On the L^p Neumann and regularity problem for the heat equation in non cylindrical domains

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1 Introduction

In this note, we briefly discuss recent work on the Neumann and regularity problems for the heat equation in certain non-smooth, time-varying domains. We shall only state our results, and describe some related history. Details of the proofs will appear elsewhere. The class of domains which we consider are those given by the region above a time-varying graph:

$$\Omega \equiv \{(x_0, x, t) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} : x_0 > A(x,t)\}. \tag{1.1}$$

Here, A is Lipschitz in x, uniformly in time, i.e.,

$$\sup_{x,t} |A(x,t) - A(x+h,t)| \leq \beta_0 |h|, \tag{1.2}$$

for some $\beta_0 < \infty$; furthermore, A satisfies a certain half-order smoothness condition in t, which we shall now describe. Following Fabes and Riviere [?], we define the half-order time derivative

$$\mathbb{D}_n \equiv \frac{\partial}{\partial t} \circ \left(\frac{\partial}{\partial t} - \Delta \right)^{-\frac{1}{2}}; \tag{1.3}$$

that is, on the Fourier transform side,

$$(\mathbb{D}_n A)^\wedge(\xi, \tau) \equiv c \frac{\tau}{\sqrt{|\xi|^2 - i\tau}} \hat{A}(\xi, \tau),$$

where, obviously, ξ and τ denote the Fourier transform variables in space and time, respectively. We shall assume that

$$\|\mathbb{D}_n A\|_* \leq \beta_1 < \infty, \tag{1.4}$$

where the parabolic BMO norm is, as usual, defined by

$$\|b\|_* = \sup_B \frac{1}{|B|} \int_B |b(x) - b_B| dz.$$

Here B denotes an arbitrary parabolic ball

$$B \equiv \{z \in \mathbb{R}^n : \|z - z_0\| < r\},$$

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and, for non-zero $z = (x, t) \in \mathbb{R}^n$, the parabolic norm $||z|| = ||(x,t)||$ is defined to be the unique positive solution ρ of the equation

$$\frac{|x|^2}{\rho^2} + \frac{t^2}{\rho^4} = 1.$$

It is well known, and easy to verify, that

$$||(x,t)|| \cong |x| + |t|^{\frac{1}{2}},$$

and that

$$||(rx, r^2 t)|| = r||(x,t)||.$$

We remark that by an argument of Strichartz [?], the conditions (??) and (??) imply, and are slightly stronger than, the following $Lip_{1,1/2}$ condition in (x,t) :

$$\sup_{x,t} |A(x,t) - A(x+h,t)| + \sup_{x,t} |A(x,t) - A(x,t+h^2)| \leq C(\beta_0 + \beta_1)|h|. \tag{1.5}$$

We also note that, together, (??) and (??) are equivalent to the L^2 boundedness of the parabolic Calderon commutator $[(\frac{\partial}{\partial t} - \ Delta)^{1/2}, A]$ (see [?]).

Having defined the class of domains which we shall consider, we are now in a position to define the parabolic Sobolev spaces on $\partial\Omega$, in which spaces we shall take our boundary data. For each fixed t, let Ω_t denote the cross-section

$$\Omega_t \equiv \{(x_0, x) \in \mathbb{R} \times \mathbb{R}^{n-1} : x_0 > A(x,t)\}.$$

By (??), Ω_t is a Lipschitz domain with Lipschitz constant no larger than β_0. We define $d\sigma_t$ to be the usual surface measure on the Lipschitz graph $\partial\Omega_t$, i.e., in graph co-ordinates,

$$d\sigma_t \equiv \sqrt{1 + |\nabla_x A(x,t)|^2} dx.$$

We then define "surface measure" $d\sigma$ on $\partial\Omega$ as

$$d\sigma \equiv d\sigma_t dt.$$

The parabolic Sobolev spaces $L^p_{1,1/2}(\mathbb{R}^n)$ are given by $L^p_{1,1/2}(\mathbb{R}^n) \equiv \left(\frac{\partial}{\partial t} - \Delta\right)^{-1/2}(L^p(\mathbb{R}^n))$, at least for $1 < p < n+1$ (in this paper, p will always lie in this range, and, typically $p \leq 2$). By parabolic singular integral theory (see [?]),

$$||f||_{L^p_{1,\frac{1}{2}}(\mathbb{R}^n)} \cong ||\nabla_x f||_{L^p(\mathbb{R}^n)} + ||\mathbb{D}_n f||_{L^p(\mathbb{R}^n)}.$$

Since A is Lipschitz, in graph co-ordinates

$$d\sigma(x,t) \equiv d\sigma_t(x) dt \cong dx dt;$$

thus, we can naturally define (following [?]) $L^p_{1,\frac{1}{2}}(\partial\Omega)$ by setting

$$||f||_{L^p_{1,\frac{1}{2}}(\partial\Omega)} \equiv ||\tilde{f}||_{L^p_{1,\frac{1}{2}}(\mathbb{R}^n)}$$

where $\tilde{f}(x,t) \equiv f(A(x,t), x, t)$.

In this paper, we consider the regularity problem

$$R_p \begin{cases} \Delta u - \frac{\partial u}{\partial t} = 0 & \text{in } \Omega \\ u|_{\partial\Omega} = f \in L^p_{1,\frac{1}{2}}(\partial\Omega) \\ N_*(\nabla u) \in L^p(\partial\Omega), \end{cases} \tag{1.6}$$

and the Neumann problem

$$N_p \begin{cases} \Delta u - \frac{\partial u}{\partial t} = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial N}|_{\partial\Omega} = f \in L^p(\partial\Omega) \\ N_*(\nabla u) \in L^p(\partial\Omega), \end{cases} \tag{1.7}$$

where for any given point $(P,t) \in \partial\Omega$, $\frac{\partial}{\partial N}$ denotes differentiation in the direction of the outer unit normal to $\partial\Omega_t$ at the point P.

Here N_* is the parabolic non-tangential maximal operator

$$N_*(F)(A(x,t),x,t) \equiv \sup_{\Gamma} |F(y_0,y,s)|,$$

and $\Gamma \equiv \Gamma(A(x,t),x,t)$ is the parabolic cone

$$\Gamma \equiv \{(y_0,y,s) : \|(x-y,t-s)\| \le \mu(y_0 - A(x,t))\}$$

It is not hard to see that for μ small enough, depending only on β_0 and β_1 in (??) and (??), one has that $\Gamma(A(x,t),x,t) \subseteq \Omega$, at every point $(A(x,t),x,t) \in \partial\Omega$. Indeed, the non-tangential accessibility follows easily from (??).

Our results are the following:

Theorem 1.8. *Given a domain Ω as in (??), which satisfies (??) and (??), then $\exists\, p_0 > 1$ such that the regularity problem R_p is uniquely solvable in the range $1 < p < p_0$. Here p_0 can be taken to depend only on β_0, β_1, and the dimension n.*

Theorem 1.9. *Let Ω be a domain of the type considered in the previous Theorem. Then, given $p \in (1,2]$, there exists $\epsilon_p \equiv \epsilon(n,p,\beta_0)$, with $\epsilon_p \to 0$ as $p \to 1$, such that the Neumann problem N_p is uniquely solvable whenever, in addition, we assume that the constant β_1 in (??) satisfies $\beta_1 \le \epsilon_p$. Furthermore, there exist counterexamples (with $\beta_0 = 0$) which show that the assumption $\epsilon_p \to 0$ as $p \to 1$, cannot, in general, be relaxed. In particular, it is impossible, in general, to obtain L^1 bounds for $N_*(\nabla u)$, when the Neumann data lies in the Hardy space $H^1(\partial\Omega)$.*

We remark that the domains in the counterexamples alluded to in the second theorem, are given by the region outside of a half-order "cusp." The key observation is that, when the domain is completely flat in the space variable, the Neumann and Dirichlet problems become essentially equivalent. We also note that, by standard functional analysis arguments, Theorem ?? extends to the case $p < 2+\delta$, $\delta = \delta(n,\beta_1)$. The case $p = 2$ has already appeared in [?].

To put these results into context, let us review a bit of recent history. The class of domains which we consider here was introduced by the second author (Lewis) and M. Murray in [?], albeit with a slightly different, but in retrospect equivalent, formulation of condition (??). In [?], the following result was proved.

Theorem 1.10. *Given a domain Ω as in Theorem ??, there exists a $q_0 < \infty$ such that the adjoint Dirichlet problem (D_q) (defined below) is uniquely solvable in the range $q_0 < q < \infty$. Here q_0 can be taken to depend only on β_0, β_1 and dimension.*

The adjoint Dirichlet problem (D_q^*) entails finding a solution v to the following problem:

$$D_q^* \begin{cases} \Delta v + \frac{\partial v}{\partial t} = 0 & \text{in } \Omega \\ v|_{\partial\Omega} = f \in L^q(\partial\Omega) & \\ N_* v \in L^q(\partial\Omega) & \end{cases}$$

(Remark—by the change of variable $t \to -t$, it is equivalent to solve D_q). The exponents p and q in Theorems ?? and ??, respectively, are dual to each other, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. Indeed, our proof of Theorem ?? is based on a technique introduced by Verchota [?] in the case of harmonic functions in a Lipschitz domain, and depends on showing that the solvability of D_q^* implies that of R_p, $\frac{1}{p} + \frac{1}{q} = 1$. To do the latter is a fairly straightforward matter of adapting Verchota's proof to our situation, except that we need to handle some bad terms which arise as a consequence of the non-cylindrical nature of our domains. We remark that the converse to Theorem ?? (namely that $R_p \to D_q^*$) is easy, and has been noted in [?]. Furthermore, in [?] it is shown that Theorem ?? is optimal, in the sense that even when $\beta_0 = 0$ (i.e., the case that $A(x,t) = A(t)$), one can construct a class of domains for which solvability of R_p, for any given $p > 1$, can be made to fail by taking β_1 large enough. I.e., one can never hope to fix a p for which R_p holds in all domains Ω of the type considered here: to do so, one must impose some restriction on the size of β_1. An optimal theorem of the latter sort was proved in [?]; namely that R_2,and also N_2 (as mentioned above) and D_2, hold for domains of the type considered here for arbitrary $\beta_0 < \infty$, if β_1 is small enough depending only on β_0 and dimension.

The results of [?] and [?] were thus extensions to the non-cylindrical case of work of Fabes and Salsa [?] and R. Brown [?], [?], who had proved that if $\beta_1 = 0$ (i.e. $A(x,t) \equiv A(x)$), then one has solvability of D_q^* ([FS]), and R_p and the L^p Neumann problem ([Br1,2]), in the optimal ranges $2-\delta < q < \infty$, $1 < p < 2+\delta$. Given these theorems in the cylindrical case, and also the prior work in the harmonic case of Verchota [?], and of Dahlberg and Kenig [?], it had been a reasonable conjecture that R_p, and also the L^p Neumann problem, should be solvable for p in the dual range to that of Theorem ??; Theorem ?? states that this is indeed true for R_p. What is surprising though, is that in contrast to [?] and [?], this is not at all the case for the Neumann problem, as the counterexamples of Theorem ?? demonstrate. An interesting feature of the theory in non-cylindrical domains, then, is the dichotomy between the regularity and Neumann problems.

In the next section, we conclude the paper by briefly sketching some ideas of the proof of Theorem ??, which is significantly more difficult than that of Theorem 1.8.

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2 Outline of the Proof of Theorem 1.9

As we have mentioned above, there is no H^1 theory for the Neumann problem in this setting. Thus, one cannot apply directly the techniques introduced by Dahlberg and Kenig [?] in the context of harmonic functions in Lipschitz domains, which techniques were subsequently extended to the case of the heat equation in cylindrical domains by R. Brown [?], and to the case of divergence form elliptic operators with L^∞ coefficients, by Kenig and Pipher (see the exposition in [?]). Nonetheless, those techniques will be of use to us, in proving estimates for solutions to a certain auxiliary equation.

The idea is as follows. We define a parabolic approximate identity

$$\begin{aligned} P_\lambda f(x,t) &\equiv \varphi_\lambda * f(x,t) \\ &\equiv \lambda^{-d} \iint \varphi\left(\frac{x-y}{\lambda}, \frac{t-s}{\lambda^2}\right) f(y,s)dyds, \end{aligned}$$

where $d = n+1$ is the homogeneous dimension of parabolic $\mathbb{R}^n$, and where $\varphi \in C_0^\infty$, supp$\varphi \subseteq \beta_1(0)$ (the unit ball), $\varphi \geq 0$, φ is even and $\int \varphi = 1$. Next, we choose a small (fixed) constant $\gamma > 0$, depending only on the constant $C(\beta_0 + \beta_1)$ in (??), such that for all $(x,t) \in \mathbb{R}^n$,

$$\frac{1}{2} \leq 1 + \frac{\partial}{\partial \lambda} P_{\gamma\lambda} A(x,t) \leq 3/2.$$

Consider the Dahlberg-Kenig-Stein mapping

$$\rho(\lambda, x, t) \equiv (\lambda + P_{\gamma\lambda} A(x,t), x, t),$$

which defines a $1-1$ mapping of the half-space $\mathbb{R}_+^{n+1} \equiv \{(\lambda, x, t) = \lambda > 0, (x,t) \in \mathbb{R}^n\}$ onto Ω, and furthermore $\rho : \partial\mathbb{R}_+^{n+1} \to \partial\Omega$. We remark that this mapping appeared first in a paper of Dahlberg [?] (although this explicit construction was due to Kenig and Stein), and has recently proven useful in our work on the L^2 versions of these theorems [?], as well as in work of Dahlberg, Kenig, Pipher and Verchota on square function estimates for constant coefficient elliptic equations and systems in Lipschitz domains.

If u is a solution of the heat equation in Ω, then the "pullback" $v \equiv u \circ \rho$ is a solution, in the half-space, of a divergence form parabolic equation

$$div(a(\lambda, x, t)\nabla v - (1 + \epsilon(\lambda, x, t))\frac{\partial v}{\partial t} + \mathbf{B}(\lambda, x, t) \cdot \nabla v \ = \ 0,$$

where

$$|\mathbf{B}(\lambda, x, t)| \leq C\,\lambda^{-1}$$

and

$$|\mathbf{B}(\lambda, x, t)|^2 \lambda d\lambda dx dt$$

is a parabolic Carleson measure (with small norm, since β_1 is small). It turns out that this singular lower order term causes the difficulties near $p = 1$; indeed, dropping the drift term, we obtain an operator to which the Dahlberg-Kenig-Pipher H^1 techniques can be applied. We then view our original operator (or rather it's pullback) as a perturbation of the one without the drift term. Since the error is controlled by a Carleson measure with small norm, we can adapt the perturbation techniques for the Neumann problem, developed for elliptic divergence form operators by Kenig and Pipher (an exposition of their method may be found in [?]).

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