B(x_i,r_i),\\ \partial\Omega\cap B(x_i,r_i) &= \{x=(x',x_n)\in\mathbf{R}^n : x_n=\phi_i(x')\}\cap B(x_i,r_i),\end{aligned}\tag{2.1}$$

[1]For preprints we refer to www.ms.uky.edu/∼john and www.math.umu.se/personal/nystrom_kaj.

in an appropriate coordinate system and for a Lipschitz function $\phi_i : \mathbf{R}^{n-1} \to \mathbf{R}$. The Lipschitz constant of Ω is defined to be $N = \max_i \| |\nabla \phi_i| \|_\infty$. Recall that the notion of NTA-domains was introduced in Definition 1.3. Moreover, if Ω is Lipschitz, then Ω is also NTA and if $\Omega \subset \mathbf{R}^n$ is a bounded NTA-domain, then M, as in Definition 1.3, is referred to as the NTA-constant of Ω. If Ω is Ahlfors regular, then C in Definition 1.10 is the Ahlfors regular constant of Ω. Throughout this paper, unless otherwise stated, c will denote a positive constant ≥ 1, not necessarily the same at each occurrence, depending only on p, n, M, N, C. In general, $c(a_1, \dots, a_m)$ denotes a positive constant ≥ 1, which may depend only on p, n, N, M, C, and $a_1, \dots, a_m$, not necessarily the same at each occurrence. If $A \approx B$ then A/B is bounded from above and below by constants which, unless otherwise stated, only depend on p, n, M and, if appropriate, N. Moreover, we let $\max_{B(z,s)} u$, $\min_{B(z,s)} u$ be the essential supremum and infimum of u on $B(z, s)$ whenever $B(z, s) \subset \mathbf{R}^n$ and whenever u is defined on $B(z, s)$. As in section 1, we put $\Delta(w, r) = \partial\Omega \cap B(w, r)$ whenever $w \in \partial\Omega$, $0 < r$, and we let σ denote the $(n-1)$-dimensional Hausdorff measure on $\partial\Omega$. Finally, $e_i, 1 \leq i \leq n$, denotes the point in $\mathbf{R}^n$ with one in the i th coordinate position and zeroes elsewhere.

2.1 Basic estimates for p-harmonic functions in NTA-domains

For proofs and for references to the proofs of Lemmas 2.2-2.6 stated below we refer to [LN].

Lemma 2.2. *Given $p, 1 < p < \infty$, let u be a positive p-harmonic function in $B(w, 2r)$. Then*

$$(i) \qquad r^{p-n} \int\limits_{B(w,r/2)} |\nabla u|^p \, dx \;\leq\; c\,(\max_{B(w,r)} u)^p,$$

$$(ii) \qquad \max_{B(w,r)} u \leq c \min_{B(w,r)} u.$$

Furthermore, there exists $\alpha = \alpha(p, n) \in (0, 1)$ such that if $x, y \in B(w, r)$, then

$$(iii) \qquad |u(x) - u(y)| \leq c \left(\frac{|x-y|}{r} \right)^\alpha \max_{B(w,2r)} u.$$

Lemma 2.3. *Let $\Omega \subset \mathbf{R}^n$ be a bounded NTA-domain and suppose that p, $1 < p < \infty$, is given. Let $w \in \partial\Omega$, $0 < r < r_0$, and suppose that u is a non-negative continuous p-harmonic function in $\bar\Omega \cap B(w, 2r)$ and that $u = 0$ on $\Delta(w, 2r)$. Then*

$$(i) \qquad r^{p-n} \int\limits_{\Omega \cap B(w,r/2)} |\nabla u|^p \, dx \;\leq\; c\,(\max_{\Omega \cap B(w,r)} u)^p.$$

Furthermore, there exists $\alpha = \alpha(p, n, M) \in (0, 1)$ such that if $x, y \in \Omega \cap B(w, r)$, then

$$(ii) \qquad |u(x) - u(y)| \leq c \left(\frac{|x-y|}{r} \right)^\alpha \max_{\Omega \cap B(w,2r)} u.$$

Lemma 2.4. *Let $\Omega \subset \mathbf{R}^n$ be a bounded NTA-domain and suppose that p, $1 < p < \infty$, is given. Let $w \in \partial\Omega$, $0 < r < r_0$, and suppose that u is a non-negative continuous p-harmonic function* in $\bar{\Omega} \cap B(w, 2r)$ and that $u = 0$ on $\Delta(w, 2r)$. There exists $c = c(p, n, M)$, $1 \leq c < \infty$, such that if $\tilde{r} = r/c$, then

$$\max_{\Omega \cap B(w,\tilde{r})} u \,\leq\, c\, u(a_{\tilde{r}}(w)).$$

Lemma 2.5. *Let $\Omega \subset \mathbf{R}^n$ be a bounded NTA-domain and suppose that p, $1 < p < \infty$, is given. Let $w \in \partial\Omega$, $0 < r < r_0$, and suppose that u is a non-negative continuous p-harmonic function in $\bar{\Omega} \cap B(w, 4r)$ and that $u = 0$ on $\Delta(w, 4r)$. Extend u to $B(w, 4r)$ by defining $u \equiv 0$ on $B(w, 4r) \setminus \Omega$. Then u has a representative in $W^{1,p}(B(w, 4r))$ with Hölder continuous partial derivatives in $\Omega \cap B(w, 4r)$. In particular, there exists $\sigma \in (0, 1]$, depending only on p, n, such that if $x, y \in B(\hat{w}, \hat{r}/2)$, $B(\hat{w}, 4\hat{r}) \subset \Omega \cap B(w, 4r)$, then*

$$c^{-1}\, |\nabla u(x) - \nabla u(y)| \,\leq\, (|x - y|/\hat{r})^{\sigma} \max_{B(\hat{w},\hat{r})} |\nabla u| \,\leq\, c\,\hat{r}^{-1}\, (|x - y|/\hat{r})^{\sigma} \max_{B(\hat{w},2\hat{r})} u.$$

Lemma 2.6. *Let $\Omega \subset \mathbf{R}^n$ be a bounded NTA-domain. Given $p, 1 < p < \infty, w \in \partial\Omega, 0 < r < r_0$, suppose that u is a positive p-harmonic function in $\Omega \cap B(w, 2r)$, continuous in $\bar{\Omega} \cap B(w, 2r)$ with $u = 0$ on $\Delta(w, 2r)$. Extend u to $B(w, 2r)$ by defining $u \equiv 0$ on $B(w, 2r) \setminus \Omega$. There exists a unique finite positive Borel measure μ on $\mathbf{R}^n$, with support in $\Delta(w, 2r)$, such that whenever $\phi \in C_0^\infty(B(w, 2r))$, then*

$$(i) \qquad \int |\nabla u|^{p-2} \,\langle \nabla u,\, \nabla\phi\rangle \, dx \;=\; -\int \phi \, d\mu.$$

Moreover, there exists $c = c(p, n, M)$, $1 \leq c < \infty$, such that if $\tilde{r} = r/c$, then

$$(ii) \qquad c^{-1} r^{p-n} \mu(\Delta(w, \tilde{r})) \leq (u(a_{\tilde{r}}(w)))^{p-1} \leq c\, r^{p-n}\, \mu(\Delta(w, \tilde{r}/2)).$$

2.2 Estimates for p-harmonic functions in Lipschitz domains

In the following we state a number of results and estimates proved in [LN], [LN1] and [LN2]. In particular, for the proof of Theorems 2.7-2.9 stated below we refer to [LN] and [LN1] and we note that Theorem 2.9 is referred to as Lemma 4.28 in [LN1] while Theorem 2.7 and Theorem 2.8 are two of the main results established in [LN] and [LN1] respectively.

Theorem 2.7. *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain with constant N. Given $p, 1 < p < \infty, w \in \partial\Omega$, $0 < r < r_0$, suppose that u and v are positive p-harmonic functions in $\Omega \cap B(w, 2r)$. Assume also that u and v are continuous in $\bar{\Omega} \cap B(w, 2r)$ and $u = 0 = v$ on $\Delta(w, 2r)$. Then there exists $c_1, 1 \leq c_1 < \infty$, depending only on p, n and N, such that if $\tilde{r} = r/c_1$, $u(a_{\tilde{r}}(w)) = v(a_{\tilde{r}}(w))$ and $y \in \Omega \cap B(w, r/c_1)$, then*

$$\frac{u(y)}{v(y)} \leq c_1.$$

Theorem 2.8. *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain with constant N. Given $p, 1 < p < \infty, w \in \partial\Omega$, $0 < r < r_0$, suppose that u and v are positive p-harmonic functions in $\Omega \cap B(w, 2r)$. Assume also that u and v are continuous in $\bar{\Omega} \cap B(w, 2r)$ and $u = 0 = v$ on $\Delta(w, 2r)$. Then* *there exist $c_2, 1 \leq c_2 < \infty$, and $\alpha \in (0,1)$, both depending only on p, n and N such that if $y_1, y_2 \in \Omega \cap B(w, r/c_2)$, then*

$$\left| \log\left(\frac{u(y_1)}{v(y_1)}\right) - \log\left(\frac{u(y_2)}{v(y_2)}\right) \right| \leq c_2 \left(\frac{|y_1 - y_2|}{r}\right)^{\alpha}.$$

Theorem 2.9. *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain with constant N. Given $p, 1 < p < \infty, w \in \partial\Omega$, $0 < r < r_0$, suppose that u is a positive p-harmonic function in $\Omega \cap B(w, 2r)$. Assume also that u is continuous in $\bar{\Omega} \cap B(w, 2r)$ and $u = 0$ on $\Delta(w, 2r)$. Then there exist $c_3, 1 \leq c_3 < \infty$, and $\hat{\lambda} > 0$, depending only on p, n and N, such that if $y \in \Omega \cap B(w, r/c_3)$, then*

$$\hat{\lambda}^{-1} \frac{u(y)}{d(y, \partial\Omega)} \leq |\nabla u(y)| \leq \hat{\lambda} \frac{u(y)}{d(y, \partial\Omega)}.$$

If $\Omega \subset \mathbf{R}^n$ is a bounded Lipschitz domain and if $w \in \partial\Omega, 0 < r < r_0$, and $0 < b < 1$, then we let, whenever $x \in \partial\Omega$,

$$\Gamma(x) = \Gamma_b(x) = \{y \in \Omega : \ d(y, \partial\Omega) > b|x - y|\} \cap B(w, 4r). \tag{2.10}$$

Given a measurable function k on $\cup_{x \in \Delta(w,2r)} \Gamma(x)$ we define the non-tangential maximal function $N(k) : \Delta(w, 2r) \to \mathbf{R}$ for k as

$$N(k)(x) = \sup_{y \in \Gamma(x)} |k|(y) \text{ whenever } x \in \Delta(w, 2r). \tag{2.11}$$

We let $L^q(\Delta(w, 2r)), 1 \leq q \leq \infty$, be the space of functions which are integrable, with respect to the surface measure, σ, to the power q on $\Delta(w, 2r)$. Furthermore, given a measurable function f on $\Delta(w, 2r)$ we say that f is of bounded mean oscillation on $\Delta(w, r)$, $f \in BMO(\Delta(w, r))$, if there exists a constant A, $0 < A < \infty$, such that whenever $x \in \Delta(w, r)$ and $0 < s < r$,

$$\int_{\Delta(x,s)} |f - f_{\Delta(x,s)}| d\sigma \ \leq \ A\sigma(\Delta(x, s)), \tag{2.12}$$

where $f_{\Delta(x,s)}$ denotes the average of f on $\Delta(x, s)$ with respect to σ. The least A for which (2.12) holds is denoted by $\|f\|_{BMO(\Delta(w,r))}$.

The following theorem is essentially Theorem 1 in [LN2].

Theorem 2.13. *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain with constant N. Given $p, 1 < p < \infty, w \in \partial\Omega$, $0 < r < r_0$, suppose that u is a positive p-harmonic function in $\Omega \cap B(w, 4r)$, u is continuous in $\bar{\Omega} \cap B(w, 4r)$ and $u = 0$ on $\Delta(w, 4r)$. Then*

$$\lim_{y \in \Gamma(x), y \to x} \nabla u(y) = \nabla u(x)$$

for σ almost every $x \in \Delta(w, 4r)$. Furthermore, there exist $q > p$ and a constant c, $1 \leq c < \infty$, which both only depend on p, n and N, such that

$$\begin{aligned}
&(i) && N(|\nabla u|) \in L^q(\Delta(w, 2r)),\\
&(ii) && \int_{\Delta(w,2r)} |\nabla u|^q d\sigma \leq c r^{(n-1)(\frac{p-1-q}{p-1})} \Bigg(\int_{\Delta(w,2r)} |\nabla u|^{p-1} d\sigma \Bigg)^{q/(p-1)},\\
&(iii) && \log|\nabla u| \in BMO(\Delta(w, r)), \quad \|\log|\nabla u|\|_{BMO(\Delta(w,r))} \leq c,\\
&(iv) && d\mu = |\nabla u|^{p-1} d\sigma \ \textit{on} \ \partial\Omega.
\end{aligned}$$

We note that (iv) in Theorem 2.13 was not stated explicitly in Theorem 1 in [LN2] since we had not introduced μ at that point. In fact (iv) can be found in [LN2, display (3.7)].

3 Estimates for p-harmonic functions in Reifenberg flat domains

In this section we first consider p-harmonic functions in an NTA-domain Ω assuming that $\Omega \in \mathcal{F}(\delta, N, r_0)$, where $\mathcal{F}(\delta, N, r_0)$ was defined in Definition 1.4. In particular, we prove Lemma 3.8 referred to in section 1. We then consider (δ, r_0)-Reifenberg flat domains and proceed towards the proof of Theorem 1. Theorem 1 is then proved at the end of the section.

3.1 Non-degeneracy of $|\nabla u|$

The following lemma is similar to Lemma 4.3 and 5.4 in [LN1] and will play a key role in the proof of the 'fundamental inequality' in (1.5) mentioned throughout section 1.

Lemma 3.1. *Let O be an open set, $w \in \partial O, r > 0$, and suppose that $\hat u, \hat v$ are positive p-harmonic functions in $O \cap B(w, 4r)$. Let $a \geq 1, x \in O$, and assume that*

$$\frac{1}{a}\frac{\hat v(x)}{d(x, \partial O)} \leq |\nabla \hat v(x)| \leq a\frac{\hat v(x)}{d(x, \partial O)}.$$

Let $\tilde\epsilon^{-1} = (ca)^{(1+\sigma)/\sigma}$, where σ is as in Lemma 2.5 and $c = c(p, n)$. Then the following statement is true for $c = c(p, n)$ suitably large. If

$$(1 - \tilde\epsilon)\hat L \leq \frac{\hat v}{\hat u} \leq (1 + \tilde\epsilon)\hat L$$

in $B(x, \frac{1}{4}d(x, \partial O))$ for some $\hat L, 0 < \hat L < \infty$, then

$$\frac{1}{ca}\frac{\hat u(x)}{d(x, \partial O)} \leq |\nabla \hat u(x)| \leq ca\frac{\hat u(x)}{d(x, \partial O)}.$$

Proof: Let $a \geq 1$ and $x \in O$ be as in the statement of the lemma. Using Lemma 2.5 and the Harnack inequality in Lemma 2.2 (ii) we see that,

$$|\nabla \hat u(z_1) - \nabla \hat u(z_2)| \leq ct^\sigma \max_{B(x, td(x,\partial O))} |\nabla \hat u(\cdot)| \leq c^2 t^\sigma \,\hat u(x)/d(x, \partial O) \tag{3.2}$$

whenever $z_1, z_2 \in \bar B(x, td(x, \partial O))$ and $0 < t \leq 1/100$. Here c depends only on p, n. Using (3.2) we see that we only have to prove bounds from below for the gradient of $\hat u$ at x. To achieve this we suppose that,

$$|\nabla \hat u(x)| \leq \zeta\, \hat u(x)/d(x, \partial O), \tag{3.3}$$

for some small $\zeta > 0$ to be chosen. From (3.2) with $z = z_1, x = z_2$ and (3.3) we then deduce that

$$|\nabla \hat u(z)| \leq [\zeta + c^2 t^\sigma]\, \hat u(x)/d(x, \partial O) \tag{3.4}$$

whenever $z \in B(x, td(x, \partial O))$. Integrating, it follows that if $y \in \partial B(x, td(x, \partial O))$, $|x-y| = td(x, \partial O)$, $t = \zeta^{1/\sigma}$, then

$$|\hat{u}(y) - \hat{u}(x)| \leq c' \zeta^{1+1/\sigma}\, \hat{u}(x). \tag{3.5}$$

The constants in (3.4),(3.5) depend only on p, n.

Next we note that (3.2) also holds with $\hat{u}$ replaced by $\hat{v}$. Let $\lambda = \frac{\nabla \hat{v}(x)}{|\nabla \hat{v}(x)|}$. Then from (3.2) for $\hat{v}$ and the non-degeneracy assumption on $|\nabla \hat{v}|$ in Lemma 3.1, we find that

$$\langle \nabla \hat{v}(z), \lambda \rangle \geq (1 - c\, a\zeta)|\nabla \hat{v}(x)| \text{ whenever } \bar{B}(x, \zeta^{1/\sigma} d(x, \partial O)),$$

for some $c = c(p, n)$. If $\zeta \leq (2ca)^{-1}$, where c is the constant in the last display, then we get from integration that

$$c^*(\hat{v}(y) - \hat{v}(x)) \geq a^{-1} \zeta^{1/\sigma} \hat{v}(x) \tag{3.6}$$

with $y = x + \zeta^{1/\sigma} d(x, \partial O)\lambda$ and where the constant c^* depends only on p, n. From (3.6), (3.5), we see that if $\tilde{\epsilon}$ is as in Lemma 3.1, then

$$\begin{aligned}(1-\tilde{\epsilon})\hat{L} \;&\leq\; \frac{\hat{u}(y)}{\hat{v}(y)} \leq \left(\frac{1 + c'\zeta^{1+1/\sigma}}{1 + \zeta^{1/\sigma}/(ac^*)}\right)\frac{\hat{u}(x)}{\hat{v}(x)} \\ &\leq\; (1+\tilde{\epsilon})\left(\frac{1 + c'\zeta^{1+1/\sigma}}{1 + \zeta^{1/\sigma}/(ac^*)}\right)\hat{L} < (1-\tilde{\epsilon})\hat{L}\end{aligned} \tag{3.7}$$

provided $1/(a\tilde{c})^{1/\sigma} \geq \zeta^{1/\sigma} \geq a\tilde{c}\,\tilde{\epsilon}$ for some large $\tilde{c} = \tilde{c}(p, n)$. This inequality and (3.6) are satisfied if $\tilde{\epsilon}^{-1} = (\tilde{c}a)^{(1+\sigma)/\sigma}$ and $\zeta^{-1} = \tilde{c}a$. Moreover, if the hypotheses of Lemma 3.1 hold for this $\tilde{\epsilon}$, then in order to avoid the contradiction in (3.7) it must be true that (3.3) is false for this choice of ζ. Hence Lemma 3.1 is true. $\Box$

Next we prove Lemma 3.8 referred to in section 1.

Lemma 3.8. *Let $\Omega \subset \mathbf{R}^n$ be a NTA-domain with constants M, r_0 and assume that $\Omega \in \mathcal{F}(\delta, N, r_0)$. Given $p, 1 < p < \infty$, $w \in \partial\Omega$, $0 < r < r_0$, suppose that u is a positive p-harmonic function in $\Omega \cap B(w, 4r)$, that u is continuous in $\bar{\Omega} \cap B(w, 4r)$, and that $u = 0$ on $\Delta(w, 4r)$. Then there exist $\delta^* = \delta^*(p, n, N, M)$, $\hat{c} = \hat{c}(p, n, N, M)$ and $\bar{\lambda} = \bar{\lambda}(p, n, N, M)$, such that if $0 < \delta \leq \delta^*$, then*

$$\bar{\lambda}^{-1}\frac{u(y)}{d(y, \partial\Omega)} \leq |\nabla u(y)| \leq \bar{\lambda}\frac{u(y)}{d(y, \partial\Omega)} \textit{ whenever } y \in \Omega \cap B(w, r/\hat{c}).$$

Proof: Let $u \equiv 0$ in $B(w, 4r) \setminus \Omega$. Then from the assumptions in Lemma 3.8 it is easily seen that $u \in W^{1,p}(B(w, 4r))$ and that u is continuous in $B(w, 4r)$. Let c_3 be as in Theorem 2.9 and choose $c' \geq 100c_3$ so that if $x \in \Omega \cap B(w, r/c'), s = 4c_3 d(x, \partial\Omega)$, and $z \in \partial\Omega$ with $|x - z| = d(x, \partial\Omega)$, then

$$\max_{B(z, 4s)} u \;\leq\; cu(x) \tag{3.9}$$

for some $c = c(p, n, M)$. This is possible since Ω is an NTA-domain with constant M. Using Definition 1.4 with w, r replaced by $z, 4s$, we see that there exists a Lipschitz graph Λ such that

$$h(\partial\Omega \cap B(z, 4s), \Lambda \cap B(z, 4s)) \leq 4\delta s. \tag{3.10}$$

For the moment we allow δ^* in Lemma 3.8 to vary but shall later fix it as a number satisfying several conditions. As in (1.6), we see that (3.10) implies

$$\{y \in \Omega \cap B(z, 2s) : d(y, \partial\Omega) \geq 8\delta s\} \subset \text{ one component of } \mathbf{R}^n \setminus \Lambda$$

whenever $0 < \delta \leq \delta^*$ and $\delta^* = \delta^*(p, n, N, M)$ is sufficiency small. Since the p-Laplacian is invariant under rotations, we may assume that $\Lambda = \{(y', \phi(y')) : y' \in \mathbf{R}^{n-1}, \phi(z') = z_n\}$ where ϕ is Lipschitz with $\||\nabla\phi|\|_\infty \leq N$, $z = (z', z_n)$ and

$$\{y \in \Omega \cap B(z, s) : d(y, \partial\Omega) \geq 8\delta s\} \subset \{y \in \mathbf{R}^n : y_n > \phi(y')\}. \tag{3.11}$$

From (3.11) and Lipschitzness of ϕ we find that if we define

$$\begin{aligned} \Lambda' &= \{(y', \phi(y')) + 2(N + 10)\delta s e_n,\ y' \in \mathbf{R}^{n-1}\}, \\ \Omega' &= \{y \in \mathbf{R}^n : y_n > \phi(y') + 2(N + 10)\delta s\}, \end{aligned}$$

then

$$\Omega' \cap \bar{B}(z, 2s) \subset \Omega \cap B(z, 2s). \tag{3.12}$$

Let v be a p-harmonic function in $\Omega' \cap B(z, 2s)$ with continuous boundary values on $\partial(\Omega' \cap B(z, 2s))$ and such that $v \leq u$ on $\partial(\Omega' \cap B(z, 2s))$. Moreover, we choose v so that

$$\begin{aligned} v(y) &= u(y) \text{ whenever } y \in \partial[\Omega' \cap B(z, 2s)] \text{ and } y_n > \phi(y') + 4(N + 10)\delta s, \\ v(y) &= 0 \quad\ \ \text{whenever } y \in \partial[\Omega' \cap B(z, 2s)] \text{ and } y_n < \phi(y') + 3(N + 10)\delta s. \end{aligned}$$

Existence of v follows easily from well known facts about continuous p-harmonic functions. By construction and the maximum principle for p-harmonic functions we have $v \leq u$ in $\Omega' \cap B(z, 2s)$. Also, since each point of $\partial[\Omega' \cap B(z, 2s)]$ where $u \neq v$ lies within $8(N + 10)\delta s$ of a point where u is zero, it follows from (3.9) and Lemma 2.3 that $u \leq v + c\delta^\alpha u(x)$ on $\partial[\Omega' \cap B(z, 2s)]$. In particular, again using the maximum principle for p-harmonic functions we conclude that

$$v \leq u \leq v + c\delta^\alpha u(x) \text{ in } \Omega' \cap B(z, 2s).$$

Using the last inequality and Harnack's inequality, we see that

$$1 \leq \frac{u(y)}{v(y)} \leq (1 - c\delta^\alpha)^{-1} \text{ whenever } y \in B(x, d(x, \partial\Omega')/4) \tag{3.13}$$

and δ is small enough. Using Theorem 2.9 and our construction, we also have

$$\hat{\lambda}^{-1} \frac{v(x)}{d(x, \partial\Omega')} \leq |\nabla v(x)| \leq \hat{\lambda} \frac{v(x)}{d(x, \partial\Omega')}. \tag{3.14}$$

for some $\hat{\lambda} = \hat{\lambda}(p, n, N)$. In particular, from (3.13), (3.14) we see for $0 < \delta < \delta^*$, and $\delta^* = \delta^*(p, n, N, M)$ suitably small, that the hypotheses of Lemma 3.1 are satisfied with $O = \Omega'$ and $a = \hat{\lambda}$. We now fix δ^* and conclude from Lemma 3.1, $d(x, \partial\Omega) \approx d(x, \partial\Omega')$, that

$$\bar{\lambda}^{-1} \frac{u(x)}{d(x, \partial\Omega)} \leq |\nabla u(x)| \leq \bar{\lambda} \frac{u(x)}{d(y, \partial\Omega)}$$

for some $\bar{\lambda} = \bar{\lambda}(p, n, N, M)$. Since $x \in \Omega \cap B(w, r/c')$ is arbitrary, the proof of Lemma 3.8 is now complete. $\Box$

Finally in this subsection we prove the following lemma.

Lemma 3.15. *Let $\Omega \subset \mathbf{R}^n$ be a (δ, r_0)-Reifenberg flat domain. Given p, $1 < p < \infty$, $w \in \partial\Omega$, $0 < r < r_0$, suppose that u is a positive p-harmonic function in $\Omega \cap B(w, 4r)$, u is continuous in $\bar{\Omega} \cap \bar{B}(w, 4r)$, and $u = 0$ on $\Delta(w, 4r)$. Then there exist, for $\epsilon > 0$ given, $\hat{\delta} = \hat{\delta}(p, n, \epsilon) > 0$ and $c = c(p, n, \epsilon)$, $1 \leq c < \infty$, such that*

$$c^{-1}\left(\frac{\hat{r}}{r}\right)^{1+\epsilon} \leq \frac{u(a_{\hat{r}}(w))}{u(a_r(w))} \leq c\left(\frac{\hat{r}}{r}\right)^{1-\epsilon}$$

whenever $0 < \delta \leq \hat{\delta}$ and $0 < \hat{r} < r/4$.

Proof: Again we extend u to $B(w, 4r) \setminus \Omega$ by putting $u \equiv 0$ in this set and note that u is now continuous in $B(w, 4r)$. Next observe from Definition 1.7 that it suffices to prove Lemma 3.15 for $\delta = \hat{\delta}$. Also, as discussed after Definition 1.7 we may assume that Ω is a NTA-domain. Moreover, since the p-Laplace equation is invariant under scaling, translation, and rotations, we can assume that $r = 4$, $w = 0$, and that

$$h(P \cap B(0, 4), \partial\Omega \cap B(0, 4)) \leq 4\hat{\delta},$$

where $P = \{x \in \mathbf{R}^n : x_n = 0\}$. Furthermore, if $\bar{\delta} = 4\hat{\delta}$ is small enough, then we may assume, as in (3.11), that

$$\begin{aligned} B(0,4) \cap \{(x', x_n) : x_n \geq 2\bar{\delta}\} &\subset \Omega, \\ B(0,4) \cap \{(x', x_n) : x_n \leq -2\bar{\delta}\} &\subset \mathbf{R}^n \setminus \Omega. \end{aligned} \tag{3.16}$$

Finally we assume, as we may, that $u(a_1(0)) = 1$. Thus, to prove Lemma 3.15 it suffices to show that

$$c^{-1}\hat{r}^{1+\epsilon} \leq u(a_{\hat{r}}(0)) \leq c\hat{r}^{1-\epsilon} \text{ whenever } 0 < \hat{r} < 1. \tag{3.17}$$

To begin the proof of (3.17) we let

$$Q_1 = \{x : |x'| < 1,\, 2\bar{\delta} < x_n < 1\},\ Q_2 = \{x : |x'| < 1,\, -2\bar{\delta} < x_n < 1\}.$$

Moreover, we introduce two auxiliary functions u^+ and u^-. In particular, we define u^+ to be p-harmonic in Q_1 with boundary values on ∂Q_1 defined as follows,

$$\begin{aligned} u^+(y) = u(y) &\qquad \text{if } y \in \partial Q_1 \cap \{y : 4\bar{\delta} \leq y_n\}, \\ u^+(y) = \frac{(y_n - 2\hat{\delta})}{2\hat{\delta}} u(y) &\qquad \text{if } y \in \partial Q_1 \cap \{y : 2\hat{\delta} < y_n < 4\bar{\delta}\}, \\ u^+(y) = 0 &\qquad \text{if } y \in \partial Q_1,\ y_n = 2\bar{\delta}. \end{aligned}$$

Similarly, u^- is the p-harmonic function in Q_2 with $u^- = u$ on ∂Q_2. Observe from the maximum principle for p-harmonic functions and (3.16) that $u^+ \leq u \leq u^-$. We also define $v^+(x) := x_n - 2\bar{\delta}$,

$v^-(x) := x_n + 2\bar{\delta}$ whenever $x \in \mathbf{R}^n$. Hence v^+ and v^- are p-harmonic functions and grow linearly in the e_n-direction.

We first focus on the right hand inequality in (3.17). Using the normalization of u stated above (3.17), Lemma 2.4, the maximum principle for p-harmonic functions, and Theorem 2.7 applied to the functions u^-, v^- in Q_2, we see for $\bar{\delta}$ small enough that

$$u(x) \leq u^-(x) \leq cv^-(x) = c(x_n + 2\bar{\delta}) \tag{3.18}$$

whenever $x \in \Omega \cap B(0, 1/(4c_1))$. Thus, $u(a_{\bar{\delta}}(0)) \leq \hat{c}\bar{\delta}$. Suppose by way of induction that we have shown,

$$u(a_{\bar{\delta}^k}(0)) \leq (\hat{c}\,\bar{\delta})^k \tag{3.19}$$

where $\hat{c}$ depends only on p, n. From Reifenberg flatness we see there exists a plane P' containing 0 for which

$$h(P' \cap B(0, 4\bar{\delta}^k), \partial\Omega \cap B(0, 4\bar{\delta}^k)) \leq 4\hat{\delta}\bar{\delta}^k.$$

Repeating the above argument with p-replaced by P' and 4 by $4\bar{\delta}^k$ we find that

$$u(a_{\bar{\delta}^{k+1}}(0)) \leq \hat{c}\,\bar{\delta}\,u(a_{\bar{\delta}^k}(0)) \leq (\hat{c}\,\bar{\delta})^{k+1}.$$

Thus by induction the above inequality is true for all positive integers k. Let $\bar{\delta}^{-\epsilon} = \hat{c}$ where $\hat{c}$ is the constant in the above display. Given $0 < \hat{r} < 1$, let k be the smallest integer such that $\bar{\delta}^k \leq \hat{r}$. Then from Lemmas 2.3 and 2.4, (3.19), and our choice of $\bar{\delta}$ we see that $u(a_{\hat{r}}(0)) \leq c\hat{r}^{1-\epsilon}$, for some $c = c(p, n, \epsilon)$. Thus the proof of Lemma 3.15 is complete in one direction. To finish the proof we focus on the left hand inequality in (3.17). Indeed, applying Theorem 2.7 to the functions u^+, v^+ in Q_1 and using the Harnack inequality we see, provided $\bar{\delta}$ is small enough, that

$$\frac{u^+(a_{32\bar{\delta}}(0))}{v^+(a_{32\bar{\delta}}(0))} \approx \frac{u^+(a_{1/2}(0))}{v^+(a_{1/2}(0))} \approx \frac{u^+(a_{1/2}(0))}{1 - \bar{\delta}}. \tag{3.20}$$

Using that $u^+(a_{1/2}(0)) \geq c^{-1}$ as well as the maximum principle we see that (3.20) implies

$$u^+(a_{32\bar{\delta}}(0)) \geq c^{-1}v^+(a_{32\bar{\delta}}(0)) \geq \bar{c}^{-1}\bar{\delta}.$$

for some $\bar{c} = c(p, n)$, $1 \leq \bar{c} < \infty$. Using this inequality and arguing as in the proof of the right hand inequality in (3.17), i.e., using dilation, translation, rotation invariance of the p Laplacian as well as induction, we get

$$u(a_{(32\bar{\delta})^k}(0)) \geq (\bar{c}^{-1}\bar{\delta})^k. \tag{3.21}$$

Let $\bar{\delta}$ be so small that $\bar{c}^{-1}\bar{\delta} \geq (32\bar{\delta})^{1+\epsilon}$ and assume that $\hat{r} \in [(32\bar{\delta})^{k+1}, (32\bar{\delta})^k]$. With $\bar{\delta}(p, n, \epsilon)$ now fixed, it follows from Harnack's inequality for p-harmonic functions that

$$u(a_{\hat{r}}(0)) \geq c^{-1}u(a_{(32\bar{\delta})^k}(0)) \geq c^{-1}(32\bar{\delta})^{k(1+\epsilon)} \geq c^{-1}\hat{r}^{(1+\epsilon)}. \tag{3.22}$$

for some $c = c(p, n, \epsilon)$. (3.22) completes the proof of (3.17) and of Lemma 3.15. □

3.2 Degenerate elliptic equations

We here consider degenerate elliptic operators in the sense of [FKS], [FJK] and [FJK1]. The idea is to make use of the results of these authors to establish Claim 1 and Claim 2, stated in the introduction, under the assumptions of Theorem 1.

Let $w \in \mathbf{R}^n$, $0 < r$ and let $\lambda(x)$ be a real valued Lebesgue measurable function defined almost everywhere on $B(w, 2r)$. $\lambda(x)$ is said to belong to the class $A_2(B(w, r))$ if there exists a constant γ such that

$$r^{-2n} \int\limits_{B(\tilde{w},\tilde{r})} \lambda \, dx \cdot \int\limits_{B(\tilde{w},\tilde{r})} \lambda^{-1} dx \leq \gamma \tag{3.23}$$

whenever $\tilde{w} \in B(w, r)$ and $0 < \tilde{r} \leq r$. If $\lambda(x)$ belongs to the class $A_2(B(w, r))$ then λ is referred to as an $A_2(B(w, r))$-weight. The smallest γ such that (3.23) holds is referred to as the constant of the weight.

In the following we let $\Omega \subset \mathbf{R}^n$ be a (δ, r_0)-Reifenberg flat domain with NTA-constant M. We let $w \in \partial\Omega$, $0 < r < r_0$, and we consider the operator

$$\hat{L} = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\hat{b}_{ij}(x)\frac{\partial}{\partial x_j}\right) \tag{3.24}$$

in $\Omega \cap B(w, 2r)$. We assume that the coefficients $\{\hat{b}_{ij}(x)\}$ are bounded, Lebesgue measurable functions defined almost everywhere on $B(w, 2r)$. Moreover, $\hat{b}_{ij} = \hat{b}_{ji}$ for all $i, j \in \{1, .., n\}$, and

$$c^{-1}\lambda(x)|\xi|^2 \leq \sum_{i,j=1}^{n} \hat{b}_{ij}(x)\xi_i\xi_j \leq c|\xi|^2\lambda(x) \tag{3.25}$$

for almost everywhere every $x \in B(w, 2r)$, where $\lambda \in A_2(B(w, r))$. By definition $\hat{L}$ is a degenerate elliptic operator (in divergence form) in $B(w, 2r)$ with ellipticity measured by the function λ. If $O \subset B(w, 2r)$ is open then we let $\tilde{W}^{1,2}(O)$ be the weighted Sobolev space of equivalence classes of functions v with distributional gradient ∇v and norm

$$\|v\tilde{\|}_{1,2}^2 = \int\limits_O v^2\lambda dx + \int\limits_O |\nabla v|^2\lambda dx < \infty.$$

Let $\tilde{W}_0^{1,2}(O)$ be the closure of $C_0^\infty(O)$ in the norm $\tilde{W}^{1,2}(O)$. We say that v is a weak solution to $\hat{L}v = 0$ in O provided $v \in \tilde{W}^{1,2}(O)$ and

$$\int\limits_O \sum_{i,j} \hat{b}_{ij}v_{x_i}\phi_{x_j} dx = 0 \tag{3.26}$$

whenever $\phi \in C_0^\infty(O)$.

The following three lemmas, Lemmas 3.27-3.29, are tailored to our situation and based on the results in [FKS], [FJK] and [FJK1].

Lemma 3.27. *Let $\Omega \subset \mathbf{R}^n$ be a NTA-domain with constant M, $w \in \partial\Omega$, $0 < r < r_0$, and let λ be an $A_2(B(w, r))$-weight with constant γ. Suppose that v is a positive weak solution to $\hat{L}v = 0$*

in $\Omega \cap B(w,2r)$. Then there exists a constant c, $1 \leq c < \infty$, depending only on n, M and γ, such that if $\tilde w \in \Omega$, $0 < \tilde r$, $B(\tilde w, 2\tilde r) \subset \Omega \cap B(w,r)$, then

$$(i) \qquad c^{-1}\tilde r^2 \int\limits_{B(\tilde w,\tilde r/2)} |\nabla v|^2 \lambda dx \leq c\Bigg(\int\limits_{B(\tilde w,\tilde r)} \lambda dx\Bigg)(\max_{B(\tilde w,\tilde r)} v)^2 \leq c \int\limits_{B(\tilde w,2\tilde r)} |v|^2\lambda dx,$$

$$(ii) \qquad \max_{B(\tilde w,\tilde r)} v \leq c \min_{B(\tilde w,\tilde r)} v.$$

Furthermore, there exists $\alpha = \alpha(n,M,\gamma) \in (0,1)$ such that if $x, y \in B(\tilde w,\tilde r)$ then

$$(iii) \qquad |v(x) - v(y)| \leq c\bigg(\frac{|x-y|}{\tilde r}\bigg)^{\alpha} \max_{B(\tilde w,2\tilde r)} v.$$

Lemma 3.28. *Let $\Omega \subset \mathbf{R}^n$ be a NTA-domain with constant M, $w \in \partial\Omega$, $0 < r < r_0$, and let λ be an $A_2(B(w,r))$-weight with constant γ. Suppose that v is a positive weak solution to $\hat L v = 0$ in $\Omega \cap B(w,2r)$ and that $v = 0$ on $\Delta(w,2r)$ in the weighted Sobolev sense. Then there exists $\tilde c = \tilde c(n,M,\gamma)$, $1 \leq \tilde c < \infty$, such that the following holds with $\tilde r = r/\tilde c$.*

$$(i) \qquad r^2 \int\limits_{\Omega\cap B(w,r/2)} |\nabla v|^2\lambda dx \leq \tilde c \int\limits_{\Omega\cap B(w,r)} |v|^2 \lambda dx,$$

$$(ii) \qquad \max_{\Omega\cap B(w,\tilde r)} v \leq \tilde c v(a_{\tilde r}(w)).$$

Furthermore, there exists $\alpha = \alpha(n,M,\gamma) \in (0,1)$ such that if $x, y \in \Omega \cap B(w,\tilde r)$, then

$$(iii) \qquad |v(x) - v(y)| \leq c\bigg(\frac{|x-y|}{r}\bigg)^{\alpha} \max_{\Omega\cap B(w,2\tilde r)} v.$$

Lemma 3.29. *Let $\Omega \subset \mathbf{R}^n$ be a NTA-domain with constant M, $w \in \partial\Omega$, $0 < r < r_0$, and let λ be an $A_2(B(w,r))$-weight with constant γ. Suppose that v_1 and v_2 are two positive weak solutions to $\hat L v = 0$ in $\Omega \cap B(w,2r)$ and $v_1 = 0 = v_2$ on $\Delta(w,2r)$ in the weighted Sobolev sense. Then there exist $c = c(n,M,\gamma)$, $1 \leq c < \infty$, and $\alpha = \alpha(n,M,\gamma) \in (0,1)$ such that if $\tilde r = r/c$, $v_1(a_{\tilde r}(w)) = v_2(a_{\tilde r}(w))$, then $v_1/v_2 \leq c$ in $\Omega \cap B(w,r/c)$ and if $y_1, y_2 \in \Omega \cap B(w,r/c)$, then*

$$\left|\frac{v_1(y_1)}{v_2(y_1)} - \frac{v_1(y_2)}{v_2(y_2)}\right| \leq c\bigg(\frac{|y_1-y_2|}{r}\bigg)^{\alpha}.$$

Let u be as in the statement of Lemma 3.15. To be able to use Lemmas 3.27-3.29 to study solutions to the operators $\tilde L$, see (1.16)-(1.18), and L, see (1.20)-(1.21), and thus to prove Claim 1 and Claim 2 in the case of Theorem 1, we first use Lemma 3.15 to prove that $|\nabla u|^{p-2}$ can be extended, locally, to an A_2-weight. More precisely we prove the following lemma.

Lemma 3.30. *Let Ω be a (δ, r_0)-Reifenberg flat domain, $w \in \partial\Omega$ and $0 < r < r_0$. Fix p, $1 < p < \infty$, and suppose that u is a positive p-harmonic function in $\Omega \cap B(w,2r)$, u is continuous in $\bar\Omega \cap B(w,2r)$, and $u = 0$ on $\Delta(w,2r)$. There exist $\delta' = \delta'(p,n)$, and $c = c(p,n) \geq 1$ such that if $0 < \delta < \delta'$, and $\hat r = r/c$, then $|\nabla u|^{p-2}$ extends to an $A_2(B(w,\hat r))$-weight with constant* *depending only on p, n.*

Proof: Our proof of Lemma 3.30 follows the lines of the proof of Lemma 2.36 in [L]. Let $Q_j = Q(x_j, r_j)$, $j = 1, 2, \ldots$ be a Whitney decomposition of $\mathbf{R}^n \setminus \bar{\Omega}$ into open cubes with center at x_j and sidelength r_j. Then $\cup_j \bar{Q}(x_j, r_j) = \mathbf{R}^n \setminus \bar{\Omega}$ and $Q(x_j, r_j) \cap Q(x_i, r_i) = \emptyset$ when $i \neq j$. We furthermore construct the Whitney cubes in such a way that $10^{-4n} d(Q_j, \partial\Omega) \leq r_j \leq 10^{-2n} d(Q_j, \partial\Omega)$. Note for δ' small, that Ω is NTA with constant $M = M(n)$. Thus there exists a constant $\tilde{c} = \tilde{c}(n)$, $1 \leq \tilde{c} < \infty$, such that if $Q_j \cap B(w, r) \neq \emptyset$, then there is a $w_j \in \Omega \cap B(w, \tilde{c}r)$ for which $d(w_j, \partial\Omega) \sim d(w_j, x_j) \sim d(x_j, \partial\Omega)$. Here $A \sim B$ means that A/B is bounded from above and below by constants which only depend on n. Let $\hat{r} = r/\max\{\tilde{c}, c\}$ where $c = c(p, n)$, $1 \leq c < \infty$, is so large that if $Q_j \cap B(w, 8\hat{r}) \neq \emptyset$, then

$$c_*^{-1} \frac{u(x)}{d(x, \partial\Omega)} \leq |\nabla u(x)| \leq c_* \frac{u(x)}{d(x, \partial\Omega)} \text{ whenever } x \in B(w_j, d(w_j, \partial\Omega)/2) \tag{3.31}$$

where $c_* = \bar{\lambda}$ is the constant in Lemma 3.8. In particular, using Lemma 3.8 we see that such a $c = c(p, n)$ exists provided δ' is small enough. Next we define $\lambda(x) = |\nabla u(x)|^{p-2}$ whenever $x \in \Omega \cap B(w, 4\hat{r})$ and we let Γ be the set of all j such that if $j \in \Gamma$ then $Q_j \cap B(w, 4\hat{r}) \neq \emptyset$. Moreover, if $j \in \Gamma$ then we choose $w_j \in \Omega \cap B(w, \tilde{c}r)$ as above and define $\lambda(x) = \lambda(w_j)$ when $x \in Q_j$. This defines λ almost everywhere on $B(w, 4\hat{r})$ with respect to Lebesgue n measure, since it follows from (3.36) that for δ small enough, $\partial\Omega \cap B(w, r)$ has Lebesgue n measure zero. Using (3.31) and the Harnack inequality for p-harmonic functions we see that

$$\lambda(x) = \lambda(w_j) \approx \lambda(z) \text{ whenever } x \in Q_j \text{ and } z \in B(w_j, d(w_j, \partial\Omega)/2). \tag{3.32}$$

To complete the proof of Lemma 3.30 we prove that the inequality in (3.23) holds for this choice of λ and for all balls $B(\tilde{w}, \tilde{r})$ such that $\tilde{w} \in B(w, \hat{r})$, $\tilde{r} \leq \hat{r}$, with $\gamma = \gamma(p, n)$. To do this we consider several cases. If $\tilde{r} < d(\tilde{w}, \partial\Omega)/2$, then (3.23) is true as we see using (3.31), (3.32) and the Harnack inequality. On the other hand, if $\tilde{r} \geq d(\tilde{w}, \partial\Omega)/2$ then we choose $\hat{w} \in \partial\Omega$ with $d(\tilde{w}, \partial\Omega) = d(\tilde{w}, \hat{w})$ and we note that, in this case, $B(\tilde{w}, \tilde{r}) \subset B(\hat{w}, 3\tilde{r}) \subset B(w, 8\hat{r})$. We now continue under the assumption that $p > 2$. From Hölder's inequality, Lemmas 2.4, 2.5, and Harnack's inequality we see that

$$\begin{aligned} \int_{B(\tilde{w}, \tilde{r})} \lambda dx &\leq \int_{B(\hat{w}, 3\tilde{r})} \lambda dx \leq c \int_{\Omega \cap B(\hat{w}, c^* \tilde{r})} |\nabla u|^{p-2} dx \\ &\leq c \left(\int_{\Omega \cap B(\hat{w}, c^* \tilde{r})} |\nabla u|^p dx \right)^{(1-2/p)} \tilde{r}^{2n/p} \leq c u(a_{\tilde{r}}(\hat{w}))^{p-2} \tilde{r}^{n+2-p}. \end{aligned} \tag{3.33}$$

Let $\eta = \min\{1, |p-2|^{-1}\}/20$. To estimate the part of (3.23) involving λ^{-1} we observe from Lemma 3.15 and Harnack's inequality, that if $y \in \Omega \cap B(\hat{w}, c^* \tilde{r})$, and δ' is small enough, then

$$c u(y) \geq u(a_{\tilde{r}}(\hat{w})) \left(\frac{d(y, \partial\Omega)}{\tilde{r}} \right)^{1+\eta}. \tag{3.34}$$

Therefore, using (3.31) and (3.34) in a deduction similar to (3.33) we see that

$$\int_{B(\tilde{w}, \tilde{r})} \lambda^{-1} dx \leq c \tilde{r}^{(1+\eta)(p-2)} u(a_{\tilde{r}}(\hat{w}))^{2-p} \int_{\Omega \cap B(\hat{w}, c^* \tilde{r})} d(y, \partial\Omega)^{-\eta(p-2)} dy. \tag{3.35}$$

To complete the estimate in (3.35) we need to estimate the integral involving the distance function. To do this we define

$$I(z,s) = \int\limits_{\Omega\cap B(z,s)} d(y,\partial\Omega)^{-\eta(p-2)}dy$$

whenever $z \in \partial\Omega \cap B(w,r)$, $0 < s < r$. Let

$$E_k = \Omega \cap B(z,s) \cap \{y : d(y,\partial\Omega) \leq \delta^k s\} \text{ for } k = 1,2,\ldots$$

Then since $\partial\Omega$ is δ-Reifenberg flat we deduce that

$$\int_{E_k} dy \leq c_+^k \delta^k s^n \tag{3.36}$$

where $c_+ = c_+(p,n)$. Indeed, from δ-Reifenberg flatness it is easily seen that this statement holds for E_1. Moreover, E_1 can be covered by at most c/δ^{n-1} balls of radius $100\delta s$ with centers in $\partial\Omega \cap B(z,s)$. We can then repeat the argument in each ball to get that (3.36) holds for E_2. Continuing in this way we get (3.36) for all positive integers k. Using (3.36) and writing $I(z,s)$ as a sum over $E_k \setminus E_{k+1}, k = 1,2,\ldots$ we get

$$I(z,s) \leq cs^{n-\eta(p-2)} + \delta^{\eta(p-2)-n} \sum_{k=1}^{\infty} (c_+^k \delta^k s)^{n-\eta(p-2)} < \tilde{c}s^{n-\eta(p-2)},$$

where $\tilde{c} = \tilde{c}(p,n)$, provided δ' is small enough. Using this estimate with $s = \tilde{r}$, we can continue our calculation in (3.35) and conclude that

$$\int\limits_{B(\tilde{w},\tilde{r})} \lambda^{-1}dx \quad \leq \quad c\tilde{r}^{n+p-2}u(a_{\tilde{r}}(\hat{w}))^{2-p}. \tag{3.37}$$

To complete the proof of Lemma 3.30, in the case $p > 2$, we simply combine (3.33) and (3.37). Note that the case $p = 2$ is trivial and in case $p < 2$ the argument above can be repeated with $p-2$ replaced by $2-p < p$. The proof of Lemma 3.30 is now complete. □

3.3 Proof of Theorem 1.

We are now ready to prove Claim 1 and Claim 2 stated in section 1. More specifically we prove the following.

Lemma 3.38. *Let $\Omega \subset \mathbf{R}^n$ be a (δ,r_0)-Reifenberg flat domain and let p, w, r, u and v be as in the statement of Theorem 1. Assume that u and v satisfy (1.11) and let $u(\cdot,\tau)$, $0 \leq \tau \leq 1$, be the p-harmonic function in $\Omega \cap B(w,4r^*)$ defined in (1.12). There exist $\delta_1 = \delta_1(p,n) > 0$ such that if $0 < \delta < \delta_1$ then Claims 1 and 2 are true in the case of Theorem 1.*

Proof: Let $\Omega \subset \mathbf{R}^n$, p, w, r, u, v and $u(\cdot, \tau), 0 \leq \tau \leq 1$, be as in the statement of the lemma. For the moment we allow δ_1 in Theorem 1 to vary. To prove Claim 1 we first consider the operator $\tilde{L} = \tilde{L}_{t,\tau}$ in (1.16) and we see, using (1.19), that

$$\sum_{i,j=1}^{n} A_{ij}(x, t, \tau)\xi_i\xi_j \approx |\xi|^2 \bigg(|\nabla u(x,t)| + |\nabla u(x, \tau)| \bigg)^{p-2} \tag{3.39}$$

whenever $\xi \in \mathbf{R}^n \setminus \{0\}$. We also note from (1.11) that for U as in (1.15) and $t > s$

$$v/2 \leq U(x, t, s) = v - u \leq s^{-1}u(x, s) \text{ whenever } x \in \partial(\Omega \cap B(w, 4r^*)).$$

Using the maximum principle for p-harmonic functions it then follows that

$$0 \leq U \leq c \text{ and } U(\cdot, t, s) \leq s^{-1}u(\cdot, s) \text{ in } \Omega \cap B(w, 4r^*) \tag{3.40}$$

when $t > s$ and $s, t \in (0, 1]$.

To verify Claim 1 and Claim 2 we first note from (3.39), (3.40) and Lemma 3.8 that there exists $\hat{\delta}_1 = \hat{\delta}_1(p, n)$ such that if $\delta < \hat{\delta}_1$ then U is, for $c = c(p, n)$ large enough, a bounded weak solution in $B(w, r/c)$ to a locally uniformly elliptic equation in divergence form. Using this fact, Lemma 2.5 applied to $u(\cdot, s)$, and (3.40) once again we see that U has a Hölder continuous extension to $B(w, r/c)$, obtained by putting $U \equiv 0$ on $B(w, r/c) \setminus \Omega$. Moreover, the Hölder exponent is independent of t, s while the Hölder constant depends only on p, n, s. The proof of Claim 1 is now complete. We then repeat the argument from (1.20) to (1.23) and note that there exists $\tilde{\delta}_1 = \tilde{\delta}_1(p, n)$ such that if $\delta < \tilde{\delta}_1$ then Claim 2 follows from Lemmas 3.29, 3.30. Hence, the proof of the lemma is complete if we let $\delta_1 = \min\{\hat{\delta}_1, \tilde{\delta}_1\}$. $\Box$

Theorem 1 now follows from Claim 1 and Claim 2 (as pointed out in section 1), when (1.11) holds. It therefore remains to remove assumption (1.11). To do this we first note that Theorem 1 is invariant under multiplication of u, v by constants. Using this note and Lemma 2.4 we see that if $r^* = r/c$, then for $c = c(p, n)$ large enough, we may assume that

$$\max_{B(x,4r^*)} h \approx h(a_{r^*}(w)) = 1 \text{ whenever } h = u \text{ or } v. \tag{3.41}$$

Let $\tilde{u}, \tilde{v}$ be the p-harmonic functions in $\Omega \cap B(w, 4r^*)$ with boundary values $\tilde{u} = \min(u, v)$ and $\tilde{v} = 2\max(u, v)$ respectively on $\partial(\Omega \cap B(w, 4r^*))$. From the maximum principle for p-harmonic functions we then have $\tilde{u} \leq u, v \leq \tilde{v}/2$ in $\Omega \cap B(w, 4r^*)$. Using this inequality and applying Theorem 1 to $\tilde{u}, \tilde{v}$ with r replaced by r^*, we get $\tilde{v}/\tilde{u} \leq c$ in $\Omega \cap B(w, \tilde{r})$. Thus the first display in Theorem 1 is valid without assumption (1.11). To get the second display in Theorem 1 we note from boundedness of the ratio in the first display that (1.11) can now be achieved in $\Omega \cap B(w, 4r^*)$, for r^* small enough, from multiplying v by a suitably large constant. The proof of Theorem 1 is complete. $\Box$

4 Estimates for p-harmonic functions in Ahlfors regular NTA-domains which are uniformly δ-approximable by Lipschitz graph domains

In this section our goal is to prove Theorem 2. Thus our main focus will be on p-harmonic functions in Ahlfors regular NTA-domains that are uniformly δ-approximable by Lipschitz graph domains.

4.1 Local strong non-degeneracy

Recall that a bounded domain $\tilde\Omega \subset \mathbf{R}^n$ is said to be starlike Lipschitz with respect to $\hat x \in \tilde\Omega$ provided

$$\partial\tilde\Omega = \{\hat x + R(\omega)\omega : \omega \in \partial B(0,1)\} \text{ where } \log R : \partial B(0,1) \to \mathbf{R} \text{ is Lipschitz on } \partial B(0,1).$$

We shall refer to $\| \log R\hat{\|}_{\partial B(0,1)}$ as the Lipschitz constant for $\tilde\Omega$. Observe that this constant is invariant under scaling about $\hat x$. In this section we first prove the following localization lemma.

Lemma 4.1. *Let $\hat\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain with constant $\hat N$. Given $p, 1 < p < \infty, w \in \partial\hat\Omega$, $0 < r < r_0$, suppose that v is a positive p-harmonic function in $\hat\Omega \cap B(w, 2r)$, v is continuous in the closure of $\hat\Omega \cap B(w, 2r)$ and $v = 0$ on $\Delta(w, 2r) = \partial\hat\Omega \cap B(w, 2r)$. Then there exists a constant $c = c(p, n, \hat N)$, $1 \le c < \infty$, such that the following is true with $r' = r/c$. There exists, for $\epsilon > 0$ given, a starlike Lipschitz domain $\tilde\Omega \subset \hat\Omega \cap B(w, r)$, with center at a point $\hat w$, $d(\hat w, \partial\tilde\Omega) \ge c^{-1}r$, and with Lipschitz constant bounded by c, such that*

$$\frac{\sigma(\partial\tilde\Omega \cap \Delta(w, r'))}{\sigma(\Delta(w, r'))} \ge 1 - \epsilon.$$

Moreover, there exists a constant $\tilde c = \tilde c(p, n, \hat N, \epsilon)$, $1 \le \tilde c < \infty$, such that if $x \in \tilde\Omega$, then

$$\tilde c^{-1} r^{-1}\, v(\hat w) \,\le\, |\nabla v(x)| \,\le\, \tilde c r^{-1}\, v(\hat w)\,.$$

Proof: In the statement of Lemma 4.1 σ is surface measure on $\partial\hat\Omega$. We note that Lemma 4.1 is somewhat stronger than Lemma 2.11 in [LN] and similar, but less refined, to Lemma 4.1 in [LN2]. Let $\eta, 0 < \eta < 10^{-6}$, be a small positive number which we for the moment allow to vary but which we later shall fix to satisfy several conditions depending on $p, n, \hat N$ and ϵ. From Theorem 2.13 (iii) with u replaced by v, we see that $\log |\nabla v| \in BMO(\Delta(w, r/2))$. Thus by definition (see (2.12)), if A denotes the average of $f = \log |\nabla v|$ with respect to surface measure, σ, over $\Delta(w, r/4)$, then

$$\frac{\sigma(\{x \in \Delta(w, r/4) : |f(x) - A| > \eta^{-2}\})}{\sigma(\Delta(w, r/4))} \le \eta^2\, (\sigma(\Delta(w, r/4))^{-1} \int\limits_{\Delta(w,r/4)} |f - A| d\sigma \le c\eta^2. \quad (4.2)$$

If $b = e^A$, then from (4.2) we see there exists a set $E \subset \Delta(w, r/4)$ such that if $\lambda = e^{\eta^{-2}}$, then $\lambda^{-1} b \le |\nabla v| \le \lambda b$ on E and with the property that if $F = \Delta(w, r/4) \setminus E$, then $\sigma(F) \le$

$c\eta^2\,\sigma(\Delta(w,r/4))$. We now define, for σ integrable functions h defined on $\Delta(w,r/4)$ and for $x\in\Delta(w,r/8)$, the maximal function

$$M(h)(x) \;=\; \sup_{0<s<r/8} \frac{1}{\sigma(\Delta(x,s))} \int\limits_{\Delta(x,s)} h\,d\sigma.$$

Let $G=\{x\in\Delta(w,r/8):\ M(\chi_F)(x)\le\eta\}$ where χ_F is the indicator functions for the set F and define $K=\Delta(w,r/8)\setminus G$. Using weak type estimates for the maximal function, it follows that

$$\sigma(K)\le c\eta\sigma(\Delta(w,r/8)). \tag{4.3}$$

Let $y\in G\cap\Delta(w,r/16)$, $0<s\le r/100$, and let ν be the measure associated with v as in Lemma 2.6 with u,μ replaced by v,ν. Then from Theorem 2.13, we deduce

$$\begin{aligned}\nu(\Delta(y,s)) &= \int\limits_{\Delta(y,s)} |\nabla v|^{p-1}d\sigma\\ &= \int\limits_{E\cap\Delta(y,s)} |\nabla v|^{p-1}d\sigma + \int\limits_{F\cap\Delta(y,s)} |\nabla v|^{p-1}d\sigma = T_1+T_2.\end{aligned} \tag{4.4}$$

From the definitions of the sets E,F,G, we see that

$$\tfrac12\,(b/\lambda)^{p-1}\sigma(\Delta(y,s))\le T_1\le 2(\lambda b)^{p-1}\sigma(\Delta(y,s)), \tag{4.5}$$

provided η is sufficiently small. Also from Hölder's inequality,

$$(\sigma(\Delta(y,s)))^{-1}T_2\le\left(\frac{1}{\sigma(\Delta(y,s))}\int\limits_{\Delta(y,s)}|\nabla v|^p d\sigma\right)^{(p-1)/p}\left(\frac{\sigma(F\cap\Delta(y,s))}{\sigma(\Delta(y,s))}\right)^{1/p}. \tag{4.6}$$

Using $y\in G$, and the reverse Hölder inequality for $|\nabla v|$ in Theorem 2.13 (ii) we get from (4.6) that

$$T_2\le c\eta^{1/p}\nu(\Delta(y,s)) \tag{4.7}$$

where c depends only on $p,n,\hat N$. Using (4.5), (4.7) in (4.4), we obtain, for $\eta=\eta(p,n,\hat N)$, sufficiently small, that

$$\tfrac14\,(b/\lambda)^{p-1}\le\frac{\nu(\Delta(y,s))}{\sigma(\Delta(y,s))}\le 4(\lambda b)^{p-1}. \tag{4.8}$$

To construct $\tilde\Omega$ we assume, as we may, that

$$\begin{aligned}\hat\Omega\cap B(w,2r) &= \{(x',x_n):x_n>\phi(x')\}\cap B(w,2r),\\ \partial\hat\Omega\cap B(w,2r) &= \{(x',x_n):x_n=\phi(x')\}\cap B(w,2r),\end{aligned} \tag{4.9}$$

where $\phi:\mathbf{R}^{n-1}\to\mathbf{R}$ is Lipschitz with $\||\nabla\phi|\|_\infty\le\hat N$. Let $\hat r=r/c, r'=\hat r/c$, $(c\ge 100)$, and $\hat w=w+\frac14\hat r\,e_n$. Let $\tilde\Omega$ be the domain obtained from drawing all open line segments from points in $B(\hat w,r')$ to points in $\Delta(w,r')\cap G$. If $c=c(p,n,\hat N)$ is large enough enough, it follows from

Lipschitzness of $\hat{\Omega}$ and elementary geometry that $\tilde{\Omega} \subset \hat{\Omega}$ is a starlike Lipschitz domain with center at $\hat{w}$ and Lipschitz constant $\tilde{N} = \tilde{N}(\hat{N})$. Now from (4.3) we see that if $\eta = \epsilon/c^*$ and $c^* = c^*(p, n, \hat{N})$ is large enough then the first display in Lemma 4.1 is valid. With η now fixed we first note from Theorem 2.9 with u replaced by v that

$$c^{-1} \frac{v(x)}{d(x, \partial\Omega)} \leq |\nabla v(x)| \leq c \frac{v(x)}{d(x, \partial\Omega)} \tag{4.10}$$

whenever $x \in \hat{\Omega} \cap B(w, \hat{r})$. Second we note that if $x \in \tilde{\Omega}$, then there exists $y \in G$ with $d(x, \partial\hat{\Omega}) \approx |x - y|$. If $s = |x - y|$, then from (4.8), Lemma 2.6 (*ii*), and Harnack's inequality, we find that

$$\tfrac{1}{4}(b/\lambda)^{p-1} \leq \frac{\nu(\Delta(y, s))}{\sigma(\Delta(y, s))} \approx \left(\frac{v(x)}{d(x, \partial\Omega)} \right)^{p-1} \leq c(\lambda b)^{p-1}. \tag{4.11}$$

From (4.10), (4.11) and the fact that $\hat{w} \in \tilde{\Omega}$ we obtain the last display in Lemma 4.1. The proof of Lemma 4.1 is now complete. $\Box$

Next we need a definition.

Definition 4.12. *Given a bounded domain $D \subset \mathbf{R}^n$ we say that ∂D contains big pieces of Lipschitz graphs from within provided there exist constants $r_1 > 0$, and $k_1, k_2 \geq 2$ such that whenever $x \in D$, with $d(x, \partial D) < r_1$, there exists a Lipschitz domain $\tilde{D}$ with*

(α) $\tilde{D} \subset B(x, 10\, d(x, \partial D)) \cap D$,

(β) *after a rotation of coordinates if necessary,*
$\tilde{D} = \{w = (w', w_n) : \psi(w') < w_n < x_n + d(x, \partial D)/2,\ |x' - w'| < d(x, \partial D)/k_1\}$,

(γ) $\|\psi\|_\infty = \sup_{z' \in \mathbf{R}^{n-1}} |\psi(z')| \leq x_n - d(x, \partial D)/k_1$,

for some $\psi : \mathbf{R}^{n-1} \rightarrow \mathbf{R}$ with

$$|\psi(y') - \psi(z')| \leq k_1\, |y' - z'| \text{ for all } y', z' \in \mathbf{R}^{n-1}.$$

Moreover,

$$\sigma\left(\{(w', \psi(w')) : |x' - w'| \leq d(x, \partial D)/k_1\} \cap \partial D\right) \geq d(x, \partial D)^{n-1}/k_2$$

where σ is surface measure on $\partial\tilde{D}$.

In our argument we need the following lemma.

Lemma 4.13. *Let $\hat{D}$ be an Ahlfors regular NTA-domain with NTA-constants M, r_0, and Ahlfors regular constant C. Then $\hat{D}$ contains big pieces of Lipschitz graphs from within in the sense of Definition 4.12 with constants k_1, k_2, only depending on n, M, C and with $r_1 = r_0/10^5$.*

Proof: Let r_1 be as above and suppose $x \in \hat{D}$ with $d(x, \partial\hat{D}) < r_1$. Choose $z \in \bar{B}(x, d(x, \partial\hat{D})) \cap \partial\hat{D}$. Let $\mathcal{S} = [B(z, 100 d(x, \partial\hat{D})) \cap \partial\hat{D}] \cup \partial B(z, 100\, d(x, \partial\hat{D}))$. Using the definition of $\hat{D}$, it is easily checked that $\mathcal{S}$ is Ahlfors regular (for radii $\leq 100 d(x, \partial\hat{D})$) and in the language of [DJ], satisfies a two balls condition. That is, given $z^* \in \mathcal{S}$, $0 < r \leq 100\, d(x, \partial\hat{D})$, there exist two balls of radii approximately equal to r which lie in different components of $\mathbf{R}^n \setminus \mathcal{S}$ and whose centers are in $B(z^*, r)$. Therefore $B(x, d(x, \partial\hat{D}))$ and some ball of approximately the same size are contained in $B(z^*, 10d(x, \partial\hat{D}))$ and lie in different components of $\mathbf{R}^n \setminus \mathcal{S}$. Lemma 4.7 now follows for k_1, k_2 suitably large, depending only on n, C, M, from a clever geometric argument in [DJ] (see the remark following display (10) in [DJ]). We omit the details. $\Box$

Using Lemmas 4.1, 4.13 we next prove the following lemma.

Lemma 4.14. *Let $\hat{D} \subset \mathbf{R}^n$ be a bounded Ahlfors regular NTA-domain with constants, r_0, M, C. Given $p, 1 < p < \infty, w \in \partial\hat{D}$, $0 < r < r_0$, suppose that v is a positive p-harmonic function in $\hat{D} \cap B(w, 2r)$, v is continuous in the closure of $\hat{D} \cap B(w, 2r)$ and $v = 0$ on $\Delta(w, 2r) = \partial\hat{D} \cap B(w, 2r)$. There exists a constant $c_* = c_*(p, n, M, C)$, $1 \leq c_* < \infty$, such that the following is true. There is a starlike Lipschitz domain $\Omega^* \subset \hat{D} \cap B(w, r)$, with center at a point $\tilde{z}$, $d(\tilde{z}, \partial\Omega^*) \geq c_*^{-1} r$, and with Lipschitz constant bounded by c_*, such that*

$$\frac{\sigma(\partial\Omega^* \cap \Delta(w, r))}{\sigma(\Delta(w, r))} \geq c_*^{-1}.$$

Moreover, if $y \in \Omega^*$ then

$$c_*^{-1} r^{-1}\, v(\tilde{z}) \;\leq\; v(y)/d(y, \partial\hat{D}) \;\leq\; c_* r^{-1}\, v(\tilde{z})\,.$$

Proof: Let $r' = r/10^7$ and let $x \in \hat{D} \cap B(w, r')$ with $d(x, \partial\hat{D}) \geq r/c$. From Lemma 4.13 we observe that there exists $\tilde{D} \subset \hat{D}$ such that $(\alpha), (\beta), (\gamma)$ in Definition 4.12 hold after a possible rotation with D replaced by $\hat{D}$. Since the p-Laplacian is invariant under rotations, we may assume that $(\beta), (\gamma)$ hold in the usual coordinate system. Let v' be the p-harmonic function in $\tilde{D} \setminus B(x, d(x, \partial\hat{D})/k_1^2)$ with continuous boundary value 0 on $\partial\tilde{D}$ and $v(x)$ on $\partial B(\, x, d(x, \partial\hat{D})/k_1^2\,)$. From the maximum principle for p-harmonic functions we have

$$v' \leq cv \text{ in } \tilde{D} \setminus \bar{B}(x, d(x, \partial\hat{D})/k_1^2). \tag{4.15}$$

From the last display in Definition 4.12 and the pigeonhole principle we see there exist, $\bar{c}_i = \bar{c}_i(n, M, C), i = 1, 2$, and $z \in \partial\tilde{D} \cap \partial\hat{D}$ such that if $\rho = r/\bar{c}_1^2$, then

$$\sigma(\partial\tilde{D} \cap \partial\hat{D} \cap B(z, \rho)) \geq \bar{c}_2^{-1}\, \sigma(\partial\tilde{D} \cap B(z, \rho)). \tag{4.16}$$

Moreover, if $\bar{c}_1$ is large enough we can apply Lemma 4.1 with $r, w, v, \hat{\Omega}$, replaced by $\bar{c}_1\rho, z, v', \tilde{D}$, respectively and $\epsilon = \bar{c}_2^{-2}$. Using this lemma and (4.16) we get $\tilde{\Omega} \subset \tilde{D} \cap B(z, \rho)$ with center $\tilde{z}, d(\tilde{z}, \partial\tilde{\Omega}) \geq r/c$, such that if $W = \partial\tilde{\Omega} \cap \partial\tilde{D} \cap \partial\hat{D}$, then

$$\begin{aligned} &(a) \qquad \sigma(W) \geq c^{-1}\sigma(\partial\hat{D} \cap B(z, \rho)),\\ &(b) \qquad v'(\tilde{z})/r \leq cv'(y)/d(y, \partial\tilde{D}) \text{ whenever } y \in \tilde{\Omega}, \end{aligned} \tag{4.17}$$

for some $c = c(p, n, M, C)$.

Let ν be the positive Borel measure associated with v as in Lemma 2.6. We put, for $y \in \partial\hat{D} \cap B(z, \rho)$,

$$M\nu(y) = \sup_{0<s<2\rho} \frac{\nu(\partial\hat{D} \cap B(y, s))}{\sigma(\partial\hat{D} \cap B(y, s))}.$$

Also, given $\lambda > 1$ we let

$$H = \left\{ y \in \partial\hat{D} \cap B(z,\rho) : M\nu(y) > \lambda \frac{\nu(\partial\hat{D} \cap B(z,\rho))}{\sigma(\partial\hat{D} \cap B(z,\rho))} \right\}.$$

Then from weak type estimates for the maximal function and Lemma 2.6 (ii) we get

$$\frac{\sigma(H)}{\sigma(\partial\hat{D} \cap B(z,\rho))} \leq c/\lambda \tag{4.18}$$

where $c = c(p,n)$. If $\lambda = \lambda(p,n,M,C)$ is large enough and $V = W \setminus H$ then from (4.17), (4.18) we get

$$\sigma(V) \geq c^{-1}\sigma(\partial\hat{D} \cap B(z,\rho)). \tag{4.19}$$

From the construction of $\tilde{\Omega}$, or just the definition of starlike Lipschitzness, we note that there exists $c_+ = c_+(p,n,M,C) \geq 2$ such that the open line segments drawn from points in $\partial\tilde{\Omega}$ to points in $B(\tilde{z},\rho/c_+)$ are always in $\tilde{\Omega}$. Finally we let $\Omega^* \subset \tilde{\Omega}$ be the starlike Lipschitz domain obtained from drawing all open line segments from points in V to points in $B(\tilde{z},\rho/c_+^2)$. From (4.19) we conclude the validity of the first display in Lemma 4.14.

Note that if $y \in \Omega^*$, then there exists $\hat{y} \in V$ with $d(y,\partial\tilde{\Omega}) \approx |y-\hat{y}|$. Using this fact we see for $y \in \Omega^*$ that

$$d(y,\partial\hat{D}) \approx d(y,\partial\tilde{\Omega}). \tag{4.20}$$

From (4.20), Lemma 2.6 (ii), and the definition of H we find as in (4.11) that

$$\frac{v(y)}{d(y,\partial\hat{D})} \leq cv(\tilde{z})/r \text{ whenever } y \in \Omega^*. \tag{4.21}$$

From barrier type estimates for v' and Harnack's inequality applied to v, v' we also see that

$$v(x) \approx v'(\tilde{z}) \approx v(\tilde{z}). \tag{4.22}$$

Using (4.22), (4.20), the fact that $\tilde{\Omega} \subset \tilde{D}$, (4.17) (b) and (4.15) we obtain

$$v(\tilde{z})/r \leq c\frac{v(y)}{d(y,\partial\hat{D})} \text{ whenever } y \in \Omega^*. \tag{4.23}$$

Combining (4.21), (4.23) we get the last display in Lemma 4.14. □

4.2 More degenerate elliptic equations

We are now in a position to use, essentially verbatim, arguments in [LN, section 2 and 3] (see also [LN1, section 2]) in order to establish the boundary Harnack inequalities needed to prove Theorem 2.

Throughout this subsection we assume that $\Omega \subset \mathbf{R}^n$ is an Ahlfors regular NTA-domain with constants M, C, r_0, which is uniformly (δ, N, r_0)-approximable by Lipschitz graph domains. Also, for p fixed, $1 < p < \infty$, $w \in \partial\Omega$, $0 < r < r_0$, suppose that $\hat{u}$ is p-harmonic in $\Omega \cap B(w,4r)$, that $\hat{u}$ is continuous in $\bar{\Omega} \cap B(w, 4r)$, and $\hat{u} = 0$ on $\Delta(w, 4r) = \partial\Omega \cap B(w, 4r)$. Let δ^* be as in Lemma 3.8 and $0 < \delta < \delta^*$. Then from Lemma 3.8 we see, for $r' = r/c$, that

$$c^{-1} \frac{\hat{u}(x)}{d(x, \partial\Omega)} \leq |\nabla \hat{u}(x)| \leq c \frac{\hat{u}(x)}{d(x, \partial\Omega)} \tag{4.24}$$

whenever $x \in \Omega \cap B(w, r')$, provided $c = c(p, n, N, M, C)$ is large enough. Let

$$b_{ij}(x) = |\nabla \hat{u}(x)|^{p-4}[(p-2)\hat{u}_{x_i}(x)\hat{u}_{x_j}(x) + \delta_{ij}|\nabla \hat{u}|^2(x)], \tag{4.25}$$

whenever $x \in \Omega \cap B(w, r')$ and $1 \leq i, j \leq n$. In the following we consider weak solutions ζ, in $\Omega \cap B(w, r')$, to the equation

$$L\zeta \;=\; \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(b_{ij}(x)\zeta_{x_j}(x) \right) = 0. \tag{4.26}$$

Using (4.25), (4.24) we see that L is a locally uniformly elliptic operator in $\Omega \cap B(w, r')$. Moreover, recall from the remark after (1.21) that $\hat{u}$ is a solution to L in $\Omega \cap B(w, r')$.

Next we outline the proof of a boundary Harnack inequality for positive solutions to L vanishing on a portion of $\Delta(w, r')$. We note that the argument follows the lines of [LN, sections 2 and 3] where the corresponding boundary Harnack inequalities were established in the setting of Lipschitz domains. To start with observe that if $z \in \Delta(w, r'/2)$ and $0 < \rho < r'/100$, then there exists, as we see using Lemma 4.14, a Lipschitz domain $\Omega^* \subset \Omega \cap B(z, \rho)$ with center $\tilde{z}$ and such that

$$|\nabla \hat{u}(x)| \approx \hat{u}(\tilde{z})/\rho \text{ whenever } x \in \Omega^*. \tag{4.27}$$

In (4.27) the constants of proportionality depend only on p, n, N, M and C and we achieve the estimate in (4.27) by combining Lemma 4.14 and (4.24). We define

$$d\gamma(x)/dx = d(x, \partial\Omega^*) \max_{B(x, \frac{1}{2}d(x, \partial\Omega^*))} \{|\nabla \hat{u}|^{2p-6} \sum_{i,j=1}^{n} \hat{u}^2_{y_i y_j}\}$$

whenever $x \in \Omega^*$. (4.27) then implies that the following is true.

Lemma 4.28. *γ is a Carleson measure on Ω^* in the sense that if $\hat{x} \in \partial\Omega^*$ and $0 < t < \rho$, then*

$$\gamma(\Omega^* \cap B(\hat{x}, t)) \;\leq\; c\, t^{n-1} \,(\hat{u}(\tilde{z})/\rho)^{2p-4}$$

for some $c = c(p, n, N, M, C)$.

Proof: Lemma 4.28 follows from Schauder type estimates for the second derivatives of $\hat{u}$, (4.27), and integration by parts. A complete proof is given in [LN, Lemma 2.54]. We omit the details.□

From Lemma 4.28 and the definition of $\{b_{ij}\}$ in (4.25) we observe that if

$$d\tilde{\gamma}(y) = \sum_{i,j=1}^{n} d(y, \partial\Omega^*) \max_{\bar{B}(y, \frac{1}{2}d(y, \Omega^*))} |\nabla b_{ij}|^2 \, dy \text{ whenever } y \in \Omega^*,$$

then $\tilde{\gamma}$ is a Carleson measure on Ω^* satisfying,

$$\sup_{0<t<\rho} \frac{\tilde{\gamma}(B(\hat{x},t)\cap\Omega^*)}{t^{n-1}} \leq c\left(\frac{\hat{u}(\tilde{z})}{\rho}\right)^{2p-4} \text{ whenever } \hat{x}\in\partial\Omega^*. \tag{4.29}$$

Let $\omega^*(\cdot)$ be elliptic measure for Ω^* defined relative to $L, \tilde{z}$. We refer to [LSW] for the definition of the elliptic measure. Then, using (4.29) and a theorem of [KP] one can prove the following lemma, Lemma 4.30. For a complete proof of Lemma 4.30 we refer to [LN, Theorem 3.11] and for equivalent characterizations of A^∞ we refer to [CF].

Lemma 4.30. *ω^* is an A^∞-weight with respect to surface measure, σ, on $\partial\Omega^*$ with constants depending only on p,n,N,M,C. In particular, there exist c, $1\leq c<\infty$, and θ_1,θ_2, both small, all depending only on p,n,N,M,C, such that whenever $\hat{x}\in\partial\Omega^*$ and t are as in (4.28) then the following is true.*

$$\omega^*(\partial\Omega^*\cap B(\hat{x},t)) \leq c\,\omega^*(\partial\Omega^*\cap B(\hat{x},t/2))$$

and if E is a Borel subset of $\partial\Omega^*\cap B(\hat{x},t)$, then

$$\frac{\sigma(E)}{\sigma(\partial\Omega^*\cap B(\hat{x},t))} \geq 1-\theta_1 \text{ implies that } \frac{\omega^*(E)}{\omega^*(\partial\Omega^*\cap B(\hat{x},t))} \geq \theta_2.$$

Next we let ω be elliptic measure defined relative to $L, a_\rho(z)$, in $\Omega\cap B(w,r')$. It then follows from Lemmas 4.30, 4.14, the maximum principle for solutions to L and Harnack's inequality, that

$$\omega(\partial\Omega\cap B(z,\rho)) \geq \omega(\partial\Omega^*\cap\partial\Omega) \geq c^{-1}\omega^*(\partial\Omega^*\cap\partial\Omega) \geq c^{-2}. \tag{4.31}$$

In [LN, Lemma 3.13] it is shown that once (4.31) is established we are able to conclude the following boundary Harnack inequality for the operator L.

Lemma 4.32. *Let $\hat{u}, L, \Omega, w, r, r'$ be as in (4.24)-(4.26) and suppose that v_1, v_2 are two positive weak solutions to $L\zeta = 0$ in $\Omega\cap B(w,r')$ with $v_1 = 0 = v_2$ continuously on $\Delta(w,r')$. There exist $c\geq 1, \alpha, 0<\alpha<1$, depending only on p,n,N,M,C such that if $\hat{r} = r'/c$, and $v_1(a_{\hat{r}}(w)) = v_2(a_{\hat{r}}(w))$, then*

$$\left|\frac{v_1(y_1)}{v_2(y_1)} - \frac{v_1(y_2)}{v_2(y_2)}\right| \leq c\left(\frac{|y_1-y_2|}{r}\right)^\alpha$$

whenever $y_1, y_2 \in \Omega\cap B(w,\hat{r})$.

4.3 Proof of Theorem 2

To prove Theorem 2 we use the same strategy as in proving Theorem 1. Analogous to Lemma 3.38 we state the following.

Lemma 4.33. *Let $\Omega\subset\mathbf{R}^n$ be an Ahlfors regular NTA-domain with constants M, C, r_0, which is uniformly (δ, N, r_0)-approximable by Lipschitz graph domains and let p, w, r, u, v, be as in the statement of Theorem 2. Assume that u and v satisfy (1.11) and let $u(\cdot,\tau)$, $0\leq\tau\leq 1$, be the p-harmonic function in $\Omega\cap B(w,4r^*)$ defined as in (1.12). Then there exist $\delta_2 = \delta_2(p,n,N,M,C) > 0$ such that if $0<\delta<\delta_2$ then Claims 1 and 2 are true in the case of*

Theorem 2.

Proof: From Lemma 3.8 we deduce that there exists $\delta_2 = \delta_2(p, n, N, M, C) > 0$ such that if $0 < \delta < \delta_2$, then (4.24) holds with $\hat{u}$ replaced by $u(\cdot, \tau)$ in $\Omega \cap B(w, r/c)$, for some $c = c(p, n, N, M, C)$ whenever $\tau \in [0, 1]$. Using this deduction and arguing as following (3.39), (3.40) we get Claim 1 under assumption (1.11). Next observe for fixed $\tau \in (0, 1]$ that if $\hat{u} = u(\cdot, \tau)$, then (1.20), (1.21) are the same as (4.25),(4.26). Using this observation and (1.22), (1.23), we see that Claim 2 is just Lemma 4.32 with $2r''$ replaced by r'. Thus Claim 2 is also valid under assumption (1.11). $\Box$

Finally to complete the proof of Theorem 2 using Lemma 4.33 we only have to get rid of assumption (1.11). But this can be done exactly as we did in the proof of Theorem 1 (see the discussion after (3.41)). The proof of Theorem 2 is therefore complete. $\Box$

5 The Martin Boundary Problem

In this section we consider the Martin boundary problem assuming that Ω is as in either Theorem 3 or Theorem 4.

5.1 Preliminary definitions

We shall need the following definitions in our discussion of the Martin boundary problem.

Definition 5.1. *Fix $p, 1 < p < \infty$. Then $\hat{u}$ is said to be a minimal positive p-harmonic function in Ω relative to $w \in \partial\Omega$, provided $\hat{u} > 0$ is p-harmonic in Ω and $\hat{u}$ has continuous boundary value* 0 *on $\partial\Omega \setminus \{w\}$. $\hat{u}$ is said to be unique up to constant multiples if $\hat{v} = \lambda\hat{u}$, for some constant λ, whenever $\hat{v}$ is a minimal positive p-harmonic function relative to $w \in \partial\Omega$.*

Definition 5.2. *We say that the p-Martin boundary of Ω can be identified with $\partial\Omega$ provided each $w \in \partial\Omega$ corresponds to a unique (up to constant multiples) minimal positive p-harmonic function.*

Definition 5.3. We call $\tilde{\Omega} \subset \Omega$ a non-tangential approach region at $w \in \partial\Omega$ if

$$\tilde{\Omega} = \{x \in \Omega : d(x, \partial\Omega) \geq \tilde{\eta}|x - w|\}$$

for some $\tilde{\eta}, 0 < \tilde{\eta} < 1$. If we wish to emphasize $w, \tilde{\eta}$ in Definition 5.3 we write $\tilde{\Omega}(w, \tilde{\eta})$.

5.2 Estimates for minimal p-harmonic functions in domains beyond Lipschitz domains

Let $\Omega \subset \mathbf{R}^n$ be a NTA-domain with constants M, r_0 and assume that $\Omega \in \mathcal{F}(\delta, N, r_0)$. Moreover, let $\hat{u}$ be a minimal positive p-harmonic function in Ω relative to $w \in \partial\Omega$. We can then apply Lemma 3.8 to conclude that there exist $\delta^* = \delta^*(p, n, N, M)$ and $\hat{c} = \hat{c}(p, n, N, M) \geq 1$ such that if $0 < \delta \leq \delta^*$, then, for each $\hat{x} \in \partial\Omega \setminus \{w\}$, the 'fundamental inequality',

$$c^{-1}\frac{\hat{u}(x)}{d(x,\partial\Omega)} \leq |\nabla \hat{u}(x)| \leq c\frac{\hat{u}(x)}{d(x,\partial\Omega)} \tag{5.4}$$

holds whenever $x \in \partial\Omega \cap B(\hat{x}, |\hat{x}-w|/c) \cap B(w,r_0)$. In particular, using this fact we see that if $0 < \delta \leq \delta^*$ in either Theorem 3 or Theorem 4, then there exist $\tilde{\eta}, c$, depending only on p, n, N, M, C such that

$$\hat{u} \text{ satisfies (5.4) in } [\Omega \setminus \tilde{\Omega}(w,\tilde{\eta})] \cap B(w,r_0). \tag{5.5}$$

Using (5.5) and Theorem 1 and 2 we can now state a weakened version of Theorem 3 in [LN1]. In particular, we have the following sufficient criteria for the conclusion that a minimal positive p-harmonic function is unique (relative to $w \in \partial\Omega$).

Lemma 5.6 *Let $0 < \delta < \min(\delta_1, \delta_2)$, Ω as in either Theorem 3 or Theorem 4, and let $\tilde{\eta}$ be as in (5.5). Fix $p, 1 < p < \infty$, and suppose that $\hat{u}$ is a minimal positive p-harmonic function relative to $w \in \partial\Omega$. If there exist $A \geq 1, \hat{r}$, such that*

$$A^{-1}\frac{\hat{u}(x)}{d(x,\partial\Omega)} \leq |\nabla \hat{u}(x)| \leq A\frac{\hat{u}(x)}{d(x,\partial\Omega)} \text{ whenever } x \in B(w,\hat{r}) \cap \tilde{\Omega}(w,\tilde{\eta}),$$

then $\hat{u}$ is unique relative to w (up to constant multiples).

Proof: To outline the proof of the lemma we let $\hat{v}$ be another minimal positive p-harmonic function relative to $w \in \partial\Omega$. To prove Lemma 5.6 the idea is to show the existence of $r_1 < \min(\hat{r}, r_0), c \geq 1$, and $0 < \alpha < 1$, depending only on p, n, N, M, C, A , such that if $0 < r < r_1/c$, then

$$\left|\log\frac{\hat{u}(y_1)}{\hat{v}(y_1)} - \log\frac{\hat{u}(y_2)}{\hat{v}(y_2)}\right| \leq c\left(\frac{r}{\min(\hat{r}, |y_1-w|, |y_2-w|)}\right)^{\alpha} \tag{5.7}$$

whenever $y_1, y_2 \in \Omega \setminus B(w, cr)$. Letting $r \to 0$ in (5.7), we then get Lemma 5.6.

To prove (5.7) we can use the same ideas as in the proof of Theorems 1, 2, except that now arguments are given in $\Omega \setminus B(w,r')$ for r' sufficiently small. To outline this proof, let $u(\cdot,\tau)$, $\tau \in [0,1]$, be the p-harmonic function in $\Omega \setminus \bar{B}(w,r')$ which has continuous boundary values $\tau\hat{v} + (1-\tau)\hat{u}$ on $\partial(\Omega \setminus \bar{B}(w,r'))$. Multiplying $\hat{v}$ by a constant if necessary and using either Theorem 1 or Theorem 2, as well as Harnack's inequality, we see that one can assume

$$c^{-1} \leq \hat{u}/\hat{v} \leq 1/2 \tag{5.8}$$

in Ω for some $c = c(p,n,N,M,C)$. Moreover, if we knew, for some $\bar{\lambda} = \bar{\lambda}(p,n,N,M,C,A) \geq 1$ and $\tilde{r}, 2r' < \tilde{r} < r/2$, that

$$\bar{\lambda}^{-1}\frac{u(x,\tau)}{d(x,\partial\Omega)} \leq |\nabla u(x,\tau)| \leq \bar{\lambda}\frac{u(x,\tau)}{d(x,\partial\Omega)} \text{ whenever } 0 < \tau < 1,\ x \in \Omega \setminus B(w,\tilde{r}), \tag{5.9}$$

then we could repeat the argument following (1.12) in section 1 to reduce the proof of (5.7) to claims analogous to Claims 1 and 2. We could then use Lemma 3.29 or Lemma 4.32 to establish these claims. We refer to Lemma 5.5 in [LN1] for a proof in a similar situation. Unfortunately though, we can not apriori assume that (5.9) holds, which is an important difference compared to the situation in Theorem 1 and 2, where we could use Lemma 3.8.

To overcome this difficulty we use a method from [LN1] based on a continuity-induction type argument. To describe this method, we first observe, using Lemma 3.1, that there exists $\epsilon_0 = \epsilon_0(p, n, N, M, C, A)$ such that if, for some $\hat{L}$ and $\rho \geq r'$, the following holds

$$(1-\epsilon_0)\hat{L} \leq \frac{u(\cdot,\tau)}{\hat{u}(\cdot)} \leq (1+\epsilon_0)\hat{L} \quad \text{in } \Omega \setminus B(w,\rho), \tag{5.10}$$

then, by the assumption on $|\nabla \hat{u}|$, (5.9) holds in $\Omega \setminus B(w, 2\rho)$ for some $\bar{\lambda}$ having the desired dependence. Now using (5.8) and the maximum principle for p-harmonic functions it is easily seen that (5.10) holds for $\tau \in [0, \epsilon_1]$ with $\hat{L} = 1$ whenever $x \in \Omega \setminus B(w, r')$ provided $\epsilon_1 = \epsilon_0/c$ and c is large enough. Thus (5.9) is valid in $\Omega \setminus B(w, 2r')$ whenever $\tau \in [0, \epsilon_1]$. Based on this we can then follow the game plan outlined below (5.9) to get (5.7) with $\hat{u}, \hat{v}$ replaced by $\hat{u}, u(\cdot, \epsilon_1)$ and with r replaced by a multiple of r'.

In the next step we use (5.7), applied to $\hat{u}, u(\cdot, \epsilon_1)$, and the assumption in (5.8) to deduce that (5.10) holds in $\Omega \setminus B(w, cr')$, provided c is large enough, whenever $\tau \in [\epsilon_1, 2\epsilon_1]$ and for suitably chosen $\hat{L}$. Then, arguing as we did below (5.10) we get (5.7) in $\Omega \setminus B(w, c'r)$, for some large c', and with $\hat{u}, \hat{v}$ replaced by $\hat{u}, u(\cdot, 2\epsilon_1)$. Moreover, iterating this argument we get, after k steps, that (5.7) holds in $\Omega \setminus B(w, c_k r)$, for some large c_k, and with $\hat{u}, \hat{v}$ replaced by $\hat{u}, u(\cdot, k\epsilon_1)$. Continuing in this manner we eventually get (5.9) after $m < 2/\epsilon_1$ steps in $\Omega \setminus B(w, c_m r)$. The proof also gives (5.7) since $u(\cdot, 1) = \hat{v}(\cdot)$.

An important part of the argument we have outlined above is to show that c_m can be chosen to have the same dependence as the constant in (5.7). Still this can be achieved and we refer to [LN1, displays (5.19)-(5.24)] for a proof in a similar situation and to [LN1, Corollary 5.25]. We omit further details. $\Box$

5.3 Proof of Theorems 3 and 4

Using Lemma 5.6 we see that to complete the proof of Theorem 3 it suffices to show that if δ_+ is sufficiently small, Ω is δ-Reifenberg flat, and $0 < \delta < \delta_+$, then there exists, for each $w \in \partial\Omega$, a minimal positive p-harmonic function $\hat{u}$, satisfying

$$\tilde{c}^{-1}\frac{\hat{u}(x)}{d(x,\partial\Omega)} \leq |\nabla \hat{u}(x)| \leq \tilde{c}\frac{\hat{u}(x)}{d(x,\partial\Omega)} \text{ whenever } x \in \Omega(w,\tilde{\eta}) \cap B(w,\hat{r}) \tag{5.11}$$

and for some $\tilde{c} = \tilde{c}(p, n, N, M, C)$. Moreover, in the case of Theorem 4 we need to obtain this conclusion when Ω has a tangent plane at $w \in \partial\Omega$. A complete proof of the existence of $\hat{u}$ satisfying (5.11), in the case of Theorem 3 as well as Theorem 4, can be given by essentially copying verbatim the proof for C^1-domains in Theorem 4 of [LN1, displays (6.4)-(6.21)]. In fact, the proof in [LN1, (6.4)-(6.21)] only uses that $\partial\Omega \cap B(w, r)$ is 'flat' if r is sufficiently small and this fact is valid in the case of C^1-domains, δ-Reifenberg flat domains, and at points $w \in \partial\Omega$ where Ω has a tangent plane.

To briefly outline the proof, let $\hat{u}$ be a minimal positive p-harmonic function relative to $w \in \partial\Omega$. Extend $\hat{u}$ to be continuous in $\mathbf{R}^n \setminus \{w\}$ by putting $\hat{u} \equiv 0$ on $\mathbf{R}^n \setminus \Omega$. We assume as we may that $w = 0$ and after a rotation, if necessary, that there exists $\tilde{r} = \tilde{r}(\sigma)$, for given $\sigma > 0$ (sufficiently small), such that if $0 < r \leq \tilde{r}$, then

$$B(0,nr)\cap\{y: y_n\geq \sigma r\}\subset\Omega,\quad B(0,nr)\cap\{y: y_n\leq -\sigma r\}\subset\mathbf{R}^n\setminus\Omega. \tag{5.12}$$

Let

$$Q=\{y: |y_i|<r, 1\leq i\leq n-1\}\cap\{y: \sigma r<y_n<r\}\setminus B(0,\sqrt{\sigma}\,r\,)$$

and let v_1 be the p harmonic function in Q with the following continuous boundary values,

$$\begin{aligned} v_1(y) &= \hat u(y) &&\text{if } y\in\partial Q\cap\{y: 2\sigma r\leq y_n\},\\ v_1(y) &= \frac{(y_n-\sigma r)}{\sigma r}\hat u(y) &&\text{if } y\in\partial Q\cap\{y: \sigma r\leq y_n<2\sigma r\}\,. \end{aligned}$$

Comparing boundary values and using the maximum principle for p-harmonic functions, we deduce

$$v_1\leq\hat u \text{ in Q}. \tag{5.13}$$

Let $\sigma(\epsilon)=\exp(-1/\epsilon)$. To prove Lemma 5.6 one shows

Lemma 5.14. *Let $0<\epsilon\leq\hat\epsilon$, let $\sigma=\sigma(\epsilon)$ be as above and let $\tilde\eta$ be as in (5.5). If $\hat\epsilon$ is small enough, then there exists $\hat\theta=\hat\theta(p,n,N,M,C), 0<\hat\theta\leq 1/2$, such that if $\hat\rho=\sigma^{1/2-\hat\theta}r$, then*

$$1\,\leq\,\hat u(y)/v_1(y)\,\leq\,1+\epsilon$$

whenever $y\in\tilde\Omega(0,\tilde\eta/16)\cap[B(0,\hat\rho)\setminus B(0,4\sqrt{\sigma}r)]$.

Lemma 5.15 *Let $v_1,\epsilon,\hat\epsilon,\hat\theta,r,\sigma,$ be as in Lemma 5.14 and let $\tilde\eta$ be as in (5.5). If $\hat\epsilon$ is small enough, then there exist $\theta=\theta(p,n,N,M,C), 0<\theta\leq\hat\theta/10$, and $\hat\delta=\hat\delta(p,n,N,M,C)>1$ such that if $\rho=\sigma^{1/2-4\theta}r, b=\sigma^{-\theta}$, then*

$$\hat\delta^{-1}\frac{v_1(x)}{d(x,\partial\Omega)}\,\leq\,|\nabla v_1(x)|\,\leq\hat\delta\frac{v_1(x)}{d(x,\partial\Omega)}$$

whenever $x\in\tilde\Omega(0,\tilde\eta/4)\cap[B(0,b\rho)\setminus B(0,\rho/b)]$ and $0<\epsilon\leq\hat\epsilon$.

Assuming Lemmas 5.14, 5.15, we can use Lemma 3.1 to deduce, for $\hat\epsilon$ sufficiently small, that

$$\tilde\lambda^{-1}\frac{\hat u(x)}{d(x,\partial\Omega)}\,\leq\,|\nabla\hat u(x)|\,\leq\tilde\lambda\frac{\hat u(x)}{d(x,\partial\Omega)}\ \text{ in }\Omega(0,\tilde\eta)\cap[B(0,b^{1/2}\rho)\setminus B(0,\rho/b^{1/2})] \tag{5.16}$$

for some $\tilde\lambda>1$, depending only on p,n,N,M,C. With $\hat\epsilon$ now fixed we conclude from arbitrariness of r that (5.11) is true.

To prove Lemma 5.14 one uses (5.13) and Lemmas 2.1 - 2.4 for $\hat u$, to make estimates. In the proof of Lemma 5.15 one compares v_1 to several auxillary functions, satisfying boundary Harnack inequalities, and eventually uses Lemmas 3.1, 3.8, once again to get the conclusion in this lemma. For more details the reader is referred to [LN1] where Lemmas 5.14, 5.15, appear as Lemmas 6.6, 6.7, respectively. □

5.4 Closing remarks.

We note that Theorem 4 can be improved upon as follows. Let Ω be as in Theorem 4, $w \in \partial\Omega$, and put

$$f(r) = \inf_P h(\partial\Omega \cap B(w,r), P \cap B(w,r))$$

where p is an $(n-1)$-dimensional plane containing w and h is Hausdorff distance defined in section 1. There exists $\zeta = \zeta(p, n, N, M, C) > 0$ such that if

$$\liminf_{r \longrightarrow 0} r^{-1} f(r) \leq \zeta,$$

then minimal positive p-harmonic functions relative to w are unique up to constants. To prove the sufficiency of the above display one needs to prove a stronger version of Lemma 5.6 similar to Theorem 3 in [LN1]. After that the argument outlined after Lemmas 5.14, 5.15, can be used with only minor changes (see Corollary 6.22 in [LN1]).

Finally we mention that the 'size' of $\delta_1, \delta_2, \delta_+$ in Theorems 1 - 3 and its dependence on p is of interest to us. For example is it true that necessarily

$$\delta_1 \rightarrow 0 \text{ as } p \rightarrow \infty. \tag{5.17}$$

In fact our proof of Theorem 1 is based on Lemma 3.30 which is most likely false for large p unless (5.17) holds.

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