Uniqueness in a Free Boundary Problem

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Abstract: We study overdetermined boundary conditions for positive solutions to the p Laplacian in a bounded domain D. We show these conditions imply uniqueness in certain free boundary problems.

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1. Introduction. Denote points in Euclidean n space $\mathbb{R}^n$ by $x = (x_1, \dots, x_n)$. Let $\langle \cdot, \cdot \rangle$ be the standard inner product on $\mathbb{R}^n$ and let $|x| = \langle x, x \rangle^{1/2}$ be the Euclidean norm of x. Set $B(x, r) = \{y \in \mathbb{R}^n : |x - y| < r\}$ whenever $x \in \mathbb{R}^n$ and $r > 0$. Let dx denote Lebesgue measure on $\mathbb{R}^n$ and define k dimensional Hausdorff measure, in $\mathbb{R}^n, 0 < k \leq n$, as follows: For fixed $\delta > 0$ and $E \subseteq \mathbb{R}^n$, let $L(\delta) = \{B(x_i, r_i)\}$ be such that $E \subseteq \bigcup B(x_i, r_i)$ and $0 < r_i < \delta, \quad i = 1, 2, ...$ Set

$$\phi_\delta^k(E) = \inf_{L(\delta)} \left(\sum \alpha(k)\, r_i^k \right),$$

where $\alpha(k)$ denotes the volume of the unit ball in $\mathbb{R}^k$. Then

$$H^k(E) = \lim_{\delta \to 0} \phi_\delta^k(E).$$

If $O \subset \mathbb{R}^n$ is open and $1 \leq q \leq \infty$, let $W^{1,q}(O)$ be the space of equivalence classes of functions f with distributional gradient $\nabla f = (f_{x_1}, \dots, f_{x_n})$, both of which are q th power integrable on O. Let

$$\|f\|_{1,q} = \|f\|_q + \|\nabla f\|_q$$

be the norm in $W^{1,q}(O)$ where $\| \cdot \|_q$ denotes the usual Lebesgue q norm in O. Let $C_0^\infty(O)$ be infinitely differentiable functions with compact support in O and let $W_0^{1,q}(O)$ be the closure of $C_0^\infty(O)$ in the norm of $W^{1,q}(O)$. Let D be a bounded domain, N an open neighborhood of ∂D, p fixed, $1 < p < \infty$, and u a positive weak solution to the p Laplace partial differential equation in $D \cap N$. That is, $u \in W^{1,p}(D \cap N), u > 0$, and

$$\int |\nabla u|^{p-2} \langle \nabla u, \nabla \theta \rangle \, dx = 0 \tag{1.1}$$

whenever $\theta \in W_0^{1,p}(D \cap N)$. Observe that if u is smooth and $\nabla u \neq 0$ in $D \cap N$, then $\nabla \cdot (|\nabla u|^{p-2} \nabla u) \equiv 0$. We shall always assume that

$$u(x) \to 0 \text{ as } x \to \partial D. \tag{1.2}$$

Put $u \equiv 0$ in $N \setminus D$. Then $u \in W^{1,p}(N)$ and it follows from (1.1), as in [HKM], that there exists a positive finite Borel measure μ on $\mathbb{R}^n$ with support contained in ∂D and the property that

$$\int |\nabla u|^{p-2} \langle \nabla u, \nabla \phi \rangle \, dx = - \int \phi \, d\mu \tag{1.3}$$

whenever $\phi \in C_0^\infty(N)$. We note that if ∂D is smooth enough, then

$$d\mu = |\nabla u|^{p-1} \, dH^{n-1}|_{\partial D}. \tag{1.4}$$

We assume throughout this paper that there exists $\beta, 0 < \beta < \infty$, for which

$$\mu[B(y, r)] \leq \beta\, r^{n-1}, \; 0 < r \leq r_0, \tag{1.5}$$

where r_0 is so small that $\bigcup_{y \in \partial D} \bar{B}(y, r_0) \subset N$. We study the implications of boundary conditions (1.2), (1.5) on D. To be more specific we first discuss a related free boundary problem. Let E be a compact set and G an open set containing E. For fixed $p, 1 < p < \infty$, set

$$K_p(E, G) = \inf \{ \int |\nabla \theta|^p \, dx \}$$

where the infimum is taken over all $\theta \in C_0^\infty(G)$ with $\theta \equiv 1$ on E. $K_p(E,G)$ is called the p capacity of E relative to G. Consider the following free boundary problem. Given $F \subset \mathbb{R}^n$ a compact convex set, $a > 0$, and $1 < p < \infty$, find a function u defined on a domain $D = D(a,p) \supset F$ with

$$\begin{aligned} &(*) \qquad \nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0 \text{ weakly in } D \setminus F. \\ &(**) \qquad u(x) \to 1 \text{ whenever } x \to y \in F \text{ and } u(x) \to 0 \text{ as } x \to y \in \partial D. \\ &(***) \quad |\nabla u(x)| \to a \text{ whenever } x \to y \in \partial D. \end{aligned} \tag{1.6}$$

This problem was solved in [HS] (see also [HS1], [HS2] for related problems). They proved

Theorem A. *If $K_p(F,G) > 0$ for some open $G \supset F$, then there exists a unique $\hat{u}, \hat{D} = \hat{D}(a,p)$ satisfying (1.6). Moreover $\hat{D}$ is convex with a smooth (C^∞) boundary.*

We remark that the above authors assume that F has nonempty interior. However, their theorem can easily be extended to more general F using the following facts. First if $K_p(F,G) > 0$ for some open $G \supset F$ then $K_p(F,G') > 0$ for any bounded open $G' \supset F$. If $K_p(F,G') > 0$, and G', F are convex, then there exists w continuous in $\bar{G}'$ with $w = 0$ on $\partial G'$ and $w = 1$ on F. Moreover, w is a weak solution to the p Laplacian in $G' \setminus F$. Taking $G' = B(0,R)$ for large $R > 0$ one can now repeat the argument in [HS, section 4.1] to get Theorem A. In view of (1.4) and Theorem A, a natural first question is whether $\hat{u}$ remains unique if $(***)$ in (1.6) is replaced by

$$\mu = a^{p-1} H^{n-1} \text{ on } \partial D? \tag{1.7}$$

Here μ corresponds to u, as in (1.3). If $p = 2$ the answer to this question is that $\hat{u}$ need not be unique. To explain why, suppose $F = B(0,1)$. Then $\hat{D}(a,p) = B(0,R)$ for some $R > 1$, as can be seen from Theorem A and the fact that radial solutions exist. On the other hand using the argument in [LV], one can construct D, u, μ satisfying $(*), (**)$, and (1.7) with $\partial D \neq$ ball. In fact $D = f(B(0,1))$ where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a K quasiconformal mapping and $K > 1$ can be arbitrarily near 1. Moreover, exploratory work by the first author indicates the likely existence of examples in $\mathbb{R}^2$ showing nonuniqueness in the two dimensional free boundary problem when $1 < p < \infty$. All of our examples have the property that $|\nabla u| \to \infty$ as $x \to \partial D$ through a certain sequence. Thus an additional assumption on μ other than (1.7) is needed to rule out these examples and insure uniqueness in our generalization of Theorem A. In this paper we prove,

Theorem 1. *Let D, u, p, a, be as in $(1.6)(*), (**)$ and let μ be the measure corresponding to u as in (1.3). If μ satisfies (1.5), (1.7), then $u = \hat{u}, D = \hat{D}(a,p)$.*

We note that (1.5) is equivalent (see Lemma 2.4) to the assumption that u has a Lipschitz extension to a neighborhood of ∂D. From this note and the above discussion we conclude that Theorem 1 is sharp. To outline our proof, in [LV1] we considered an overdetermined boundary value problem similar to Theorem 1 when $p = 2$ for the Green's function, g, with pole at $0 \in D$ and harmonic measure, ν, with respect to 0. We proved,

Theorem B. *If g satisfies (1.2) and ν satisfies (1.5), (1.7) with u, μ replaced by g, ν, then D is a ball with center at the origin.*

The key step in proving Theorem B is to show that

$$\limsup_{x \longrightarrow \partial D} |\nabla g(x)| \leq a. \tag{1.8}$$

Theorem B then follows easily from (1.8) and a barrier type argument. In retrospect, the proof of (1.8) made important use of the fact that g is a solution (away from 0) and $|\nabla g|^2$ is a subsolution to the same partial differential equation (Laplace's equation). Using this fact we could first prove an important square function type estimate and then use Green's formula on certain level sets to get a boundary type integral for which estimates could be made in terms of g, ν. To make this method work in the proof of Theorem 1, it was first necessary to find a suitable divergence form partial differential equation for which u is a solution and $|\nabla u|^2$ is a subsolution. For this purpose note for u as in (1.1) that if $\eta \in \mathbb{R}^n$ with $|\eta| = 1$, while ∇u is nonzero and sufficiently smooth in $D \cap N$, then $\zeta = \langle \nabla u, \eta \rangle$, is a strong solution in $D \cap N$ to

$$L\zeta \;=\; \nabla \cdot [(p-2)|\nabla u|^{p-4}\langle \nabla u, \nabla \zeta\rangle \, \nabla u + |\nabla u|^{p-2}\nabla \zeta] = 0 \tag{1.9}$$

as follows from differentiating the p Laplace partial differential equation for u with respect to η. Clearly,

$$Lu \;=\; (p-1)\, \nabla \cdot \left[\, |\nabla u|^{p-2}\, \nabla u \,\right] = 0 \; \text{ in } D \cap N. \tag{1.10}$$

(1.9) can be rewritten in the form

$$L\zeta \;=\; \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left[\, b_{ij}(x) \zeta_{x_j}(x) \,\right] = 0, \tag{1.11}$$

where at $x \in D \cap N$,

$$b_{ij}(x) = |\nabla u|^{p-4}[(p-2)u_{x_i}u_{x_j} + \delta_{ij}|\nabla u|^2](x), \; 1 \leq i, j \leq n, \tag{1.12}$$

and δ_{ij} is the Kronecker δ. It is easily checked that $|\nabla u|^2$ is a subsolution to (1.11), (1.12) (see (2.6)). Using (1.11), (1.12) in place of Laplace's equation we are able to follow the general outline of the proof in [LV1] and after suitable modifications involving some delicate asymptotics eventually obtain (1.8) with g replaced by u. Theorem 1 then follows from (1.8), Theorem A, and the fact that H^{n-1} measure decreases under the nearest point projecton onto a closed convex set (see section 4). As for the plan of this paper, in section 2 we state some known results for subsolutions to the p Laplacian and then use these results to show in Lemma 2.4 that u has a Lipschitz extension to a neighborhood of ∂D. Next we prove in Lemma 2.5 an important square function estimate. In section 3 we use this square function estimate and the divergence theorem, as outlined above, to first estimate in a quantitative way the set of points near ∂D where $|\nabla u|$ is near a. Second we show that this estimate implies (1.8) for u. In section 4 we show that (1.8) implies Theorem 1.

Finally we remark that Theorem 1 is part of our general strategy whose goal is to obtain uniqueness or symmetry in a large class of overdetermined boundary problems under minimal boundary assumptions on $u, \partial D$. More specifically in [S] the following theorem was proved.

Theorem C. *Suppose that the region D has a C^2 boundary. If there is a positive solution $u \in C^2(\bar{D})$ to the elliptic equation*

$$h(u, |\nabla u|^2)\, \Delta u + k(u, |\nabla u|^2) \sum_{i,j=1}^{n} u_{x_i}\, u_{x_j}\, u_{x_i x_j} \;=\; l(u, |\nabla u|^2) \tag{1.13}$$

where h, k, l are continuously differentiable everywhere with respect to their arguments and if u satisfies the boundary conditions

$$u = 0 \text{ and } |\nabla u| = \alpha = \text{ constant on } \partial D \tag{1.14}$$

then D is a ball and u is radially symmetric about the center of D.

Now (1.13) can be written in divergence form as

$$\nabla \cdot \big(A(u, |\nabla u|^2) \nabla u \big) = C(u, |\nabla u|^2) \tag{1.15}$$

and this equation can be further rewritten in weak form as,

$$\int_{D\cap N} [\, \langle A(u, |\nabla u|^2) \nabla u, \nabla \phi \rangle + C(u, |\nabla u|^2)\, \phi\,]\, dx = 0 \tag{1.16}$$

whenever $\phi \in C_0^\infty(D \cap N)$. If (1.2) holds for u, then we can put $u \equiv 0$ on $N \setminus D$ and deduce as in (1.3) that there exists a positive Borel measure μ satisfying

$$\int_{D\cap N} [\langle A(u, |\nabla u|^2) \nabla u, \nabla \phi \rangle + C(u, |\nabla u|^2) \phi]\, dx \;=\; - \int \phi\, d\mu \tag{1.17}$$

whenever $\phi \in C_0^\infty(N)$. Consequently, a weak formulation of (1.14) would be that (1.2) holds and

$$\mu = b\, H^{n-1} \text{ on } \partial D \text{ where } \alpha\, A(0, \alpha^2) = b. \tag{1.18}$$

Thus one can conjecture that the conclusion of Theorem C is valid under appropriate structure assumptions on A when u satisfies (1.2) and μ as in (1.17) satisfies (1.5), (1.18). As noted in our outline for the proof of Theorem 1, a first step in this generalization is to prove (1.8) with a, g replaced by α, u. Unfortunately, however, the p Laplace partial differential equation seems to be essentially the only divergence form partial differential equation of the above form with the property that a solution, u, and its partial derivatives, u_{x_i}, both satisfy the same divergence form partial differential equation (see (1.10) - (1.11)). To see why, suppose $A(u, |\nabla u|^2) = A(|\nabla u|^2)$ and $C \equiv 0$ in (1.15). Suppose that u is a smooth solution to the new version of (1.15) at $x \in D, \nabla u(x) \neq 0$, and $A \in C^\infty[(0, \infty)]$. Then as in (1.9) we deduce for $\zeta = \langle \nabla u, \eta \rangle$ that at x,

$$\tilde{L} \zeta = \nabla \cdot [2A'(|\nabla u|^2) \langle \nabla u,\, \nabla \zeta \,\rangle\, \nabla u \,+\, A(|\nabla u|^2) \nabla \zeta] = 0.$$

Furthermore, from (1.15) we see that

$$\tilde{L} u = \nabla \cdot [2A'(|\nabla u|^2) |\nabla u|^2\, \nabla u]$$

at x and this equation is only obviously zero if $A(t) = a t^\lambda$ for some real a, λ.

Without such an equation for $u, |\nabla u|^2$, we are not able to use u to make estimates. Instead, in order to carry through the argument in section 3, it appears that one is forced to consider some rather delicate questions, concerning the absolute continuity of elliptic measure with respect to H^{n-1} measure on ∂D. However, in [LV2] we were able to answer these questions provided μ satisfies (1.5), (1.18) and the additional assumption that

$$\mu(B(y,r)) \geq \gamma\, r^{n-1}, 0 < r \leq r_0.$$

In place of (1.10)-(1.11) we were able to use a complicated stopping time argument together with theorems from [KP], [BL] in order to obtain absolute continuity of our elliptic measures. After proving (1.8) for u, α in [LV2] we still needed to make an additional assumption on ∂D (Reifenberg flatness) in order to conclude uniqueness or symmetry for weak solutions to (1.15) in our generalization of [S]. In general the proof in [LV2] is much more complicated and the results obtained, considerably weaker than Theorem 1 when $A(t) = t^{p/2-1}, 0 < t < \infty$.

2. Basic Estimates. In the sequel, unless otherwise stated, c will denote a positive constant ≥ 1 (not necessarily the same at each occurrence), depending only on the ' data. ' The data may include $n, p, r_0, \beta, \mu(\partial D), a$, and D. In general, $c(a_1, \dots, a_n)$ denotes a positive constant ≥ 1, which depends on the data and $a_1, \dots, a_n$, not necessarily the same at each occurrence. To begin this section, we state some interior and boundary estimates for u satisfying (1.1), (1.2). That is, u is a positive weak solution to the p Laplacian in $D \cap N$ and u has continuous boundary value 0 on ∂D. As in section 1, we extend u to N by putting $u \equiv 0$ on $N \setminus D$. Let $\max\limits_{B(z,s)} u$, $\min\limits_{B(z,s)} u$ be the essential supremum and infimum of u on $B(z,s)$ whenever $B(z,s) \subset N$.

Lemma 2.1. *Let u be as (1.1), (1.2) and $B(w,4r) \subset N$. Then*

$$c^{-1} r^{p-n} \int_{B(w,r/2)} |\nabla u|^p \, dx \leq \max_{B(w,r)} u^p \leq c\, r^{-n} \int_{B(w,2r)} u^p \, dx.$$

If $B(w,2r) \subset D$, then

$$\max_{B(w,r)} u \leq c \min_{B(w,r)} u.$$

Proof: The first display in Lemma 2.1 is a standard subsolution estimate while the second display is a standard weak Harnack estimate for positive weak solutions to nonlinear partial differential equations of p Laplacian type (see [S1]). $\square$

Lemma 2.2. *Let u be as in (1.1), (1.2). Then u has a representative in $W^{1,p}(N)$ (also denoted u) with Hölder continuous partial derivatives in $D \cap N$. That is, for some $\sigma \in (0,1)$ we have*

$$c^{-1}\, |\nabla u(w_1) - \nabla u(w_2)| \leq (|w_1 - w_2|/r)^\sigma \max_{B(w,r)} |\nabla u| \leq c\, r^{-1}\, (|w_1 - w_2|/r)^\sigma \max_{B(w,2r)} u$$

whenever $w_1, w_2 \in B(w,r)$ and $B(w,4r) \subset D \cap N$.

Proof: The proof of Lemma 2.2 can be found in [D], [L] or [T]. In $\mathbb{R}^2$ the best Hölder exponent in Lemma 2.2 is known when $p > 2$ while for $1 < p \leq 2$ a solution has continuous second partials (see

[IM]). □

Next if $w \in \partial D$, let μ correspond to u as in (1.3) and set

$$W(w,r) = \int_0^r \left(\frac{\mu[B(w,t)]}{t^{n-p}} \right)^{1/(p-1)} \frac{dt}{t},$$

whenever $B(w,r) \subset N$.

Lemma 2.3. *Let u, μ be as in (1.1) - (1.3) and $w \in \partial D$. Then for some c_1 depending only on p, n, we have*

$$c_1^{-1} \left(\frac{\mu(B(w,r/2))}{r^{n-p}} \right)^{1/(p-1)} \leq \max_{B(w,r)} u \leq c_1 W(w, c_1 r) \text{ provided } B(w, c_1 r) \subset N.$$

Proof: The lefthand inequality in the above display is easily proved by choosing, $\phi \in C_0^\infty(B(w,r))$ with $\phi \equiv 1$ on $B(w,r/2)$ in (1.3) and simple estimates, using Lemma 2.1. The righthand inequality is proved in [KZ]. □

If E_1, E_2 are sets, let $d(E_1, E_2)$ denote the distance between E_1 and E_2. We write, $d(x)$ for $d(\{x\}, \partial D)$. Next we prove

Lemma 2.4. *Let u, μ, β be as in (1.1)-(1.3), (1.5). Then there exists an open neighborhood N_1 of ∂D with $N_1 \subset N$ and c_2 (depending only on p, n with)*

(*a*) $u(x) \leq c_2 \beta^{1/(p-1)} d(x)$ *when* $x \in N_1$,

(*b*) $|\nabla u(x)| \leq c_2 \beta^{1/(p-1)}$ *for* $x \in D \cap N_1$,

(*c*) *If* $x \in D \cap N_1, \lambda > 0$, *and either* $u(x) \geq \lambda d(x)$ *or* $|\nabla u(x)| \geq \lambda$, *then for some* $c(\lambda)$, $d(x)^{n-1} \leq c(\lambda)\, \mu[B(x, 2c_1 d(x))]$,

(*d*) *If* $|\nabla u(x)| \geq \lambda > 0, x \in D \cap N_1$, *then for some* $\tilde{c}(\lambda)$, *we have* $u \in C^\infty \left(B(x, \frac{d(x)}{\tilde{c}(\lambda)}) \right)$ *and* $\max_{B(x, \frac{d(x)}{\tilde{c}(\lambda)})} \sum_{i,j=1}^n |\, u_{y_i y_j} \,| \leq \frac{\tilde{c}(\lambda)}{d(x)}$.

Proof: Choose $w \in \partial D$ with $|x - w| = d(x)$. Using (1.5) in the integral defining W and integrating we see that

$$u(x) \leq \max_{B(w, 2d(x))} u \leq c_1 W(w, 2c_1 d(x)) \leq c\, \beta^{1/(p-1)} d(x)$$

provided $B(w, 2c_1 d(x)) \subset N$. Thus Lemma 2.4 ($a$) is true. ($b$) of Lemma 2.4 follows from (a) of Lemma 2.4 and Lemma 2.2. To prove (c) of Lemma 2.4 note from Lemma 2.4 (b), Lemma 2.2, and Lemma 2.1 that if $|\nabla u(x)| \geq \lambda$, then $u(x) \geq \lambda' d(x)$ where $\lambda' = \lambda'(\lambda)$. Thus we assume that $u(x) \geq \lambda d(x)$. Again we choose $w \in \partial D$ with $d(x) = |x - w|$. Then from Lemma 2.3 and (1.5) we see as in the proof of Lemma 2.4 (a) that if $\lambda_1 = \lambda_1(\lambda)$ is small enough, then $2c_1 W(w, \lambda_1 d(x)) \leq \lambda d(x)/2$. Using this fact and Lemma 2.3 again, we conclude that

$$\begin{aligned} \lambda d(x)/2 \, &\leq \, u(x)/2 \\ &\leq \, 2c_1[W(w, 2c_1 d(x)) - W(w, \lambda_1 d(x))] \, \leq \, c(\lambda) \, [\mu(B(w, 2c_1 d(x)) d(x)^{p-n}]^{1/(p-1)} \, . \end{aligned}$$

This inequality clearly implies Lemma 2.4 (c). To prove Lemma 2.4 (d) note from Lemma 2.4 (b) and Lemma 2.2, that there exists $\hat{c} = \hat{c}(\lambda)$ with $c\beta^{1/(p-1)} \geq |\nabla u| \geq \hat{c}^{-1}$ in $B(x, d(x)/\hat{c})$. Using this note, we can now argue as in [D], [L], or [T] to first conclude that u has distributional second partials which are square integrable in $B(x, d(x)/\hat{c})$ $(u \in W^{2,2}[B(x, d(x)/\hat{c})])$ and

$$\sum_{i,k=1}^{n} a_{ik}(y) \, u_{y_i y_k}(y) = 0 \text{ for } H^n \text{ almost every } y \in B(x, d(x)/\hat{c}).$$

Here $a_{ik} = |\nabla u|^{2-p} \, b_{ik}$, where $b_{ik}, 1 \leq i, k \leq n$, are as in (1.12). Using Lemma 2.2, it is easily checked that (a_{ik}) are Hölder continuous and uniformly elliptic in $B(x, d(x)/\hat{c})$ with Hölder norm and ellipticity constant depending only on λ and the data. Second, using these facts, Calderón - Zygmund and Schauder theory (see [GT], sec 9.6), we deduce that u has continuous second derivatives in $B(x, d(x)/\hat{c})$ with L^∞ norm in $B(x, \frac{1}{2} d(x)/\hat{c})$ bounded by, $c(\lambda) \, d(x)^{-1}$. Third, from a bootstrap argument it follows that $u \in C^\infty(B(x, d(x)/\hat{c}))$. The proof of Lemma 2.4 is now complete. □.

We note that Lemma 2.4 (b) implies u is Lipschitz in $N_1 \subset N$, as is easily seen. Next we prove our main lemma.

Lemma 2.5. *Let u, μ be as in (1.1)-(1.3), (1.5). There exists an open neighborhood N_2 of ∂D with $\bar{N}_2 \subset N_1 \subset N$ and the following property. If $\delta > 0$, then*

$$\int_{G(\delta) \cap N_2} u \sum_{i,j=1}^{n} (u_{x_i x_j})^2 \, dx \, \leq \, c(\delta, N_2),$$

where $G(\delta) = \{x \in D \cap N_1 : |\nabla u(x)| > 2\delta\}$.

Proof: Recall the definition of $(b_{ij}), L$ from (1.9) - (1.12). If $x \in D \cap N_1$ and $\nabla u(x) \neq 0$, then from Lemma 2.4, (1.9), (1.11) we see that at x,

$$\begin{aligned} L(|\nabla u|^2) \, &= \, 2 \sum_{i,j,k=1}^{n} \frac{\partial}{\partial x_i} [\, b_{ij} u_{x_k} u_{x_j x_k} \,] \\ &= \, 2 \sum_{i,j,k=1}^{n} b_{ij} u_{x_i x_k} u_{x_j x_k} \, \geq \, c^{-1} \, |\nabla u|^{p-2} \sum_{i,j=1}^{n} (u_{x_i x_j})^2. \end{aligned} \tag{2.6}$$

Let $v = \max(|\nabla u|^2 - \delta^2, 0)$ and let $\{\bar{Q}_i = \bar{Q}_i(y_i, r_i)\}$ be a Whitney cube decomposition of D with center y_i and radius r_i. We choose this sequence so that

$$
\begin{aligned}
&(a) \qquad Q_i \cap Q_j = \emptyset, i \neq j, \\
&(b) \qquad 10^{-5n} d(Q_i, \partial D) \leq r_i \leq 10^{-n} d(Q_i, \partial D), \\
&(c) \qquad \textstyle\bigcup_i \bar{Q}_i = D.
\end{aligned} \tag{2.7}
$$

Next let η_i be a partition of unity adapted to $\{Q_i\}$. That is

$$
\begin{aligned}
&(i) \qquad \sum_i \eta_i \equiv 1 \\
&(ii) \qquad \text{The support of } \eta_i \text{ is contained in the union of } \bar{Q}_i \text{ and } \textstyle\bigcup \{Q_j : \bar{Q}_j \cap \bar{Q}_i \neq \emptyset\}. \\
&(iii) \qquad \eta_i \text{ is infinitely differentiable with } \eta_i \geq c^{-1} \text{ on } Q_i \text{ and} \\
&\qquad\qquad \max\{r_i^{-1} |(\eta_i)_{x_k}|, \, r_i^{-2} |(\eta_i)_{x_k x_l}|\} \leq c, \text{ for } 1 \leq k, l \leq n.
\end{aligned} \tag{2.8}
$$

Let

$$
N_2 = \{x \in N_1 : c_3 \, d(x, \partial D) < d(\partial D, \partial N_1)\},
$$

where c_3 is a large positive constant, depending only on the data, For the moment we allow c_3 to vary but shall fix this constant after (2.12). Let $\xi < 10^{-20} \, d(\partial D, \partial N_2)$, $\Lambda = \{j : \bar{Q}_j \cap N_2 \neq \emptyset \text{ and } r_j \geq \xi\}$, and set $\Omega = \bigcup_{i \in \Lambda} Q_i$. Using (2.6), as well as the fact that $\nabla v = \nabla(|\nabla u|^2)$ when $v > 0$, and integrating by parts we find

$$
\begin{aligned}
I_1 = c(\delta)^{-1} \int_\Omega u \, v \sum_{i,j=1}^n (u_{x_i x_j})^2 \, dx &\leq \sum_{m \in \Lambda} \int u \, v \, L(|\nabla u|^2) \, \eta_m \, dx \\
&= -\sum_{m \in \Lambda} \int v \sum_{i,j=1}^n b_{ij} u_{x_i} v_{x_j} \eta_m \, dx \\
&\quad - \sum_{m \in \Lambda} \int u \sum_{i,j=1}^n b_{ij} v_{x_i} v_{x_j} \, \eta_m dx \\
&\quad - \sum_{m \in \Lambda} \int u \, v \sum_{i,j=1}^n b_{ij} v_{x_j} \, (\eta_m)_{x_i} \, dx \\
&= -I_2 - I_3 - I_4.
\end{aligned} \tag{2.9}
$$

To estimate I_4, let Λ_1 be the set of all i for which there exists Q_j, Q_k with $k \notin \Lambda, j \in \Lambda$ and $\bar{Q}_i \cap \bar{Q}_j \neq \emptyset$, $\bar{Q}_k \cap \bar{Q}_i \neq \emptyset$. Then from (2.8), Lemma 2.4, and (1.12) we see that

$$
|I_4| \leq c(\delta) \sum_{m \in \Lambda_1} \int_{Q_m} u \, v \sum_{i,j=1}^n |u_{x_i x_j}| \, |\nabla \eta_m| \, dx. \tag{2.10}
$$

To handle the righthand side of (2.10), we divide the integers in Λ_1 into two subsets, say $\Lambda_{11}, \Lambda_{12}$, where Λ_{11} consists of all i in Λ_1 for which $\bar{Q}_i$ touches a closed cube containing points not in N_2 while $\Lambda_{12} = \Lambda_1 \setminus \Lambda_{11}$ contains integers i for which $\bar{Q}_i$ touches a closed cube Q_j with $r_j \leq \xi$. From Lemma 2.4 and the definition of N_2 we see that

$$\sum_{m\in\Lambda_{11}} \int_{Q_m} u\, v \sum_{i,j=1}^{n} |u_{x_i x_j}|\, |\nabla \eta_m|\, dx \leq c(\delta, N_2). \tag{2.11}$$

Observe that the integral in (2.10) is zero unless $|\nabla u| > \delta$ at some point in Q_m. Otherwise if $m \in \Lambda_{12}$ we can apply Lemma 2.4 with $r = r_m$ and use (2.8) to get

$$\begin{aligned} \sum_{m\in\Lambda_{12}} \int_{Q_m} u\, v \sum_{i,j=1}^{n} |u_{x_i x_j}|\, |\nabla \eta_m|\, dx &\leq c(\delta) \sum_{m\in\Lambda_{12}} r_m^{n-1} \\ &\leq c(\delta) \sum_{m\in\Lambda_{12}} \mu[B(y_m, 10^{10n} c_1 r_m)] \end{aligned} \tag{2.12}$$

provided c_3 is so large that

$$\bigcup_{m\in\Lambda} B(y_m, 10^{40n} c_1 r_m) \subset N_1.$$

Fix c_3 subject to this requirement. From (2.7) $(a), (b)$ and the definition of Λ_{12} we see for fixed $m \in \Lambda_{12}$ that the cardinality of the set of integers $l \in \Lambda_{12}$ for which $B(y_m, 10^{40n}\, c_1\, r_m) \cap B(y_l, 10^{40n} c_1 r_l) \neq \emptyset$ has cardinality $k_0 < \infty$ where k_0 depends only on c_1 and n, so only on the data. Using this fact in (2.12) we conclude that

$$\sum_{m\in\Lambda_{12}} \int_{Q_m} u\, v \sum_{i,j=1}^{n} |u_{x_i x_j}|\, |\nabla \eta_m|\, dx \leq c(\delta)\, \mu(\partial D) \leq c(\delta). \tag{2.13}$$

Adding (2.11), (2.13), we deduce from (2.10) that

$$|I_4| \leq c(\delta, N_2). \tag{2.14}$$

Next we use the fact that u is a solution to (1.11) and $2v\, v_{x_j} = (v^2)_{x_j}$, as well as integration by parts in the integral defining I_2. We obtain,

$$\begin{aligned} 2\, I_2 &= \sum_{m\in\Lambda} \int \sum_{i,j=1}^{n} b_{ij}\, u_{x_i}\, (v^2)_{x_j}\, \eta_m\, dx \\ &= -\sum_{m\in\Lambda} \int \sum_{i,j=1}^{n} b_{ij}\, u_{x_i}\, v^2\, (\eta_m)_{x_j} dx \leq c(\delta, N_2), \end{aligned} \tag{2.15}$$

by the same estimate as we used for I_4. Finally from positive definiteness of (b_{ij}) we see that $I_3 \geq 0$. Using this inequality, (2.14), (2.15) in (2.9), we deduce that $I_1 \leq c(\delta, N)$. Letting $\xi \to 0$ in this inequality and observing that $v \geq 3\,\delta^2$ on $G(\delta)$, we find that

$$3\,\delta^2 \int_{G(\delta)\cap N_2} u \sum_{i,j=1}^{n} (u_{x_i x_j})^2\, dx \leq \int_{G(\delta)\cap N_2} u\, v \sum_{i,j=1}^{n} (u_{x_i x_j})^2\, dx \leq c(\delta, N_2). \tag{2.16}$$

Thus Lemma 2.5 is true. □

3. Preliminary Reductions for Theorem 1. We now begin the proof of Theorem 1. Let u, μ, a, F satisfy the hypotheses of Theorem 1. We assume, as we may, that $F \cap N_2 = \emptyset$. If

$$\limsup_{x \longrightarrow \partial D} |\nabla u| = b, \tag{3.1}$$

then from Lemma 2.4 (b) we see that $b \leq c_2 \beta^{1/(p-1)}$. We first show that

$$b \leq a. \tag{3.2}$$

If $b = 0$, then (3.2) is trivial. Otherwise, let $\phi \in C_0^\infty[(b/100, 2c_2\beta^{1/(p-1)})]$, $0 \leq \phi \leq 1$, with $\phi \equiv 1$ on $[b/50, c_2\beta^{1/(p-1)}]$. Let $\zeta \in C_0^\infty(N_1), 0 \leq \zeta \leq 1$, with $\zeta \equiv 1$ on $\bar{N}_2$. Next let (b_{ij}) be as in (1.12) and put

$$\hat{b}_{ij}(x) = \delta_{ij} + \phi(|\nabla u(x)|)\, \zeta(x)(b_{ij}(x) - \delta_{ij}), \;\; 1 \leq i, j \leq n, \tag{3.3}$$

when $x \in D \cap N_1$ and $\hat{b}_{ij}(x) = \delta_{ij}(x)$, otherwise in D. Here δ_{ij} is the Kronecker δ. From Lemma 2.4 and (1.12) we see that

(α) $\hat{b}_{ij} \in C^\infty[G(b/100)], 1 \leq i, j \leq n,$ and $b_{ij} = \hat{b}_{ij}$ on $N_2 \cap G(b/100)$,

(β) $(\hat{b}_{ij})$ is uniformly elliptic on D with ellipticity constants depending only on b and the data,

(γ) $\hat{b}_{ij} = \hat{b}_{ji}, 1 \leq i, j \leq n.$

(3.4)

From [LSW], (3.3), (3.4), and Schauder theory, we see there exists Green's function,

$$g(\cdot,\cdot) : G(b/100) \times G(b/100) \rightarrow \mathbb{R} \cup \infty$$

with the following properties.

(a') $g(x, y) = g(y, x)$ when $x \neq y, x, y \in G(b/100)$,

(b') $\zeta\, g(\cdot, y) \in C^\infty(G(b/100) \setminus \bar{B}(y, \epsilon)) \bigcap W_0^{1,2}[G(b/100) \setminus \bar{B}(y, \epsilon)]$
whenever $\epsilon > 0, y \in G(b/100),$ and $\zeta \in C_0^\infty(\mathbb{R}^n \setminus \bar{B}(y, \epsilon))$,

(c') If $\theta \in C_0^\infty(G(b/100))$, then $\theta(x) = \displaystyle\int \sum_{i,j=1}^{n} \hat{b}_{ij}(y)\, \theta_{y_i}(y)\, g_{y_j}(x, y)\, dy$ (3.5)
whenever $x \in G(b/100)$,

(d') If $n > 2$, then, $g(x, y) \leq c(b)|x - y|^{2-n}$ when $x \neq y, x, y \in G(b/100)$,
and for $n = 2$, $g(x, y) \leq c(b) \log(\frac{2 \text{ diam } D}{|x-y|})$ when $x \neq y, x, y \in G(b/100)$.

From (3.5) we see that $g(\cdot, y), g(y, \cdot)$ are strong solutions in $G(b/100) \setminus \{y\}$ to

$$\hat{L}w \;=\; \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} [\, \hat{b}_{ij}(x) w_{x_j}(x)\,] = 0. \tag{3.6}$$

Also from (3.5) (d') and Harnack's inequality we find for some $c^* = c^*(u, b, N_2)$ that $g(x, \cdot) \leq c^*\, u$ on $\partial N_2 \cap D$ provided $d(x) = d(x, \partial D) < 10^{-2} d(\partial D, \partial N_2)$. Using this inequality, (3.4) (α), (3.5), comparing boundary values, and using a weak maximum principle for $\hat{L}$ in (3.6), we deduce that if $x \in G(b/100) \cap N_2$ with $d(x) < 10^{-2} d(\partial D, \partial N_2)$, then

$$g(x, y) \leq \tilde{c}\, u(y),\ y \in G(b/100) \cap N_2 \setminus B(x, \tfrac{1}{2}\, d(x)), \tag{3.7}$$

where $\tilde{c} = \tilde{c}(u, b, N_2, x)$.

Next, fix τ, $0 < \tau < 10^{-3} b$, and put $h(x) = [\max\{|\nabla u|^2 - (b - \tau)^2, 0\}]^2$. Choose N_3 an open neighborhood of ∂D with

$$|\nabla u|(x) \leq b + \tau \text{ and } d(x) < 10^{-2} d(\partial D, \partial N_2) \text{ whenever } x \in N_3. \tag{3.8}$$

We now argue as in the proof of Lemma 2.5. Let $\eta_i, \bar{Q}_i$ be as in (2.7), (2.8). Observe from Lemma 2.4 that h has partial derivatives which are locally Lipschitz in $G(b/100) \cap N_1$ and $h(x) = 0$ in a neighborhood of each $x \in \partial G(b/100) \cap N_1$. From these facts and an approximation argument using (3.5) (d') we see that (3.5) (c') holds whenever $\theta = h\, \eta_m$ and $\bar{Q}_m \cap N_3 \neq \emptyset$. Let $x_0 \in N_3$ with $d(x_0) < 10^{-3}\, d(\partial D, \partial N_3)$ and

$$|\nabla u(x_0)| > b - \tau/2. \tag{3.9}$$

Let $\xi \leq 10^{-20n}\, d(x_0)$, put $\Lambda = \{j : \bar{Q}_j \cap N_3 \neq \emptyset \text{ and } r_j \geq \xi\}$, while $\Omega = \cup_{i \in \Lambda} Q_i$. Using (3.4), (3.5) ($c'$), the fact that $h\, \eta_m$ has support in $N_2 \cap G(b/100)$ whenever $m \in \Lambda$ and integrating by parts we find

$$\begin{aligned} b^2\tau^2/4 < h(x_0) &= \int_\Omega \sum_{i,j=1}^n \hat{b}_{ij} (h\eta_m)_{y_i}\, g(x_0, \cdot)_{y_j} dy \\ &= -2 \sum_{m \in \Lambda} \int \sum_{i,j=1}^n b_{ij}\, h_{y_i} (\eta_m)_{y_j}\, g(x_0, \cdot)\, dy \\ &\quad - \sum_{m \in \Lambda} \int \sum_{i,j=1}^n (b_{ij} h_{y_i})_{y_j}\, \eta_m\, g(x_0, \cdot) dy \\ &\quad - \sum_{m \in \Lambda} \int h \sum_{i,j=1}^n [b_{ij}\, (\eta_m)_{y_i}]_{y_j}\, g(x_0, \cdot)\, dy \\ &= -J_1 - J_2 - J_3. \end{aligned} \tag{3.10}$$

To estimate J_1, J_3, let Λ_1 be the set of all i for which there exists Q_j, Q_k with $k \notin \Lambda, j \in \Lambda$ and $\bar{Q}_i \cap \bar{Q}_j \neq \emptyset,\ \bar{Q}_k \cap \bar{Q}_i \neq \emptyset$. Then from Lemma 2.4 and (2.8) we see as in (2.10) that

$$|J_3| + |J_1| \leq c(b) \sum_{m \in \Lambda_1} \int_{Q_m} [h\, r_m^{-2} + |\nabla h|\, r_m^{-1}]\, g(x_0, \cdot)\, dy. \tag{3.11}$$

To handle the righthand side of (3.11), we divide the integers in Λ_1 into two subsets, say $\Lambda_{11}, \Lambda_{12}$, where Λ_{11} consists of all i in Λ_1 for which $\bar{Q}_i$ touches a closed cube containing points not in N_3

while $\Lambda_{12} = \Lambda_1 \setminus \Lambda_{11}$ contains integers i for which $\bar{Q}_i$ touches a closed cube $\bar{Q}_j$ with $r_j \le \xi$. From (3.7) with x, y replaced by y, x_0, respectively, and Lemma 2.4 we see that

$$\sum_{m\in\Lambda_{11}} \int_{Q_m} [h\, r_m^{-2} + r_m^{-1} |\nabla h|]\, g(x_0, \cdot)\ dy \le\ c_+ u(x_0) \le c\, c_+ d(x_0), \tag{3.12}$$

where $c_+ = c_+(b, N_3)$ and c depends only on the data. Let $\Lambda' = \Lambda'(\xi)$ denote the set of all $m \in \Lambda_{12}$ such that $h \not\equiv 0$ on Q_m. Using Lemma 2.4 as in (2.12) and (3.7) once again, we get

$$\begin{aligned}\sum_{m\in\Lambda_{12}} \int_{Q_m} & [hr_m^{-2} + |\nabla h|\, r_m^{-1}]\, g(x_0, \cdot)\, dy \\ &\le c_- \sum_{m\in\Lambda'} \int_{Q_m} [hr_m^{-2} + |\nabla h|\, r_m^{-1}]\, u(y)\, dy \le c_-^2\, \tau \sum_{m\in\Lambda'} r_m^{n-1}\end{aligned} \tag{3.13}$$

for $c_- = c_-(b, N_3, x_0)$ large enough. Next from (2.6) and (3.4) (α), we deduce that $J_2 \ge 0$. Combining this inequality with (3.11)- (3.13) and using the resulting inequality in (3.10) we conclude that

$$\begin{aligned}b^2\tau^2/4 &\le cc_+ d(x_0) + c_-^2\, \tau \sum_{m\in\Lambda'} r_m^{n-1} \\ &\le b^2\tau^2/8 + c_-^2\, \tau \sum_{m\in\Lambda'} r_m^{n-1}\end{aligned} \tag{3.14}$$

provided x_0 is near enough ∂D. With x_0 now fixed let $\xi_k = 10^{-20kn} d(x_0)$ for $k = 1, 2, \ldots$ From (3.14) we see there exists small $\gamma > 0$ with

$$\gamma \le \sum_{m\in\Lambda'(\xi_k)} r_m^{n-1} \text{ for } k = 1, 2, \ldots, \tag{3.15}$$

where γ depends on various quantities (x_0, τ, b, N_3, and the data), but is independent of $k = 1, 2, \ldots$. If $m \in \Lambda'(\xi_k), k = 1, 2, \ldots$, we now choose $\tilde{y}_m \in \bar{Q}_m$ with $h(\tilde{y}_m) \ne 0$ or equivalently, $|\nabla u(\tilde{y}_m)| > b - \tau$.

Given $\epsilon, \delta > 0$ with $0 < \delta < 10^{-6} \min(b, 1)$ and $0 < \epsilon < \delta^{100}$, we say that $B(x, d(x))$ is a good ϵ, δ tangent ball provided

$$\begin{aligned}&(+) \quad |\nabla u(x)| \ge \delta, \\ &(++) \ \text{ If } x \text{ can be joined to } \hat{x} \text{ by a chain of at most } 1/\epsilon \text{ balls, } \{B(z_i, d(z_i)/4)\} \\ &\qquad \text{with } B(z_i, d(z_i)) \subset D \cap N_3 \text{ and } \epsilon^3 d(x) \le d(z_i) \le \epsilon^{-3} d(x), \text{ then} \\ &\qquad\qquad \max_{z\in B(\hat{x},(1-\epsilon^{12})d(\hat{x}))} |\nabla u(z) - \nabla u(x)| \le \epsilon^{12}.\end{aligned} \tag{3.16}$$

By a chain of balls connecting x to $\hat{x}$ we mean that successive balls have nonempty intersection. Also x is the center of the first ball and $\hat{x}$ the center of the last ball in the chain. For given ϵ, δ as above, let $\Lambda''(\xi_k)$ be the subset of all $m \in \Lambda'(\xi_k)$ with the property that whenever $z \in B(\tilde{y}_m, d(\tilde{y}_m)/\epsilon)$ with $d(z) \geq \epsilon^2 d(\tilde{y}_m)$, and $|\nabla u(z)| \geq \delta$, then $B(z, d(z))$ is a good ϵ, δ tangent ball. We claim there exists $k_1 = k_1(\epsilon, \delta, u)$, a positive integer, $\epsilon_0 = \epsilon_0(\delta) > 0$, and $\gamma' > 0$, having the same dependence as γ in (3.15), such that if $0 < \epsilon \leq \epsilon_0$ and $k \geq k_1$, then

$$\gamma' \leq \sum_{m \in \Lambda''(\xi_k)} r_m^{n-1}. \tag{3.17}$$

To prove this claim, let

$$\Gamma(\tilde{y}_m) = \{w \in D : |w - \tilde{y}_m| \leq \epsilon^{-6} d(\tilde{y}_m), |\nabla u(w)| \geq \delta/4, \text{ and } d(w) \geq \epsilon^{20} d(\tilde{y}_m)\}.$$

If $m \in \Lambda'(\xi_k) \setminus \Lambda''(\xi_k)$, then we first show for some $c(\epsilon, \delta)$ that

$$r_m^{n-1} \leq c(\epsilon, \delta) \int_{\Gamma(\tilde{y}_m)} u \sum_{i,j=1}^{n} (u_{x_i x_j})^2 \, dx. \tag{3.18}$$

To prove (3.18) we need to sort through some definitions. If $m \in \Lambda'(\xi_k) \setminus \Lambda''(\xi_k)$, then there exists a chain, $\{B(z_i, d(z_i)/4)\}_1^l$ connecting z_1 to z_l with $\epsilon^5 d(\tilde{y}_m) \leq d(z_i) \leq \epsilon^{-5} d(\tilde{y}_m)$ for $1 \leq i \leq l$. Also, $z_1 \in B(\tilde{y}_m, d(\tilde{y}_m)/\epsilon), |\nabla u(z_1)| \geq \delta$, and

$$\max_{z \in B(z_l, (1-\epsilon^{12}) d(z_l))} |\nabla u(z) - \nabla u(z_1)| \geq \epsilon^{12}.$$

We assume as we may that

$$\max_{z \in B(z_j, (1-\epsilon^{12}) d(z_j))} |\nabla u(z) - \nabla u(z_1)| \leq \epsilon^{12}$$

when $1 < j \leq l - 1$ and $l > 1$, since otherwise we shorten the chain. Using the triangle inequality and our chain assumption we observe that $z_{j+1} \in B(z_j, 3d(z_j)/4)$ for $1 \leq j \leq l - l$. This observation, the above inequality, and $\epsilon < \delta^{100}$, imply

$$|\nabla u(z_j)| \geq |\nabla u(z_1)| - \epsilon^{12} \geq \delta/2, \text{ for } 1 \leq j \leq l. \tag{3.19}$$

Also from the triangle inequality we have

$$\epsilon^{12} \leq \max_{z \in B(z_l, (1-\epsilon^{12}) d(z_l))} |\nabla u(z) - \nabla u(z_l)| + \sum_{j=1}^{l-1} |\nabla u(z_{j+1}) - \nabla u(z_j)|$$

which implies from weak type estimates and $l \leq \epsilon^{-1}$ (see (3.16) (++)) that

$$(*) \quad \text{either} \max_{z \in B(z_l, (1-\epsilon^{12}) d(z_l))} |\nabla u(z) - \nabla u(z_l)| \geq \epsilon^{14}$$

$$(**) \quad \text{or for some } j, 1 \leq j \leq l - 1, \ |\nabla u(z_{j+1}) - \nabla u(z_j)| \geq \epsilon^{14}.$$

In either case using Lemma 2.4, (3.19), it follows that there exists a parallelpiped $\subset \Gamma(\tilde{y}_m)$ with sidelength $\geq d(\tilde{y}_m)/c(\epsilon)$ and the property that $|\nabla u(z) - \nabla u(w)| \geq \epsilon^{15}$ whenever z is in the base and w in the top of the parallelpiped. From this fact, the fact that $u \geq d(y_m)/c(\epsilon)$ in $\Gamma(\tilde{y}_m)$ and Poincáre's inequality we conclude that (3.18) is valid for some $c(\epsilon, \delta)$. Summing (3.18) over $m \in \Lambda'(\xi_k) \setminus \Lambda''(\xi_k)$ it follows from (3.15) that if (3.17) is false, then

$$\gamma \leq \hat{c}(\epsilon, \delta) \sum_{m \in \Lambda'(\xi_k) \setminus \Lambda''(\xi_k)} \int_{\Gamma(\tilde{y}_m))} u \sum_{l,j=1}^{n} (u_{x_l x_j})^2 \, dx. \tag{3.20}$$

We temporarily write $\tilde{y}_{m,k}$ for $\tilde{y}_m$ when $m \in \Lambda'(\xi_k)$. From (2.7) we deduce that there exists $\tilde{c} = \tilde{c}(\epsilon, \delta)$, a positive integer, such that each point in $\bigcup_{m,k} \Gamma(\tilde{y}_{m,k})$ lies in at most $\tilde{c}$ of $\{\Gamma(\tilde{y}_{m,k})\}$. Using this fact and summing the righthand side of (3.20) over $k \geq k_1$, we find from Lemma 2.5 that the resulting sum is bounded by

$$c^*(\epsilon, \delta) \int_{N_2 \cap G(\delta/8)} u \sum_{i,j=1}^{n} (u_{x_i x_j})^2 \, dx < \infty.$$

From the above inequality we see that (3.20) can only hold for a finite number of k. Thus claim (3.17) is true.

To continue our proof of (3.2) we need the fact that for H^{n-1} almost every $y \in \partial D$, we have

$$\limsup_{\rho \longrightarrow 0} \; \alpha(n-1)^{-1} \rho^{-(n-1)} H^{n-1}[B(z, \rho) \cap \partial D] \leq 1 \tag{3.21}$$

which follows from $H^{n-1}(\partial D) < \infty$ (by (1.7)) and basic measure theory arguments (see [Ma, Thm 6.6]). From (3.21) we see for given $\eta > 0$ that there exists $\rho = \rho(\eta) > 0$ such that

$$\frac{H^{n-1}[B(z, r)]}{\alpha(n-1)\, r^{(n-1)}} \leq (1 + \tau), 0 < r < \rho, \tag{3.22}$$

for all $z \in \partial D$, except on a set $E \subset \partial D$ of H^{n-1} measure at most η. We assert for $\eta = \eta(\epsilon, \delta, \tau)$, small enough and all $k \geq k_1$ that there exists,

$$m \in \Lambda''(\xi_k), \hat{w} \in \partial D \setminus E, \text{ and } w \in \partial D \cap \bar{B}(\tilde{y}_m, d(\tilde{y}_m)), \text{ with } |\hat{w} - w| < \epsilon^8 r_m. \tag{3.23}$$

Indeed, if $m \in \Lambda''(\xi_k)$, then $B(\tilde{y}_m, d(\tilde{y}_m))$ is a good tangent ball since $|\nabla u(\tilde{y}_m)| \geq b - \tau > \delta$, so as in the proof of Lemma 2.4 (c) we see that $u(\tilde{y}_m) \geq c(\delta)^{-1} d(\tilde{y}_m)$. From Harnack's inequality (Lemma 2.1), we then get $u(z) \geq c(\epsilon, \delta)^{-1} r_m$ at a point z on the ray from $\tilde{y}_m$ to $w \in \bar{B}(\tilde{y}_m, d(\tilde{y}_m)) \cap \partial D$ with $|z - w| = \epsilon^9 r_m$. Applying Lemma 2.4 (c) and (1.7), we deduce for some $c_+ = c_+(\epsilon, \delta)$ that

$$c_+ H^{n-1}(B(w, \epsilon^8 r_m) \cap \partial D) \geq r_m^{n-1}.$$

Summing this inequality over $m \in \Lambda''(\xi_k)$, and using (2.7), (3.17), we see that (3.23) must hold for η sufficiently small, since otherwise E would be too large. Fix $\eta > 0$ so that (3.23) is satisfied.

Next we study the geometric implications of a good tangent ball. From the definition of a good tangent ball and Lemma 2.4 (b), we see for some c depending only on the data that

$$|u(x) - u(\tilde{y}_m) - \langle \nabla u(\tilde{y}_m), x - \tilde{y}_m \rangle| \leq c\epsilon^{12} d(\tilde{y}_m) \tag{3.24}$$

whenever $x \in B(\tilde{y}_m, d(\tilde{y}_m))$. Choose $w \in \partial D \cap \bar{B}(\tilde{y}_m, d(\tilde{y}_m)$ and $\hat{w} \in \partial D \setminus E$ with $|\hat{w} - w| \leq \epsilon^8 r_m$. To simplify matters suppose that $w = 0$ and $\tilde{y}_m = d(\tilde{y}_m) e_n$, where $e_n = (0, \dots, 1)$. We also write

y for $\tilde{y}_m$. Then (3.24) implies u is within $c\epsilon^{12}d(y)$ of a linear function in $B(y, d(y))$. Since $u > 0$ in $B(y, d(y))$ and $u(0) = 0$, we must have

$$||\nabla u(y)| - u_{y_n}(y)| \leq c'\epsilon^{12},$$

which implies that

$$\begin{aligned} &(\alpha) \quad |\nabla u(y) - |\nabla u|(y)\, e_n| \;\leq\; c''\epsilon^6 \\ &(\beta) \quad |u(x) - u(y) - |\nabla u(y)|\,(x_n - y_n)\,| \;\leq\; c''\epsilon^6 d(y). \end{aligned} \tag{3.25}$$

where we have also used (3.24) in (3.25)(β). Next suppose that $\hat{x}$ is as in (3.16) (++) with x replaced by y. Then from (3.25) we see that

$$||\nabla u(y)|e_n - \nabla u(w)| \leq c^*\epsilon^6 \tag{3.26}$$

whenever $w \in B(\hat{x}, (1-\epsilon^{12})d(\hat{x}))$, where the constant in (3.26) (as in (3.24), (3.25)) depends only on the data. Now (3.26) holds whenever $\hat{x}$ can be connected to y by a chain of at most ϵ^{-1} balls as in (3.16). Therefore (3.26) holds with $\hat{x}$ replaced by a center of a ball in the chain. Using this fact and choosing a curve σ contained in the chain connecting w to 0 we deduce from (3.26), Lemma 2.4 that

$$|u(w) - |\nabla u(y)|\, w_n| \leq c\; H^1(\sigma)\epsilon^6 \text{ for } w \in \bar{B}(\hat{x}, d(\hat{x})). \tag{3.27}$$

Note from (3.27) that if $u(w) = 0$, then

$$|w_n| \leq \; c\,\delta^{-1}H^1(\sigma)\epsilon^6 \;\leq\; cH^1(\sigma)\epsilon^{11/2}. \tag{3.28}$$

We claim for $\delta > 0$ small enough (depending only on the data, b, τ), and ϵ, say $0 < \epsilon \leq \epsilon_0(\delta) < \delta^{100}$, that every point $\hat{x} \in \bar{O}$, where

$$O = B(y, d(y)/\epsilon^2) \cap \{w : w_n > \epsilon^2 d(y)\}$$

can be joined by a chain of at most $1/\sqrt{\epsilon}$ balls as in (3.16). In fact if $t \in (\epsilon^3 d(y), (2\epsilon)^{-3}d(y))$ then from (3.27), (3.28), and an iterative type argument we see that $(0, \ldots, t)$ can be joined to $(0, \ldots, d(y))$ by a chain of at most $c\log(1/\epsilon)$ balls as in (3.16), where c depends only on the data. Moreover (3.27) holds with $H^1(\sigma)$ replaced by $c\epsilon^{-3}d(y)$. If $w' \in \mathbb{R}^{n-1}$ with $|w'| \leq \epsilon^{-3}d(y)/2$, we can then join $(0, \ldots, (2\epsilon)^{-3}d(y))$ to $(w', (2\epsilon)^{-3}d(y))$ by a chain of at most 1000 balls and then join this point to $(w', \epsilon^2 d(y))$ by a chain with the desired properties. Thus our claim is true. From our claim we deduce first that

$$\partial D \cap O = \emptyset \tag{3.29}$$

and second that

$$B(\tilde{w}, 4\epsilon^2 d(y)) \cap \partial D \neq \emptyset. \text{ whenever } \tilde{w} \in \partial O \cap B(y, d(y)/\epsilon^2). \tag{3.30}$$

In fact if (3.30) is false, then we can join y to $\tilde{w}$ as in (3.16) with $\tilde{w}, y$ playing the role of $\hat{x}, x$. It would then follow from (3.27) and the discussion above (3.29) that if $\zeta = (0, \ldots, -\epsilon^2 d(y))$, then

$$\epsilon^2\, |\nabla u(y)|\, d(y) \;\leq\; |u(\zeta) - |\nabla u(y)|\zeta_n| \;\leq\; c\epsilon^6\, d(y).$$

Clearly, this inequality is impossible for δ small enough, since $|\nabla u(y)| \geq \delta$ and $\epsilon < \delta^{100}$. Thus (3.30) is true. Next suppose there exists $\hat{v} \in D \cap B(0, d(y)/2)$ with $\hat{v}_n \leq 0$, $d(\hat{v}) \geq \epsilon d(y)$, and $|\nabla u(\hat{v})| \geq \delta$. Then from the definition of $\Lambda''(\xi_k)$ above (3.17) and (3.23) we see that $B(\hat{v}, d(\hat{v}))$ is a good tangent ball. Hence we can repeat the above analysis leading to (3.29). That is, we can find $v^* \in \bar{B}(\hat{v}, d(\hat{v})) \cap \partial D$. and η with $|\eta| = 1$ so that if $O_1 = \{w : \langle (w - v^*), \eta \rangle > \epsilon^2 d(\hat{v})\} \cap B(\hat{v}, d(\hat{v})/\epsilon^2)$, then

$$\partial D \cap \bar{O}_1 = \emptyset \text{ and } B(\tilde{w}, 4\epsilon^2 d(\hat{v})) \cap \partial D \neq \emptyset \text{ when } \tilde{w} \in \partial O_1 \cap B(\hat{v}, d(\hat{v})/\epsilon^2). \tag{3.31}$$

Moreover

$$O_1 \cap O \cap B(y, d(y)/\epsilon^{1/2}) = \emptyset, \tag{3.32}$$

since otherwise using $d(\hat{v}) \geq \epsilon d(y)$ and high school geometry, we could easily get a contradiction to (3.29) and/or (3.30). Finally we claim that if $\zeta \in D \cap B(0, d(y)/\epsilon^{1/2})$, $d(\zeta) \geq \epsilon d(y)$, and $|\nabla u(\zeta)| \geq \delta$, then

$$\zeta \text{ lies within } \epsilon\, d(y) \text{ of a point in } \partial O \cup \partial O_1. \tag{3.33}$$

Indeed otherwise we could repeat the above argument once again, getting an open set O_2 with the same properties as O, O_1. These three open sets could then not intersect in $B(0, 2d(y)/\epsilon^{1/2})$ for sufficiently small ϵ which is clearly impossible.

Finally we suppose that k is so large that $\epsilon^{-6} d(y) < \rho$ where ρ is as in (3.22). We divide the proof of (3.2) into three cases.

Case 1. Let $S = \{x : |x_n| \leq \epsilon^{1/2} d(y)/2\} \cap B(0, d(y)/2)$ and suppose that

$$\partial D \cap B(0, d(y)/2) \subset S. \tag{3.34}$$

Given $\eta, 0 < \eta \leq 1/2$, let $\psi \in C_0^\infty[(-1,1)]$ be an even function with $\psi = 1$ on $(\eta - 1, 1 - \eta)$ and $|\psi'| \leq c/\eta$. Put

$$\phi(x', x_n) = \psi[4\,|x'|/d(y)]\,\psi[x_n/(\epsilon^{1/2} d(y))] \text{ whenever } x' \in \mathbb{R}^{n-1} \text{ and } x_n \in \mathbb{R}$$

$$H = \{(x', x_n) : |x'| < d(y)/4, |x_n| < \epsilon^{1/2}\, d(y)\}$$

$$T_1 = \partial H \cap \{x : x_n = \epsilon^{1/2}\, d(y)\}$$

$$T_2 = \partial H \cap \{x : x_n = -\epsilon^{1/2}\, d(y)\}$$

$$T_3 = \partial H \cap \{x : |x'| = d(y)/4\}.$$

Note that supp $\phi \subset H$ and $0 \leq \phi \leq 1$. Also $T_1 \subset O$ (as in (3.29)) for $\epsilon_0 = \epsilon_0(\delta)$ sufficiently small. Using these notes, (1.3), (3.23), (3.22), and (1.7) we obtain

$$\begin{aligned} M_1 = |\int_H |\nabla u|^{p-2} \langle \nabla u\,,\, \nabla \phi \rangle\; dy| &= \int \phi\, d\mu \\ &\leq \mu[B(\hat{w}, (1 + 4\epsilon^{1/2}) d(y)/4)] \leq a^{p-1}\, \alpha(n-1)\,(1+\tau)\,[(1/4 + \epsilon^{1/2}) d(y)]^{n-1} \end{aligned} \tag{3.35}$$

Moreover,

$$|\int_H |\nabla u|^{p-2}\, u_{x_n}\, \phi_{x_n}\, dx| \,-\, |\int_H |\nabla u|^{p-2} \langle \nabla' u, \nabla' \phi \rangle dx\,| \,=\, M_2 - M_3 \,\leq\, M_1 \tag{3.36}$$

where ∇' denotes the gradient in $x' = (x_1, \dots, x_{n-1})$ only. From (3.34) we see that either (a) $\hat{O} = \{w : w_n < -\epsilon^{1/2} d(y)/2\} \cap B(0, d(y)/2) \subset \mathbb{R}^n \setminus D$ or (b) $\hat{O} \subset D$. In either case using Lemma 2.2 and letting $\eta \to 0$ we find

$$M_2 \to \left| \int_{T_1} |\nabla u|^{p-2} u_{x_n} \, dH^{n-1}x - \int_{T_2} |\nabla u|^{p-2} u_{x_n} \, dH^{n-1}x \right| = M_4 \tag{3.37}$$

where the integral involving T_2 is zero in case (a). In case (b) we see from (3.23), (3.34) that either $(+)\, |\nabla u(z)| \leq \delta$ for all $z \in T_2$ or $(-)\, B(z, d(z))$ is a good tangent ball for some $z \in T_2$ with $d(z) \geq \epsilon^{1/2} d(y)/2$. If $(+)$ occurs then clearly

$$\left| \int_{T_2} |\nabla u|^{p-2} u_{x_n} \, dH^{n-1}x \right| \leq c\, \delta^{p-1} \, d(y)^{n-1}. \tag{3.38}$$

If (-) occurs, then from basic geometry and the argument after (3.25) (see (3.32)), it follows that necessarily $u_{x_n} < 0$ on T_2. Since $u_{x_n} \geq 0$ on T_1 we conclude that in either scenario,

$$M_5 = \left| \int_{T_1} |\nabla u|^{p-2} u_{x_n} \, dH^{n-1}x \right| \leq M_4 + c\, \delta^{p-1} \, d(y)^{n-1}. \tag{3.39}$$

Using (3.25) and the fact that $|\nabla u(y)| \geq b - \tau$ we deduce from (3.39) that

$$b^{p-1} \alpha(n-1) [d(y)/4]^{n-1} \leq M_4 + (\epsilon^6 b^{p-2} + \tau + \delta^{p-1})\, c\, d(y)^{n-1}. \tag{3.40}$$

Finally,

$$\limsup_{\eta \to 0} M_3 \leq c\, \epsilon^{1/2} \, d(y)^{n-1}, \tag{3.41}$$

as we see from boundedness of $|\nabla u|$ and simple estimates. Combining (3.35) - (3.41), we get after dividing by $\alpha(n-1)\, (d(y)/4)^{n-1}$

$$b^{p-1} \leq a^{p-1} + c \left[\epsilon^6 b^{p-2} + \tau + \delta^{p-1} + \epsilon^{1/2} \right]. \tag{3.42}$$

Since ϵ, τ, δ can be arbitrarily small we conclude that (3.2) is true when case 1 occurs.

Case 2: $\partial D \cap B(0, d(y)/2) \not\subset S$ and $u(x) \leq 4\delta |x_n|$ for all $x \in B(0, d(y)/2) \cap \{z : z_n \leq -\epsilon^{1/2} d(y)\}$.

In case 2 we observe from Lemma 2.1 that if $w \in B(0, d(y)/2)$ with $w_n \leq -4\epsilon^{1/2} d(y)$, then

$$\begin{aligned} \int_{B(w, |w_n|/4)} |\nabla u|^{p-1} \, dx &\leq c\, |w_n|^{n/p} \left(\int_{B(w, |w_n|/4)} |\nabla u|^p \, dx \right)^{1-1/p} \\ &\leq c\, |w_n|^{n/p} \left(|w_n|^{-p} \int_{B(w, |w_n|/2)} u^p \, dx \right)^{1-1/p} \leq c\, \delta^{p-1} \, |w_n|^n. \end{aligned} \tag{3.43}$$

Again we write $x = (x', x_n)$ where $x' = (x_1, \dots, x_{n-1})$ and put

$$B_r(x') = \{z' \in \mathbb{R}^{n-1} : |z' - x'| < r\}$$

$$f(x') = \int_{4\epsilon^{1/2} d(y)}^{d(y)/4} t^{1-n} \left(\int_{B_t(x')} |\nabla u|^{p-1}(w', -t) \, dH^{n-1} w' \right) dt.$$

Writing the outside integral in the definition of f as a sum and using (3.43) to make estimates we see first that $f \leq c\delta^{p-1}d(y)$ in $B_{\frac{d(y)}{8}}(0)$ and second, using this inequality as well as the Fubini theorem to interchange the order of integration, that

$$
\begin{aligned}
&\int_{4\epsilon^{1/2}d(y)}^{d(y)/32}\left(\int_{B_{\frac{d(y)}{32}}(0)} |\nabla u|^{p-1}(w', -t)\, dH^{n-1}w'\right) dt \\
&\leq c\int_{B_{\frac{d(y)}{8}}(0)} f(x')dH^{n-1}x' \leq c\delta^{p-1}\, d(y)^n.
\end{aligned}
\tag{3.44}
$$

Given $s \in (\frac{d(y)}{64}, \frac{d(y)}{32})$, let ψ be as in case 1 and put $\phi(x) = \psi(|x|/s)$. Using the Lebesgue differentiation theorem and arguing as in case 1 we see for H^1 almost every $s \in (\frac{d(y)}{64}, \frac{d(y)}{32})$ that

$$
\begin{aligned}
\mu[B(0,s)] &= \int_{\partial B(0,s)} |\nabla u|^{p-2}(x)\,\langle\,\nabla u(x)\,,\,\tfrac{x}{|x|}\,\rangle\, dH^{n-1}x \\
&= \int_{P_1}\cdots + \int_{P_2}\cdots + \int_{P_3}\cdots
\end{aligned}
\tag{3.45}
$$

where

$$P_1 = \partial B(0,s) \cap \{x : x_n \geq \epsilon^{1/2}\, d(y)\}$$

$$P_2 = \partial B(0,s) \cap \{x : -4\epsilon^{1/2}\, d(y) \leq x_n \leq \epsilon^{1/2}\, d(y)\}$$

$$P_3 = \partial B(0,s) \cap \{x : x_n \leq -4\epsilon^{1/2}\, d(y)\}$$

From (3.44) and weak type estimates we see there exists $s \in (\frac{d(y)}{64}, \frac{d(y)}{32})$ for which

$$\left|\int_{P_3} |\nabla u|^{p-1}\, dH^{n-1}x\right| \leq c\delta^{p-1}\, d(y)^{n-1}. \tag{3.46}$$

Moreover from Lemma 2.4 we have

$$\left|\int_{P_2} |\nabla u|^{p-1}\, dH^{n-1}x\right| \leq c\epsilon^{1/2}\, d(y)^{n-1}. \tag{3.47}$$

From (1.10) and the divergence theorem we see that if

$$T' = \{x : x_n = \epsilon^{1/2}d(y)\} \cap B(0,s),$$

then

$$
\begin{aligned}
&\int_{P_1} |\nabla u|^{p-2}(x)\,\langle\,\nabla u(x)\,,\,\tfrac{x}{|x|}\,\rangle\, dH^{n-1}x \\
&= \int_{T'} |\nabla u|^{p-2}u_{x_n}\, dH^{n-1}x \,\geq\, [b^{p-1} - (\epsilon^6 b^{p-2} + \tau + \epsilon^{1/2})c]\,\alpha(n-1)s^{n-1},
\end{aligned}
\tag{3.48}
$$

thanks to (3.26). Using (3.46) - (3.48) and (1.7), (3.22), (3.23) in (3.45) we see that

$$a^{p-1}\,\alpha(n-1)\,(1+\tau)\,s^{n-1} \;\geq\; \mu[B(0,s)] \;\geq\; [b^{p-1} \;-\; (b^{p-2}\epsilon^{6}+\tau+\epsilon^{1/2}+\delta^{p-1})c]\,\alpha(n-1)s^{n-1}.$$

Hence, (3.42) is valid and (3.2) is proved in case 2.

Case 3: $\partial D \cap B(0, d(y)/2) \not\subset S$ and $u(x) > 4\delta|x_n|$ for some $x \in B(0, d(y)/2)$ with $x_n \leq -\epsilon^{1/2} d(y)$.
In this case we note for x as above that $u(x) \geq 2\delta\, d(x)$, thanks to (3.30). Let $\tilde{x} \in \partial D \cap \bar{B}(x, d(x))$ and choose x^* on the line segment from x to $\tilde{x}$ with $u(x^*) = \delta\, d(x)$. From the mean value theorem of calculus we see that $|\nabla u(z)| \geq \delta$ at some point on the line segment from x to x^* and from Lemma 2.4 (a) we find that

$$d(z) \geq d(x^*) \geq [c_2\beta^{1/(p-1)}]^{-1}\;\delta\, d(x)$$

$$\geq [c_2\beta^{1/(p-1)}]^{-2}\;\delta^2\,\epsilon^{1/2}\,d(y) > \epsilon d(y)$$

for $\epsilon \leq \epsilon_0$. From (3.23) we see that $B(z, d(z))$ is a good tangent ball. To complete the proof of (3.2) in case 3 we consider two subcases. First suppose that .

$$\partial D \cap B(0, d(y)/\epsilon^{1/2}) \subset \{w : |w_n| \leq\, d(y)/\epsilon^{1/4}\}.$$

In this subcase we can argue exactly as in case 1 with $B(0, d(y))$ replaced by $B(0, d(y)/\epsilon^{1/2})$ to conclude the validity of (3.2). If $\partial D \cap B(0, d(y)/\epsilon^{1/2}) \not\subset \{w : |w_n| \leq d(y)/\epsilon^{1/4}\}$, we claim that $u(x) \leq 4\delta|x_n|$ for all $x \in B(0, d(y)/\epsilon^{1/2})$ with $x_n \leq -d(y)/\epsilon^{1/4}$. In fact otherwise it would follow as in case 2 that for some $z \in B(0, d(y)/\epsilon^{1/2})$ we have $|\nabla u(z)| \geq \delta$ and $d(z) \geq \epsilon d(y)$. From (3.23) we then conclude that $B(z, d(z))$ is a good tangent ball and thereupon from the discussion after (3.33) that we have arrived at a contradiction. Using our claim we can now argue exactly as in case 2 with $B(0, d(y))$ replaced by $B(0, d(y)/\epsilon^{1/2})$. Thus (3.2) holds in this subcase also. The proof of (3.2) is now complete.

4. Proof of Theorem 1. From (3.2) we have

$$\limsup_{x \longrightarrow \partial D} |\nabla u|(x) \;\leq\; a. \tag{4.1}$$

Also, from the hypotheses of Theorem 1 we see that

$$u(x) \longrightarrow 1 \text{ as } x \longrightarrow F \text{ and } u(x) \longrightarrow 0 \text{ as } x \longrightarrow \partial D. \tag{4.2}$$

Extend u to $\mathbb{R}^n$ by putting $u \equiv 1$ on F while $u \equiv 0$ on $\mathbb{R}^n \setminus D$ and note that $u \in W^{1,p}_0(\mathbb{R}^n)$. From well known arguments in the calculus of variations it is easily seen that

$$\int_{\mathbb{R}^n} |\nabla u|^p\, dx = K_p(F, D) > 0. \tag{4.3}$$

From (4.3) and Theorem A we deduce for given $b > 0$, the existence of a unique $\hat{D}(b, p), \hat{u} = u_{b,p}$, satisfying (1.6) with a replaced by b. Given two sets E_1, E_2 let

$$h(E_1, E_2) = \sup\{d(w_2, E_1) \;+\; d(w_1, E_2) : w_1 \in E_1, w_2 \in E_2\}$$

which is $\approx$ the Hausdorff distance between E_1, E_2. We claim that

$$\begin{aligned}
&(a) \quad h[\,\hat{D}(b_2,p), \hat{D}(b_1,p)] \longrightarrow 0 \text{ as } |b_1 - b_2| \longrightarrow 0,\\
&(b) \quad \hat{D}(b_2,p) \cup \partial\hat{D}(b_2,p) \subset \hat{D}(b_1,p) \text{ whenever } 0 < b_1 < b_2 < \infty,\\
&(c) \quad \lim_{b\longrightarrow 0}\, [\inf\{|y| : y \in \partial\hat{D}(b,p)\}] = \infty,\\
&(d) \quad \lim_{b\longrightarrow\infty} h\,[\,\hat{D}(b,p), F\,] = 0.
\end{aligned} \tag{4.4}$$

$(b), (c), (d)$ follow from the fact (see [HS, Thms 1.1 and 3.6]) that if w is the p capacitary function for F and a convex domain G (i.e w is a weak solution to the p Laplacian in $G \setminus F$, with $w = 1$ on $F, w = 0$ on ∂D, continuously), then

$$\begin{aligned}
&\hat{D}(b,p) \cup \partial\hat{D}(b,p) \subset G \text{ whenever } b > \limsup_{x\longrightarrow\partial G} |\nabla w(x)| \text{ and}\\
&\bar{G} \subset \hat{D}(b,p) \text{ whenever } b < \liminf_{x\longrightarrow\partial G} |\nabla w(x)|.
\end{aligned} \tag{4.5}$$

(b) is a direct consequence of (4.5). To prove (d), given $\epsilon > 0$ choose G a convex domain with C^∞ boundary and $h(F, \partial G) < \epsilon$. From barrier type estimates and (4.5) we see that $\hat{D}(b,p) \subset G$ for b large enough. To prove (c) let $G = B(0,R), R$ large and observe once again from (4.5) and barrier type estimates that $B(0,R) \subset \hat{D}(b,p)$ for b small enough. Finally to prove (a) fix b_1 and suppose $b \in (b_1/2, 3b_1/2)$. We observe from Lemma 2.3 that

$$\alpha_1\, d(x, \partial\hat{D}) \;\leq\; \hat{u}(x) \;\leq\; \alpha_2\, d(x, \partial\hat{D}) \tag{4.6}$$

where $\hat{u} = \hat{u}_{b,p}$ and $0 < \alpha_i = \alpha_i(b_1, p, F, n)$ for $i = 1, 2$. Also from Lemma 2.2 and [HS, Lemma 2.2] we find for given $t, 0 < t < 1/4$, that

$$\text{there exists } \epsilon = \epsilon(b_1, p, F, n, t) > 0 \text{ such that } |\nabla\hat{u}| \geq b + \epsilon \text{ on } \{x : \hat{u}(x) = t\}, \tag{4.7}$$

where $\hat{u} = u_{b,p}$ and $b \in (b_1/2, 3b_1/2)$. From (4.6) and convexity of $\hat{D}(b,p)$, we see for given $\delta > 0$ that there exists $t > 0$ such that

$$h[\{\hat{u}_{b,p} = t\}, \partial\hat{D}(b,p)] \leq \delta \text{ whenever } b \in (b_1/2, 3b_1/2). \tag{4.8}$$

Also we note from Theorem A and [L1] that $\{x : \hat{u}_{b,p}(x) > t\}$ is convex whenever $0 < t < 1$ and $b \in (b_1/2, 3b_1/2)$. Using this note, (4.7), and (4.5) we see for t as in (4.8) that

$$\partial\hat{D}(b_1,p) \subset \{x \in \hat{D}(b,p) : 0 < \hat{u}(x) < t\} \tag{4.9}$$

whenever $\hat{u} = \hat{u}_{b,p}$ and $0 < b_1 - b < \epsilon/2$. Also,

$$\partial\hat{D}(b,p) \subset \{x : 0 < \hat{u}_{b_1,p}(x) < t\} \text{ for } 0 < b - b_1 < \epsilon/2. \tag{4.10}$$

From (4.6) and (4.8)-(4.10) we conclude (a) of (4.4).

From (4.4) and a continuity argument we see there exists $b > 0$ with $\hat{D}(b,p) \subset D$ and $z \in \partial\hat{D}(b,p) \cap \partial D$. If $\hat{u} = \hat{u}_{b,F}$ then by the maximum principle we have $u \geq \hat{u}$ in $\hat{D}$. From smoothness of $\partial\hat{D}(b,p)$ and Schauder type estimates we see that $\lim_{x\longrightarrow z}|\nabla\hat{u}(x)| = b$. Using these facts, (4.1), smoothness of $\partial\hat{D}(b,p)$, and the mean value theorem from differential calculus we conclude that

$$b = \lim_{x\longrightarrow z} |\nabla\hat{u}|(x) \leq \limsup_{x\longrightarrow z} |\nabla u|(x) \leq a. \tag{4.11}$$

Next we claim that

$$a^{p-1} H^{n-1}(\partial D) \leq b^{p-1} H^{n-1}(\partial\hat{D}(b,p)) \tag{4.12}$$

where strict inequality holds unless $\hat{u} \equiv u$. To prove this inequality we observe by the same argument as in the proof of (1.3) applied to $1-u$ that there exists ν a positive Borel measure for which

$$\int |\nabla u|^{p-2}\nabla u \cdot \nabla\phi\, dx = \int \phi\, d\nu \text{ whenever } \phi \in C_0^\infty(D).$$

If $\psi \in C_0^\infty(\mathbb{R}^n)$ then we can write ψ as $\psi_1 + \psi_2$ where $\psi_1 \in C_0^\infty(D)$ and $\psi_2 \in C_0^\infty(\mathbb{R}^n \setminus F)$. Using this fact and (1.3) for u we conclude that

$$\int |\nabla u|^{p-2}\nabla u \cdot \nabla\psi dx = \int \psi\, d\nu - \int \psi\, d\mu. \tag{4.13}$$

Choosing $\psi \in C_0^\infty(\mathbb{R}^n)$ with $\psi \equiv 1$ on $\bar{D}$ we find that

$$\mu(\partial D) = \nu(F). \tag{4.14}$$

Finally since $u \in W_0^{1,p}(\mathbb{R}^n)$ we can use u in (4.13) to obtain from (4.14), (1.7) that

$$a^{p-1}H^{n-1}(\partial D) = \nu(F) = \int |\nabla u|^p\, dx. \tag{4.15}$$

Now the measure associated with $\hat{u}$ is $b^{p-1}H^{n-1}|_{\partial\hat{D}(b,p)}$ so by the same argument as in (4.15) we get

$$b^{p-1}H^{n-1}(\partial\hat{D}(b,p)) = \int |\nabla\hat{u}|^p\, dx. \tag{4.16}$$

From uniqueness of the minimizer in (4.3) we deduce that

$$\int |\nabla u|^p\, dx \leq \int |\nabla\hat{u}|^p\, dx \tag{4.17}$$

with equality only if $\hat{u} \equiv u$. Combining (4.15)-(4.17) we find that (4.12) is true. To conclude the proof of Theorem 1 we note that

$$H^{n-1}(\partial D) \geq H^{n-1}(\partial\hat{D}(b,p)) \tag{4.18}$$

This inequality is proved using the following facts. First the nearest point projection onto $\partial\hat{D}(b,p)$ is a Lipschitz mapping with norm ≤ 1 (see [LU]). Second Hausdorff measure does not increase under such a mapping (see [Ma]). In view of (4.18), (4.12), and (4.11) we see that we have reached a contradiction unless $b = a, \hat{u} = u$, and $D = \hat{D}(b,p)$. The proof of Theorem 1 is now complete. □

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