On Wolff Snowflakes

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ABSTRACT: In this paper we study the dimension of harmonic measure for Wolff Snowflakes.

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1. Introduction.[1] Denote points in Euclidean n-space $\mathbb{R}^n$ by $X = (x,t)$, $x = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}, t \in \mathbb{R}$ and let $dx,\ dX = dxdt$ denote Lebesgue measure on $\mathbb{R}^{n-1}, \mathbb{R}^n$ respectively. Let $|X|$ be the usual Euclidean norm of X and set $B(X,r) = \{Y \in \mathbb{R}^n : |X - Y| < r\}$. Define k dimensional Hausdorff measure, $0 < k \leq n$, in $\mathbb{R}^n$ as follows: For fixed $\delta > 0$ and $E \subseteq \mathbb{R}^n$, let $L(\delta) = \{B(X_i, r_i)\}$ be such that $E \subseteq \bigcup B(X_i, r_i)$ and $0 < r_i < \delta, \quad i = 1, 2, ...$ Set

$$\phi_\delta^k(E) = \inf_{L(\delta)} \left(\sum \alpha(k)\, r_i^k \right),$$

where $\alpha(k)$ denotes the volume of the unit ball in $\mathbb{R}^k$. Then

$$H^k(E) = \lim_{\delta \to 0} \phi_\delta^k(E), \quad 0 < k \leq n.$$

A domain is a connected open subset of $\mathbb{R}^n$. Suppose $D \subset \mathbb{R}^n, n \geq 2$, is a domain, $x_0 \in D$, and that μ is harmonic measure for D with respect to x_0, defined as usual by way of the Riesz representation theorem for continuous functions on ∂D and a method of Perron-Wiener-Brelot (see [H]). In this paper we study the Hausdorff dimension of μ (denoted H-dim μ) defined as follows:

$$\text{H-dim } \mu = \inf\{k : \text{ there exists } E \subset \partial D \text{ with } H^k(E) = 0 \text{ and } \mu(E) = 1\}.$$

To outline previous work, we note that Makarov [M] showed for a simply connected domain $D \subset \mathbb{R}^2$ that H-dim $\mu = 1$. Carleson [C] showed that if $D = \mathbb{R}^2 \setminus K$ where K is a certain scale invariant Cantor set, then H-dim $\mu \leq 1$. He was also the first to recognize the importance of $\int_{\partial D_n} |\nabla g_n| \log |\nabla g_n| \, dH^1$ where g_n is the Green's function for D_n with pole at x_0 and (D_n) is an increasing sequence of smooth domains whose union is D. Jones and Wolff [JW] proved that H-dim $\mu \leq 1$ when $D \subset \mathbb{R}^2$ and μ exists. We also mention results of Batakis [Ba], Kaufmann-Wu [KW], and Volberg [V] who showed for certain fractal domains and domains whose complements are Cantor sets that

$$\text{Hausdorff dimension of } \partial D = \inf\{k : H^k(\partial D) = 0\} > \text{H-dim } \mu.$$

In higher dimensions ($n \geq 3$) Bourgain [B] showed that μ is absolutely continuous with respect to $H^{n-\epsilon}$ on ∂D for some $\epsilon = \epsilon(n) > 0$ and any domain D for which μ exists. Wolff [W] used Carleson's idea and brilliant ideas of his own to construct 'Wolff snowflakes' in $\mathbb{R}^3$ for which either H-dim $\mu > 2$ or H-dim $\mu < 2$. Here we outline his construction. Take the unit cube, $D_0 = \{(x,t) : |x_i| \leq 1/2, i = 1, 2, \text{ and } |t| \leq 1/2\}$, in $\mathbb{R}^3$ and decompose each face into Whitney squares whose sidelengths are proportional to their distance from the edges of ∂D_0. Next one graphs above each face (i.e in the normal direction to a face) a certain scaling and translation of a fixed piecewise linear and compactly supported function, say $\psi(y) = \psi(y_1, y_2)$ in such a way that the resulting graph function has support well inside each square. This can be done in a unique manner provided one first specifies what sides on each square are to correspond to the positive y_1 and y_2 axes under the scaling and translation. Let D_1 be the domain whose boundary is the union of the above surfaces. We say that D_1 is obtained from D_0 by adding 'blips' to each subface. Next one divides the faces of ∂D_1 into Whitney squares whose sidelengths are proportional to their distances from the edges of ∂D_1 and then repeats the above construction in each of the new Whitney squares,

[1] Dedicated to the memory of Thomas H. Wolff. In writing this paper it is fair to say - in the words of Newton - that we stood on the shoulders of a mathematical giant.

obtaining a domain D_2 for which ∂D_2 coincides with the new surface. Iterating this process gives $D = \cup_{m=1}^{\infty} D_m$. This construction is essentially unchanged in $\mathbb{R}^n$. Thus we assume $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$ for some $n \geq 3$ so that $D \subset \mathbb{R}^n$. Finally let μ be harmonic measure for D relative to $(0, 0)$.

Put $D_\infty = \{(x, t) : t > \psi(x)\}$. Let g_∞ denote Green's function with pole at ∞, defined as the harmonic function in D_∞ which is zero on ∂D_∞ and equal to t plus a bounded harmonic function in a neighborhood of ∞. Existence of g_∞ follows from Schwarz reflection, and the fact that ψ has compact support. Using rate theorems for NTA domains to make estimates and applying an ergodic theorem of Shannon-McMillan-Breiman, Wolff [W, p 345] proved for $n = 3$:

Theorem A. *Suppose ψ has sufficiently small Lipschitz norm and set*

$$L = \int_{\partial D_\infty} |\nabla g_\infty| \log |\nabla g_\infty| dH^{n-1}.$$

If $L < 0$ then the Wolff snowflake D obtained from ψ by adding ' blips ' satisfies H-dim $\mu > n - 1$, while if $L > 0$ then H-dim $\mu < n - 1$

Again the proof is unchanged if $D \subset \mathbb{R}^n$. It is not difficult to show that one can approximate each $C_0^\infty(\mathbb{R}^{n-1})$ function ψ on $\mathbb{R}^{n-1}$ by a sequence of piecewise linear functions, (ψ_n), with uniformly bounded supports in such a way that $\psi_n \to \psi$ as $n \to \infty$ uniformly in Lipschitz norm. Also using results of Dahlberg (see [W, Lemma 2.11]) it follows easily that L_n defined relative to ψ_n converges to L as $n \to \infty$. Thus to construct Wolff snowflakes where H-dim $\mu > n - 1$ (H-dim $\mu < n - 1$) it suffices to first find a $C_0^\infty(\mathbb{R}^{n-1})$ function ψ where $L < 0$ ($L > 0$) and then approximate this function by a piecewise linear function. Thus we assume $\psi \in C_0^\infty(\mathbb{R}^{n-1})$.

To continue, let $\phi(x, \epsilon) = \epsilon\psi(x) + \epsilon^2\theta(x)$, where $\psi, \theta \in C_0^\infty(\mathbb{R}^{n-1})$, and let $g(x, t, \epsilon)$ be Green's function for $D(\epsilon) = \{(x, t) \in \mathbb{R}^n : t > \phi(x, \epsilon)\}$ with pole at ∞. Clearly $g(x, t, 0) = t$ and $D(0) = \{(x, t) : t > 0\}$. Put

$$I(\epsilon) = \int_{\partial D(\epsilon)} |\nabla g(\cdot, \epsilon)| \log |\nabla g(\cdot, \epsilon)| dH^{n-1}. \tag{1.1}$$

Using $g(x, \phi(x, \epsilon), \epsilon) = 0$ on $\partial D(\epsilon)$ and Schauder theory or just basic smoothness results about harmonic functions one sees that $I(\epsilon)$ is infinitely differentiable. When $\theta \equiv 0$ Wolff shows that (see Section 3 for details) $I'(0) = 0, I''(0) = 0$

$$I'''(0) = \int_{\mathbb{R}^{n-1}} \left[\left(\tfrac{\partial \mathcal{P}\psi}{\partial t}\right)^3 - 3|\bar{\nabla}\psi|^2 \, \tfrac{\partial \mathcal{P}\psi}{\partial t} \right] dx \tag{1.2}$$

where $\mathcal{P}\psi$ denotes the bounded harmonic extension of ψ to $D(0), \nabla$ denotes the gradient in x, t, and $\bar{\nabla}$ denotes the gradient in x only.

Given a Wolff snowflake D, let $\tilde{\mu}$ be harmonic measure for $\mathbb{R}^n \setminus \bar{D}$ relative to a fixed point in $\mathbb{R}^n \setminus \bar{D}$. Using Taylors Theorem with remainder one sees from (1.2) that if $I'''(0) < 0$ then one can approximate $\epsilon\,\psi$ for $0 < \epsilon \leq \epsilon_0$, by a piecewise linear function for which Wolff's construction gives a Wolff snowflake with H-dim $\mu > n - 1$ and H-dim $\tilde{\mu} < n - 1$ while if $I'''(0) > 0$ one gets a Wolff snowflake with H-dim $\mu < n - 1$ and H-dim $\tilde{\mu} > n - 1$. Wolff obtained such examples for a piecewise linear approximation to a $C_0^\infty(\mathbb{R}^n)$ approximation of

$$\psi(x) = -\frac{1}{(|x|^2 + 1)^{(n/2-1)}}, x \in \mathbb{R}^{n-1}, \tag{1.3}$$

when $n = 3$. For this ψ it is not difficult to show that $I'''(0) < 0$ when $n \geq 3$. However for a given $\psi \in C_0^\infty(\mathbb{R}^{n-1})$ and $n \geq 3$ the sign of the integral in (1.2) seemed rather mysterious (the integral is zero when $n = 2$ as follows from Cauchy's theorem). To shed some light on this problem, we have the following theorem which is proved in section 2.

Theorem 1 *Let $\psi \in C_0^\infty(\mathbb{R}^{n-1}), \psi \leq 0$ be radial (i.e $\psi(|x|) = \psi(x)$) and nondecreasing as a function of $|x|$ on $[0, \infty)$. If $\psi \not\equiv 0$, $\theta \equiv 0$, then $I'''(0) < 0$.*

To sum up once again, if ψ is as in Theorem 1 and $\epsilon > 0$ is small enough, then $\epsilon\psi$ can be approximated arbitrarily closely in Lipschitz norm by a piecewise linear function in such a way that the corresponding Wolff snowflake has H-dim $\mu > n - 1$ while H-dim $\tilde{\mu} < n - 1$. If $\epsilon < 0$ is small enough, then the opposite inequalities hold. The proof of Theorem 1 is quite simple. Unfortunately it was only after a considerable effort that we conceived of Theorem 1 and its proof. For a discussion of the motivation behind Theorem 1 and the efforts involved in its proof we refer the reader to the remark following the proof of Theorem 1.

We note that if $I'''(0) = 0$ for a given $\psi, \theta \in C_0^\infty(\mathbb{R}^{n-1})$, then the sign of $I''''(0)$ becomes important in Wolff's construction. In this case Wolff notes (again when $\theta \equiv 0$) that if $I''''(0) < 0$ (see [W, Remark, p 363]), then there is a piecewise linear function for which the corresponding Wolff snowflake satisfies H-dim $\mu > n - 1$ and H-dim $\tilde{\mu} > n - 1$. If $I''''(0) > 0$, then the opposite inequalities hold for H-dim μ, H-dim $\tilde{\mu}$. It is conjectured in [W] that there exist examples where the fourth derivative has either sign, although Wolff notes that he has not computed $I''''(0)$ and this derivative looks rather complicated. Thus he conjectures the existence of Wolff snowflakes in $\mathbb{R}^n, n \geq 3$, for which harmonic measure on both sides of the boundary can have Hausdorff dimension either $< n - 1$ or $> n - 1$. In section 3 we compute the first four derivatives of I at $\epsilon = 0$ relative to $\psi, \theta \in C_0^\infty(\mathbb{R}^{n-1})$ (see (3.20)). In section 4 we begin the proof of the following theorem :

Theorem 2. *If $n \geq 3$, $\psi(x) = (2 - n)\, x_1\, (|x|^2 + 1)^{-n/2}$, $x \in \mathbb{R}^{n-1}$, $\theta \equiv 0$ and*

$$A = \frac{3n\pi^{n/2}(n-2)^4\Gamma(3n/2 + 1/2)2^{-2n-1}}{\Gamma(n)\Gamma(n + 1/2)(n + 1)}$$

then $I'''(0) = 0$ and

$$A^{-1}I''''(0) = \left\{ \frac{-1}{16} \frac{\Gamma(n + \frac{1}{2})\Gamma(n - \frac{1}{2})\Gamma(\frac{n}{2} - 1)\Gamma(\frac{3n}{2} + 1)}{\Gamma(n + 1)^2\Gamma(\frac{n}{2} - \frac{1}{2})\Gamma(\frac{3n}{2} + \frac{1}{2})} \times \right.$$

$$\left. \frac{n(16 + 16n - 72n^2 - 44n^3 - 11n^4 + 12n^6)}{(n + 1)^2(n - 1)} + \frac{3n^3 + 2n^2 + 7n + 2}{2n + 1} \right\} > 0.$$

In Theorem 2, it is understood that $I'''(0), I''''(0)$ denote the integrals in (1.2) and (3.20) rather than the third and fourth derivatives of $I(\epsilon)$ in (1.1) (defined relative to the above ψ) since for derivatives it is necessary that $\psi \in C_0^\infty(\mathbb{R}^{n-1})$. Let $\psi_m = \phi_m\psi$, where $0 \leq \phi_m \in C_0^\infty(\mathbb{R}^{n-1})$ is a radial bump function with $\phi_m(x) \equiv 1$ when $|x| < m$ and $\phi_m(x) = 0$ when $|x| > 2m$. Also ψ is as in Theorem 2. We note that the integral in (1.2) corresponding to ψ_m is still zero. Indeed ψ_m is an odd function of x_1 so using difference quotients, we deduce first that $\frac{\partial}{\partial t}\mathcal{P}\psi_m$ is odd in x_1 and second that the integrand in (1.2) is odd in x_1. Thus our claim is true. Also, choosing (ϕ_m) appropriately, one sees from Hardy space theory (see [S, ch 7]) that the integral in (3.20) defining $I''''(0)$ relative to ψ_m converges to the corresponding integral defined relative to ψ as $m \to \infty$. Thus there exists a $C_0^\infty(\mathbb{R}^{n-1})$ approximation to ψ in Theorem 2 for which $I'''(0) = 0$ and $I''''(0) > 0$. We can now argue as in [W] to get the following corollary.

Corollary 1. *There exist Wolff snowflakes in* $\mathbb{R}^n, n \geq 3$, *with H-dim* $\mu < n-1$ *and H-dim* $\tilde{\mu} < n-1$.

We do not know whether there are examples in $\mathbb{R}^n$ of graph domains $D(\epsilon)$ with $\psi \in C_0^\infty(\mathbb{R}^{n-1}), \theta \equiv 0$ where $I'''(0) = 0, I''''(0) < 0$. We tried numerous examples in $\mathbb{R}^4$ and all gave positive fourth derivative (see section 6). However, by considering more general graph domains we can answer Wolff's main question. We first prove in section 6.

Theorem 3. *If* ψ *is as in Theorem 2 and* $\theta = \psi\psi_t$, *then* $I'''(0) = 0$ *and*

$$I''''(0) = -\frac{3\,\pi^{n/2}\,\Gamma(\frac{3n}{2}+1/2)(n-2)^4\,n^2\,(3n^3+2n^2-13n-6)}{\Gamma(n+\frac{1}{2})\Gamma(n)\,2^{2n+1}\,(n+1)(2n+3)(2n+1)} < 0.$$

We note that $I'''(0)$ (see (3.14)) is independent of the choice of θ. Using this fact and repeating the argument after Theorem 2 (except now one approximates $\epsilon\psi + \epsilon^2\theta$) we obtain

Corollary 2. *There exist Wolff snowflakes in* $\mathbb{R}^n, n \geq 3$, *with H-dim* $\mu > n-1$ *and H-dim* $\tilde{\mu} > n-1$.

In view of Corollaries 1 and 2 we conclude that Wolff's main conjecture is valid. Finally we mention that in section 6 we shall make three conjectures related to Theorems 2, 3 and Corollaries 1, 2, as well as discuss some examples in $\mathbb{R}^4$ where the first two conjectures are valid.

2. Proof of Theorem 1. To lighten the notation we write $\psi = \psi(x,t)$ for the bounded harmonic extension $\mathcal{P}\psi(x,t)$ of $\psi(x)$ to $D(0) = \{(x,t) : t > 0\}$. Thus $\psi(x,0)$ satisfies the hypotheses of Theorem 1 and we can write $\psi(r,t)$ for $\psi(x,t)$, where $r = |x|$. Note that ψ satisfies Laplace's equation when $t > 0$ so in cylindrical coordinates

$$r^{2-n}(r^{n-2}\psi_r)_r + \psi_{tt} = 0. \tag{2.1}$$

We shall need the following facts :

$$\lim_{r \to 0} \psi_r(r,t) = 0, 0 \leq t < \infty, \tag{2.2}$$

$$\psi_r(r,t) \geq 0 \text{ when } r,t > 0. \tag{2.3}$$

(2.2) is more or less obvious as one sees using the fact that $\psi(|x|,t)$ is infinitely differentiable at $x = 0$ when $t \geq 0$. One can also prove (2.2) for $t > 0$ by differentiating under the integral sign in the Poisson integral formula for ψ and taking a limit as $r \to 0$. As for (2.3) assume the contrary. Then from (2.2), the fact that $\psi(r,0)$ is nondecreasing on $[0,\infty)$ and the fact that ψ is infinitely differentiable in $\bar{D}(0)$ (as follows from Schauder theory) we deduce that ψ_r has a negative minimum (absolute) for some

(r,t) with $r>0, t>0$. At this value of (r,t) we have $\psi_{rr}=0$ and $\psi_{rrr}+\psi_{rtt}\geq 0$ by the second derivative test from elementary calculus. Now (2.1) can be written as $\psi_{rr}+(n-2)r^{-1}\psi_r+\psi_{tt}=0$. Differentiating this equation with respect to r we get $\psi_{rrr}+(n-2)\psi_{rr}r^{-1}-(n-2)r^{-2}\psi_r+\psi_{rtt}=0$ which leads to a contradiction since from the above remark this expression must be positive. Thus (2.3) is valid.

Now if $\Lambda=(0,\infty)\times(0,\infty)$, we have

$$\begin{aligned}\int_0^\infty [\psi_t^3-3\psi_r^2\psi_t](r,0)\, r^{n-2}dr &= \int_\Lambda \frac{d}{dt}[-\psi_t^3+3\psi_r^2\psi_t]r^{n-2}dr\,dt\\ &= \int_\Lambda [-3\psi_t^2\psi_{tt}+6\psi_r\psi_t\psi_{tr}]r^{n-2}dr\,dt+3\int_\Lambda \psi_r^2\psi_{tt}r^{n-2}dr\,dt = I_1+I_2.\end{aligned}\tag{2.4}$$

To handle I_1 in (2.4) we use (2.1) and integrate by parts in the r variable holding t fixed. We get

$$I_1=\int_\Lambda 3\psi_t^2\ (r^{n-2}\psi_r)_r\, dr\,dt+\int_\Lambda 6\psi_r\psi_t\psi_{tr}r^{n-2}dr\,dt=0. \tag{2.5}$$

As for I_2 we again use (2.1) and write

$$\begin{aligned}I_2 &= -3\int_\Lambda \psi_r^2(r^{n-2}\psi_r)_r\, dr\,dt\\ &= -\int_\Lambda (r^{n-2}\psi_r^3)_r\, dr\,dt-2(n-2)\int_\Lambda r^{n-3}\psi_r^3\, dr\,dt<0\end{aligned}\tag{2.6}$$

since the first integral is zero as follows from integrating in r holding t fixed while the second integral is negative as follows from (2.2) and our assumption that $\psi\not\equiv 0$. We note that in obtaining (2.5), (2.6) we have integrated by parts numerous times assuming ψ and its derivatives have rapid enough decay at ∞ to insure there are no boundary terms. This procedure is justified as we see from simple estimates on the Poisson integral of ψ and its derivatives in view of the fact that $\psi(\cdot,0)\in C_0^\infty(\mathbb{R}^{n-1})$. Using (2.5), (2.6) in (2.4) we conclude that $I'''(0)<0$. The proof of Theorem 1 is complete. □

Remark. Our interest in Theorem 1 stems from work in [LV], [LV1] where ‘ blips ’ were added to a ball in such a way as to produce an example of a bounded domain $D\subset\mathbb{R}^n$, $n\geq 3$ ($D\neq$ ball) with $\mu=H^{n-1}$ on ∂D. Here μ denotes harmonic measure defined relative to $0\in D$. In [LV] we showed that D can be chosen homeomorphic to a ball while in [LV1] we improved this result by showing that D can be the image of a ball under a quasiconformal mapping of $\mathbb{R}^n$. In this construction the sign of the integral in (1.1) was crucial. In [LV] certain multiples, scalings, translations, and $C_0^\infty(\mathbb{R}^{n-1})$ approximations of the function in (1.3) (restricted to $\mathbb{R}^{n-1}$) were used to construct our ‘ blips ’ while in [LV1] it was necessary (for construction of our quasiconformal mapping) to add spherical ‘ blips.’ As a consequence we had to prove Theorem 1 when $\psi(x)=-\max(1-|x|^2,0)$, $x\in\mathbb{R}^{n-1}$. To appreciate the simple proof in Theorem 1 we briefly outline our efforts to prove Theorem 1 for this ψ. We first used Maple to obtain strong evidence that $I'''(0)<0$ when $n=2,4,6,8,10$. Then using separation of variables and integrals of Bessel functions we found a representation of ψ_t via hypergeometric functions. If $r=|x|$, then for $0<r<1$,

$$-\psi_t(r,0)\ =\ -c_n\, F(n/2,-1/2,(n-1)/2,r^2),$$

where $F(a,b,c,z)$ is the hypergeometric function, $c_n = 2\Gamma(n/2)/[\Gamma((n-1)/2)\,\Gamma(3/2)]$ and Γ is the Euler Gamma function. Using properties of hypergeometric functions we found many nice formulas involving ψ_t. For example, if $n = 2k$ where $k \geq 2$ is an integer then

$$\psi_t(r,0) = \frac{2(-1)^{k-2}}{\pi(k-1)!(1-r^2)} \cdot \frac{d^{k-2}}{d(r^2)^{k-2}}\left[(1-r^2)^{k-1}\left\{\ln\left(\frac{1+r}{1-r}\right)\left(\frac{1}{2r}-\frac{3r}{2}\right)+3\right\}\right]$$

whenever $0 < r < 1$. However none of our formulas were of much help in calculating the integral in (1.2). Finally we tried asymptotic estimates as $n \to \infty$ of $I'''(0)$ using

$$-F(n/2,-1/2,(n-1)/2,r^2) \geq -1 + \frac{n}{n-1}[1-(1-r^2)^{1/2}].$$

From this estimate and Stirling's formula we found after a lot of work that $I'''(0) < 0$ for $n \geq 10$. Based on this example, we formulated the statement

If $-\psi$ is radial, positive, and smooth on $\mathbb{R}^{n-1}$ with compact support, then $I'''(0) < 0$,

which was disproved with the following example. For fixed $(x,t) \in \mathbb{R}^4$ and $0 < a, a \neq 1$, let

$$-\psi(x,t) = \frac{1}{|x|^2+(t+1)^2} - \frac{a^2}{|x|^2+(t+a)^2}\,. \tag{2.7}$$

Then clearly ψ is harmonic in $\mathbb{R}^n \setminus \{(0,-1),(0,-a)\}$ and $\psi(x,0)$ is radial. Also for $0 < a < 1$ one can show that $-\psi(x,0) > 0$ and that the corresponding integral in (1.2) is 0. Although $\psi(x,0)$ does not have compact support it turns out that for a given a, this function can be approximated by an infinitely differentiable nonnegative radial function with compact support in such a way that the integral in (1.2) remains zero (see section 6 for more discussion of this example). Reworking the statement above, in light of this example, we formulated Theorem 1. The simple proof was only found after trying several more complicated arguments.

3. The Fourth Derivative. In this section we find the fourth derivative of $I(\epsilon)$ at $\epsilon = 0$ (defined as in (1.1) for $\psi, \theta \in C_0^\infty(\mathbb{R}^{n-1})$). To begin, as in section 1, put $\phi(x,\epsilon) = \epsilon\psi(x) + \epsilon^2\theta(x)$, set $D(\epsilon) = \{(x,t) : t > \phi(x,\epsilon)\}, \epsilon \in [-1,1]$, and let $g(\cdot,\epsilon)$ be the corresponding Green's function for $D(\epsilon)$ with pole at ∞. Following Wolff [W, pp 360-362] we observe from Schauder type estimates that $g(\cdot,\epsilon)$ is infinitely differentiable in $\bar{D}(\epsilon)\times[-1,1]$ as a function of (x,ϵ). Also, since $g(x,\phi(x,\epsilon),\epsilon) \equiv 0$ when $x \in \mathbb{R}^{n-1}$ we get from differentiating with respect to $x_i, 1 \leq i \leq n-1$,

$$\bar{\nabla} g(x,\phi(x,\epsilon),\epsilon) = -g_t(x,\phi(x,\epsilon),\epsilon)(\epsilon\bar{\nabla}\psi(x) + \epsilon^2\bar{\nabla}\theta(x)). \tag{3.1}$$

where once again $\bar{\nabla}$ denotes the gradient in x only. Differentiating with respect to ϵ and using the chain rule gives, for all $x \in \mathbb{R}^{n-1}$ and $\epsilon \in [-1,1]$

$$g_\epsilon(x,\phi(x,\epsilon),\epsilon) + g_t(x,\phi(x,\epsilon),\epsilon)\,(\psi(x) + 2\epsilon\theta(x)) = 0 \tag{3.2}$$

where $g_\epsilon(x,t,\epsilon)$ denotes the partial of $g(x,t,\epsilon)$ with respect to ϵ. Evaluating (3.2) at $\epsilon = 0$ yields

$$g_\epsilon(x,0,0) = -g_t(x,0,0)\psi(x) = -\psi(x) \tag{3.3}$$

since $g(x,t,0) = t$. Furthermore since $g_\epsilon(x,t,\epsilon)$ is harmonic in (x,t) we see that $g_\epsilon(x,t,0) = -\mathcal{P}\psi(x,t)$ where $\mathcal{P}\psi$ as defined earlier denotes the bounded harmonic extension of ψ to $D(0)$, i.e the Poisson integral of ψ. As in section 2 we write $\psi(x,t)$ for $\mathcal{P}\psi(x,t)$ and shall also write $\theta(x,t)$ for $\mathcal{P}\theta(x,t)$. Taking another derivative with respect to ϵ in (3.2) gives for all $x \in \mathbb{R}^{n-1}$ and $\epsilon \in [-1,1]$,

$$\begin{aligned} g_{tt}(x,\phi(x,\epsilon),\epsilon)\,(\psi(x)+2\epsilon\theta(x))^2 + 2g_{t\epsilon}(x,\phi(x,\epsilon),\epsilon)\,(\psi(x)+2\epsilon\theta(x)) \\ + g_{\epsilon\epsilon}(x,\phi(x,\epsilon),\epsilon) + 2\theta(x)g_t(x,\phi(x,\epsilon),\epsilon) = 0. \end{aligned} \tag{3.4}$$

Evaluating at $\epsilon = 0$ yields

$$g_{\epsilon\epsilon}(x,0,0) = 2\psi(x)\,\psi_t(x,0)\; - 2\theta(x),\; x \in \mathbb{R}^{n-1}. \tag{3.5}$$

From this equality and harmonicity of $g_{\epsilon\epsilon}$ we conclude

$$g_{\epsilon\epsilon}(x,t,0) = 2\mathcal{P}(\psi\psi_t - \theta)(x,t),\; (x,t) \in D(0). \tag{3.6}$$

Next observe that

$$I(\epsilon) = \int_{\mathbb{R}^{n-1}} |\nabla g(x,\phi(x,\epsilon),\epsilon)|\;\log|\nabla g(x,\phi(x,\epsilon),\epsilon)|\sqrt{1+|\bar{\nabla}\phi(x,\epsilon)|^2}dx. \tag{3.7}$$

Also $(-\bar{\nabla}\phi(x,\epsilon),1)\,(1+|\bar{\nabla}\phi|^2)^{-1/2}$ is the inner normal to $\partial D(\epsilon)$ at $(x,\phi(x,\epsilon))$ and $\nabla g(x,\phi(x,\epsilon),\epsilon)$ is parallel to this inner normal. Thus at $(x,\phi(x,\epsilon))$,

$$|\nabla g| = \frac{-\langle\bar{\nabla}\phi,\bar{\nabla}g\rangle + g_t}{\sqrt{1+|\bar{\nabla}\phi|^2}}$$

where $\langle\cdot,\cdot\rangle$ denotes the usual inner product on $\mathbb{R}^{n-1}$. Substituting this expression into the integrand in (3.7), cancelling the square roots, and taking logarithms we obtain

$$\left[-\langle\bar{\nabla}g,\bar{\nabla}\phi\rangle + g_t\right]\,\left\{\log(-\langle\bar{\nabla}g,\bar{\nabla}\phi\rangle + g_t) - \tfrac{1}{2}\log(1+|\bar{\nabla}\phi(x,\epsilon)|^2)\right\}$$

Using (3.1) and the above equality the integrand in (3.7) now becomes at $(x,\phi(x,\epsilon)), x \in \mathbb{R}^{n-1}$,

$$\left(1+|\bar{\nabla}\phi(x,\epsilon)|^2\right)\,g_t\,\left\{\log(g_t) + \tfrac{1}{2}\log(1+|\bar{\nabla}\phi(x,\epsilon)|^2)\right\}.$$

We expand this expression in ϵ at 0 through powers of ϵ^4, since our goal is to find $I''''(0)$. For this purpose let

$$g_t(x,\phi(x,\epsilon),\epsilon) = 1 + a_1\epsilon + a_2\epsilon^2 + a_3\epsilon^3 + a_4\epsilon^4 + \cdots$$

and set

$$w = a_1\epsilon + a_2\epsilon^2 + a_3\epsilon^3 + a_4\epsilon^4 + \cdots$$

The logarithm of g_t then becomes $w - w^2/2 + w^3/3 - w^4/4 + \cdots$. Also

$$|\bar{\nabla}\phi(x,\epsilon)|^2 = \epsilon^2|\bar{\nabla}\psi|^2(x) + 2\epsilon^3\langle\bar{\nabla}\psi(x),\bar{\nabla}\theta(x)\rangle + \epsilon^4|\bar{\nabla}\theta(x)|^2$$

and

$$\log(1+|\bar\nabla\phi(x,\epsilon)|^2) = |\bar\nabla\phi(x,\epsilon)|^2 - \tfrac12\left(|\bar\nabla\phi(x,\epsilon)|^2\right)^2 + \cdots$$
$$= \epsilon^2|\bar\nabla\psi(x)|^2 + 2\epsilon^3\langle\bar\nabla\psi(x),\bar\nabla\theta(x)\rangle + \epsilon^4(|\bar\nabla\theta(x)|^2 - |\bar\nabla\psi(x)|^4/2) + \ldots + .$$

Using these equalities, multiplying out the above expression and collecting powers of ϵ up to order 4 we have for the integrand in (3.7)

$$\begin{aligned}
&\epsilon a_1 + \tfrac{\epsilon^2}{2}\left(2a_2 + a_1^2 + |\bar\nabla\psi|^2\right) + \tfrac{\epsilon^3}{6}\left(6a_3 + 6a_1a_2 - a_1^3 + 9a_1|\bar\nabla\psi|^2 + 6\langle\bar\nabla\psi,\bar\nabla\theta\rangle\right) + \\
&\tfrac{\epsilon^4}{24}\left[24a_4 + 24a_1a_3 + (36|\bar\nabla\psi|^2 - 12a_1^2 + 12a_2)a_2 + (2a_1^2 + 12|\bar\nabla\psi|^2)a_1^2\right. \\
&\left.+6|\bar\nabla\psi|^4 + 12|\bar\nabla\theta|^2 + 72\langle\bar\nabla\psi,\bar\nabla\theta\rangle\, a_1\right] + O(\epsilon^5),
\end{aligned} \tag{3.8}$$

where the last term denotes a bounded term in ϵ^5 on [-1, 1]. Now from Taylor's theorem, as well as smoothness of g and its derivatives on $\bar D(\epsilon)$, we deduce that $k!\,a_k = k!\,a_k(x)$ is the k th derivative of $g_t(x,\phi(x,\epsilon),t)$ evaluated at $\epsilon = 0$. Moreover this k th total derivative in ϵ is easily seen to be equal to the operator $(\psi + 2\epsilon\theta)\frac{\partial}{\partial t} + \frac{\partial}{\partial\epsilon}$ applied k times to g_t at $(x,\phi(x,\epsilon))$. Using this fact and (3.3) we deduce at $x\in\mathbb{R}^{n-1}$ and $\epsilon = 0$ that

$$a_1 = g_{\epsilon t} = -\psi_t. \tag{3.9}$$

From (3.7) - (3.9) we conclude that

$$I'(0) = \int_{\mathbb{R}^{n-1}} a_1 = -\int_{\mathbb{R}^{n-1}} \psi_t(x,0)\,dx = 0 \tag{3.10}$$

as follows from Green's first identity (the integral is the normal derivative of a harmonic function) using smoothness of derivatives of ψ_t in $\bar D(0)$ and the fact that $\partial^k\psi_t(x,t) = \mathrm{O}\left((|x|^2+t^2)^{-(n+k)/2}\right.$ in a neighborhood of ∞. Here ∂^k denotes an arbitrary k th partial of ψ_t in either x or t. Also, using harmonicity of g_ϵ and (3.3) we get at $x\in\mathbb{R}^{n-1}$ and $t = 0 = \epsilon$,

$$2a_2 = \psi^2\, g_{t^3} + 2\theta\, g_{t^2} + 2\psi\, g_{\epsilon t^2} + g_{\epsilon^2 t} = 2\psi\bar\Delta\psi + g_{\epsilon^2 t}. \tag{3.11}$$

Using (3.11), (3.7)-(3.9), Green's first identity and integrating by parts we find in view of the above decay estimates,

$$\begin{aligned}
I''(0) = \int_{\mathbb{R}^{n-1}}(a_1^2 + 2a_2 + |\bar\nabla\psi|^2) &= \int_{\mathbb{R}^{n-1}}[\psi_t^2 + 2\psi\bar\Delta\psi + g_{\epsilon^2 t} + |\bar\nabla\psi|^2]dx \\
&= \int_{\mathbb{R}^{n-1}}[\psi_t^2 - |\bar\nabla\psi|^2]dx = 0.
\end{aligned} \tag{3.12}$$

The last integral is zero from a Rellich type identity (see [S, ch 7]). Next we note from (3.3) that at $x\in\mathbb{R}^{n-1}, t = 0 = \epsilon$,

$$\begin{aligned}
6a_3 &= \psi^3\, g_{t^4} + 6\psi\,\theta g_{t^3} + 3\psi^2\, g_{\epsilon t^3} + 3\psi g_{\epsilon^2 t^2} + g_{\epsilon^3 t} + 6\theta\, g_{\epsilon t^2} \\
&= 3\psi^2\,\bar\Delta\psi_t - 3\psi\,\bar\Delta g_{\epsilon^2} + g_{\epsilon^3 t} + 6\theta\,\bar\Delta\psi.
\end{aligned} \tag{3.13}$$

Using (3.13), (3.11), (3.7)- (3.9), integrating by parts using harmonicity of derivatives of g_t, as well as Green's first and second identities (to handle the terms in $g_{\epsilon^3 t}$ and $g_{\epsilon^2 t}$), we conclude that

$$\begin{aligned} I'''(0) &= \int_{\mathbb{R}^{n-1}}(6a_3+6a_1a_2-a_1^3+9a_1|\bar\nabla\psi|^2\ +6\langle\bar\nabla\psi,\bar\nabla\theta\rangle\ )dx \\ &= \int_{\mathbb{R}^{n-1}}\big[3\psi^2\,\bar\Delta\psi_t-3\psi\bar\Delta g_{\epsilon^2}+g_{\epsilon^3 t}-3\psi_t(2\psi\bar\Delta\psi+g_{\epsilon^2 t}) \\ &\quad +\psi_t^3-9\psi_t|\bar\nabla\psi|^2+6\theta\bar\Delta\psi+6\langle\bar\nabla\psi,\bar\nabla\theta\rangle\,\big]\,dx \\ &= \int_{\mathbb{R}^{n-1}}\big(\psi_t^3\ -\ 3\psi_t\,|\bar\nabla\psi|^2\,\big)\,dx. \end{aligned} \tag{3.14}$$

We note that the derivatives in (3.10), (3.12), (3.14) agree with those in [W] even when $\theta\not\equiv 0$. Finally observe for $x\in\mathbb{R}^{n-1}, t=0=\epsilon$, that

$$\begin{aligned} 24a_4 &= \psi^4 g_{t^5}+4\psi^3 g_{\epsilon t^4}+12\theta\psi^2 g_{t^4}\ +\ 12\theta^2 g_{t^3}+ \\ &\quad 6\psi^2\,g_{\epsilon^2 t^3}+4\psi g_{\epsilon^3 t^2}+g_{\epsilon^4 t}+24\theta\psi g_{\epsilon t^3}+12\theta g_{t^2\epsilon^2} \\ &= -4\psi^3\,\bar\Delta\bar\Delta\psi-6\psi^2\bar\Delta g_{\epsilon^2 t}+\ 4\psi g_{\epsilon^3 t^2}+g_{\epsilon^4 t}+24\theta\psi\bar\Delta\psi_t-12\theta\bar\Delta g_{\epsilon^2}. \end{aligned} \tag{3.15}$$

Using (3.15), (3.13), (3.11), (3.7)-(3.9) we deduce that at $x\in\mathbb{R}^{n-1}, \epsilon=0$ we have

$$\begin{aligned} I''''(0) &= \int_{\mathbb{R}^{n-1}}\big(24a_4+24a_1a_3+[36|\bar\nabla\psi|^2-12a_1^2+12a_2]a_2+\ (2a_1^2+12|\bar\nabla\psi|^2)a_1^2 \\ &\quad +6|\bar\nabla\psi|^4+12|\bar\nabla\theta|^2\ -\ 72\langle\bar\nabla\psi,\bar\nabla\theta\rangle\,\psi_t\big)\,dx \\ &= \int_{\mathbb{R}^{n-1}}\big\{-4\psi^3\,\bar\Delta\bar\Delta\psi-6\psi^2\bar\Delta g_{\epsilon^2 t}+4\psi g_{\epsilon^3 t^2}+g_{\epsilon^4 t}+24\theta\psi\bar\Delta\psi_t-12\theta\bar\Delta g_{\epsilon^2} \\ &\quad -\ 4\psi_t[3\psi^2\,\bar\Delta\psi_t-3\psi\bar\Delta g_{\epsilon^2}+g_{\epsilon^3 t}+6\theta\bar\Delta\psi] \\ &\quad +[18|\bar\nabla\psi|^2-6\psi_t^2+3(2\psi\bar\Delta\psi+g_{\epsilon^2 t})](2\psi\bar\Delta\psi+g_{\epsilon^2 t}) \\ &\quad +2\psi_t^4+12|\bar\nabla\psi|^2\psi_t^2+6|\bar\nabla\psi|^4+12|\bar\nabla\theta|^2-72\langle\bar\nabla\psi,\bar\nabla\theta\rangle\psi_t\big\}\,dx \end{aligned} \tag{3.16}$$

Using Green's first identity we see that the term in $g_{\epsilon^4 t}$ integrates to zero. Also using Green's second identity we deduce that the terms in $g_{\epsilon^3 t^2}$ and $g_{\epsilon^3 t}$ taken together integrate to zero. Thus the only terms left involve at most two partials in ϵ. As for these terms we observe that

$$\int_{\mathbb{R}^{n-1}}\big[(12\psi\bar\Delta\psi+18|\bar\nabla\psi|^2\ -\ 6\psi_t^2)g_{\epsilon^2 t}-6\psi^2\bar\Delta g_{\epsilon^2 t}\big]\,dx=\int_{\mathbb{R}^{n-1}}6(|\bar\nabla\psi|^2-\psi_t^2)\,g_{\epsilon^2 t}\,dx\ . \tag{3.17}$$

Also combining the two terms in $\bar\Delta g_{\epsilon^2}$ we see from (3.5) that the sum can be written as $6g_{\epsilon^2}\bar\Delta g_{\epsilon^2}$. Using this fact, (3.5), integration by parts, and the Rellich type formula mentioned after (3.12) we see that

$$\int_{\mathbb{R}^{n-1}} \left[6g_{\epsilon^2}\,\bar\Delta g_{\epsilon^2} + 3(g_{\epsilon^2 t})^2\right] dx \;=\; \int_{\mathbb{R}^{n-1}} \left[-6|\bar\nabla g_{\epsilon^2}|^2 \;+\; 3(g_{\epsilon^2 t})^2\right] dx \;=\; \int_{\mathbb{R}^{n-1}} -\,3|\bar\nabla g_{\epsilon^2}|^2 dx$$

$$= -12\int_{\mathbb{R}^{n-1}} \big(\,\psi^2|\bar\nabla\psi_t|^2 \;+\; \psi_t^2|\bar\nabla\psi|^2 \;+\; 2\psi\psi_t\,\langle\bar\nabla\psi,\bar\nabla\psi_t\rangle \tag{3.18}$$

$$+\,|\bar\nabla\theta|^2 \;-\; 2\langle\bar\nabla\psi,\bar\nabla\theta\rangle\psi_t \;-\; 2\psi\langle\bar\nabla\psi_t,\bar\nabla\theta\rangle\,\big)\,dx.$$

Next note that

$$\int_{\mathbb{R}^{n-1}} \left(-4\psi^3\bar\Delta\bar\Delta\psi + 12\psi^2(\bar\Delta\psi)^2\right)\,dx \;=\; \int_{\mathbb{R}^{n-1}} -24\psi|\bar\nabla\psi|^2\,\bar\Delta\psi dx. \tag{3.19}$$

Using (3.17)-(3.19) in (3.16) we get

$$I''''(0) = \textstyle\int_{\mathbb{R}^{n-1}} \big\{12\psi|\bar\nabla\psi|^2\bar\Delta\psi + 24\psi\psi_t\langle\bar\nabla\psi,\bar\nabla\psi_t\rangle \;+\; 6(|\bar\nabla\psi|^2 - \psi_t^2)\,g_{\epsilon^2 t}$$
$$+\,2\psi_t^4 + 12|\bar\nabla\psi|^2\psi_t^2 + 6|\bar\nabla\psi|^4 \;-\; 24\langle\bar\nabla\psi,\bar\nabla\theta\rangle\psi_t\big\}\;dx \tag{3.20}$$

which we use for computations in sections 4 and 5.

4. The Bad Integral. Let $h(x,t) = (|x|^2+(t+1)^2)^{1-n/2}$ be the fundamental solution to Laplace's equation in $\mathbb{R}^n$ with pole at $(0,-1)$ and recall that $\theta\equiv 0$, while

$$\psi(x,t) = \frac{\partial h(x,t)}{\partial x_1} = \frac{(2-n)x_1}{[|x|^2+(t+1)^2]^{n/2}} \tag{4.1}$$

is the function in Theorem 2 when $t=0$. Also as pointed out in section 1, the integrand in (1.2) or (3.14) for this ψ is odd in x_1. Thus, $I'''(0)=0$. In this section we begin the calculation of the integral defining $I''''(0)$ in (3.20) for ψ in (4.1), $\theta\equiv 0$, by finding the ' bad integral, '

$$\int_{\mathbb{R}^{n-1}} 6(|\bar\nabla\psi|^2 - \psi_t^2)\,g_{\epsilon^2 t}\,dx. \tag{4.2}$$

We begin with computation of $g_{\epsilon\epsilon}(x,0)$. To do this we note that the solution F to the Neumann problem for the Laplacian in $D(0)=\{(x,t)\in\mathbb{R}^n : t>0\}$ with boundary function f (see [S, ch 7]) can be written as

$$F(x,t) = c_n\int_{\mathbb{R}^{n-1}} (|x-y|^2+t^2)^{1-n/2} f(y)dy \tag{4.3}$$

provided f is sufficiently smooth and has rapid enough decay at ∞. Here

$$(n-2)c_n = \Gamma(\tfrac{n}{2})\,\pi^{-n/2}$$

where Γ is the Euler Gamma function. We shall often use the fact that if $x,y>0$, then

$$\int_0^{\pi/2} \sin^{2x-1}\phi\,\cos^{2y-1}\phi\,d\phi = \frac{\Gamma(x)\Gamma(y)}{2\Gamma(x+y)}. \tag{4.4}$$

Taking $f = 2\psi\psi_t$ we see that

$$f \in C^\infty(\mathbb{R}^{n-1}) \text{ and } |f(x)| \leq c(1+|x|)^{1-2n} \text{ for } x \in \mathbb{R}^{n-1} \tag{4.5}$$

and some $c \geq 1$. Using (4.3), (4.5) and Hardy space theory (see [S, ch 7]) we deduce that

$$\begin{aligned} \frac{\partial g_{\epsilon\epsilon}}{\partial t}(x,0) &= -2c_n \lim_{t \longrightarrow 0} \frac{\partial^2}{\partial t^2} \int_{\mathbb{R}^{n-1}} (|x-y|^2+t^2)^{1-n/2} \psi\psi_t(y,0)dy \\ &= 2c_n \lim_{t \longrightarrow 0} \int_{\mathbb{R}^{n-1}} \bar{\Delta}[(|x-y|^2+t^2)^{1-n/2}]\, \psi\psi_t(y,0)dy \\ &= 2c_n \int_{\mathbb{R}^{n-1}} |y|^{2-n} \bar{\Delta}(\psi\psi_t)(x-y,0)dy. \end{aligned} \tag{4.6}$$

We note that

$$\begin{aligned} [n(n-2)^2]^{-1}\, \bar{\Delta}(\psi\psi_t)(z,0) &= \tfrac{\partial^2}{\partial z_1^2}[-\tfrac{n+1}{2n}\,(1+|z|^2)^{-n}] + \tfrac{\partial^2}{\partial z_1^2}(1+|z|^2)^{-(n+1)} \\ &\quad - (n+3)(1+|z|^2)^{-(n+1)} + (2n+2)(1+|z|^2)^{-(n+2)} \\ = &\quad (\tfrac{\partial^2 f_1}{\partial z_1^2} + \tfrac{\partial^2 f_2}{\partial z_1^2} + f_3 + f_4)(z). \end{aligned} \tag{4.7}$$

Using (4.7) in (4.6) we see that at $(x,0)$,

$$[2n(n-2)^2]^{-1} \frac{\partial g_{\epsilon\epsilon}}{\partial t} = \frac{\partial^2}{\partial x_1^2}(F_1+F_2) + F_3 + F_4 \tag{4.8}$$

where for $1 \leq i \leq 4$,

$$F_i(x) = c_n \int_{\mathbb{R}^{n-1}} |y|^{2-n} \bar{f}_i(x-y)dy \tag{4.9}$$

and to get F_1, F_2 on the righthand side of (4.8) we have interchanged the order of differentiation and integration in the corresponding integrals, which is permissible as we see from (4.5). We now introduce spherical coordinates in $\mathbb{R}^{n-1}$. Let $r = |x|, \rho = |y|$ and $y_1 = \rho\cos\theta$. From rotational symmetry we deduce that each F_i is radial, i.e, $F_i(x) = F_i(\hat{x})$ when $|\hat{x}| = r$. Using this fact and writing $F_i(r)$ for $F_i(x), 1 \leq i \leq 4$, we see for $n \geq 3$ that each F_i has the form

$$F_i(r) = d_i/\pi \int_0^\pi \int_0^\infty \frac{\sin^{n-3}\theta}{(1+r^2-2r\rho\cos\theta+\rho^2)^{k_i}} d\rho d\theta \tag{4.10}$$

where

$$\begin{aligned} d_1 &= -(n+1)/(2n), \quad k_1 = n \\ d_2 &= 1, \quad k_2 = n+1 \\ d_3 &= -(n+3), \quad k_3 = n+1 \\ d_4 &= (2n+2), \quad k_4 = n+2. \end{aligned} \tag{4.11}$$

To evaluate the integrals in (4.10) we shall use contour integration and complex notation. Let $z = x + iy$ and let Λ_R be the boundary in the complex plane (oriented counterclockwise) of $\{z = x + iy : y > 0\} \cap \{z : |z| < R\}$. If $\hat{f}(z, r, \theta) = (1 + r^2 - 2rz\cos\theta + z^2)^{-k}$ and $k \geq n$ is a positive integer, then from the residue theorem and simple estimates we have

$$\begin{aligned}
\pi I(r) &= \int_0^\pi \int_0^\infty \frac{\sin^{n-3}\theta}{(1 + r^2 - 2r\rho\cos\theta + \rho^2)^k} d\rho d\theta \\
&= \int_0^{\pi/2} \sin^{n-3}\theta \lim_{R\to\infty} \int_{\Lambda_R} \hat{f}(z, r, \theta) dz d\theta \qquad (4.12) \\
&= 2\pi i \int_0^{\pi/2} \sin^{n-3}\theta \text{ Residue } \hat{f}(\cdot, r, \theta)|_{z_1} d\theta
\end{aligned}$$

where z_1 is the pole of $\hat{f}(\cdot, r, \theta)$ with Im $(z_1) > 0$. To find this residue we note from the quadratic formula that

$$\hat{f}(z, r, \theta) = (z - z_1)^{-k}(z - z_2)^{-k}$$

where

$$z_1 = r\cos\theta + i\sqrt{1 + r^2\sin^2\theta} \text{ and } z_2 = \bar{z}_1.$$

Thus

$$\text{Residue } \hat{f}(\cdot, r, \theta)|_{z_1} = \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}}(z - z_2)^{-k}$$

$$= (-1)^{k-1} \frac{\Gamma(2k-1)}{\Gamma(k)^2}(z_1 - z_2)^{1-2k} = -i2^{1-2k} \frac{\Gamma(2k-1)}{\Gamma(k)^2}(1 + r^2\sin\theta)^{1/2-k}.$$

Using this equality in (4.12) we get

$$\begin{aligned}
I(r) &= 2^{2-2k} \frac{\Gamma(2k-1)}{\Gamma(k)^2} \int_0^{\pi/2} \frac{\sin^{n-3}\theta}{(1 + r^2\sin^2\theta)^{k-1/2}} d\theta \\
&= (1 + r^2)^{(1/2-k)} 2^{2-2k} \frac{\Gamma(2k-1)}{\Gamma(k)^2} \int_0^\pi \sin^{n-3}\theta \left(1 - \frac{r^2}{1 + r^2}\cos^2\theta\right)^{(1/2-k)} d\theta.
\end{aligned} \qquad (4.13)$$

Next we expand the integrand in (4.13) in a series, using the binomial theorem, and then use (4.4) to integrate term by term. If $(a)_0 = 1$ and $(a)_m = a(a+1)\ldots(a+m-1)$, for m a positive integer, we find that

$$\int_0^\pi \sin^{n-3}\theta \; (1 - \frac{r^2}{1 + r^2}\cos^2\theta)^{(1/2-k)} d\theta =$$

$$\sum_{m=0}^\infty \left[\frac{(k - 1/2))_m}{m!} \left(\frac{r^2}{1 + r^2}\right)^m \int_0^{\pi/2} \sin^{n-3}\theta \cos^{2m}\theta d\theta \right] \qquad (4.14)$$

$$= (1/2) \frac{\Gamma(1/2)\Gamma(n/2 - 1)}{\Gamma(n/2 - 1/2)} F(1/2, k - 1/2, n/2 - 1/2, \frac{r^2}{1 + r^2})$$

where again $F(a,b,c,w) = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{m!(c)_m} w^m$ is the Hypergeometric function. Using (4.14) and (4.13) we conclude that

$$I(r) = A(k)(1+r^2)^{(1/2-k)}\ F\left(1/2, k-1/2, n/2-1/2, \frac{r^2}{1+r^2}\right) \tag{4.15}$$

where

$$A(k) = 2^{1-2k} \frac{\Gamma(1/2)\Gamma(n/2-1)\Gamma(2k-1)}{\Gamma(n/2-1/2)\,\Gamma(k)^2}.$$

Next from (4.15), (4.11), (4.10) we deduce that

$$\begin{aligned}
F_1(r) &= -\tfrac{n+1}{2n} A(n)(1+r^2)^{(1/2-n)}\ F\left(1/2, n-1/2, n/2-1/2, \frac{r^2}{1+r^2}\right) \\
F_2(r) &= A(n+1)(1+r^2)^{(-n-1/2)}\ F\left(1/2, n+1/2, n/2-1/2, \frac{r^2}{1+r^2}\right) \\
F_3(r) &= -(n+3)A(n+1)(1+r^2)^{(-n-1/2)}\ F\left(1/2, n+1/2, n/2-1/2, \frac{r^2}{1+r^2}\right) \\
F_4(r) &= (2n+2)A(n+2)(1+r^2)^{(-n-3/2)}\ F\left(1/2, n+3/2, n/2-1/2, \frac{r^2}{1+r^2}\right).
\end{aligned} \tag{4.16}$$

To compute derivatives of $I(r)$ we note that if $z = r^2/(1+r^2)$, then $I(r)$ can be written as a constant multiple of $(1-z)^b F(a,b,c,z)$ for suitably chosen a, b, c. Then $\frac{\partial z}{\partial x_1} = 2x_1(1-z)^2$ and

$$\begin{aligned}
\tfrac{\partial}{\partial x_1}[(1-z)^b F(a,b,c,z)] &= 2x_1(1-z)^{b+1}[-bF(a,b,z) + (1-z)\tfrac{\partial F}{\partial z}(a,b,c,z)] \\
&= 2x_1\,(1-z)^{b+1}\,(a-c)\,(b/c)\,F(a,b+1,c+1).
\end{aligned}$$

Differentiating this equality once again we find with $x_1 = r\cos\theta$ that

$$\begin{aligned}
\frac{\partial^2}{\partial x_1^2} I(r) &= B(k)(1+r^2)^{-k-1/2}\,F\left(1/2, k+1/2, n/2+1/2, \frac{r^2}{1+r^2}\right) \\
&\quad + C(k)\cos^2\theta\ r^2(1+r^2)^{-k-3/2}\,F\left(1/2, k+3/2, n/2+3/2, \frac{r^2}{1+r^2}\right)
\end{aligned}$$

where

$$B(k) = -(2k-1)2^{1-2k}\frac{\Gamma(1/2)\Gamma(n/2)\Gamma(2k-1)}{\Gamma(n/2+1/2)\,\Gamma(k)^2}$$

$$C(k) = (4k^2-1)2^{1-2k}\frac{\Gamma(1/2)\Gamma(n/2+1)\Gamma(2k-1)}{\Gamma(n/2+3/2)\,\Gamma(k)^2}.$$

From these equalities we see that

$$\frac{\partial^2}{\partial z_1^2}F_1(r) = \frac{(n+1)}{2n}\,B(n)(1+r^2)^{-n-1/2}\,F\left(1/2, n+1/2, n/2+1/2, \frac{r^2}{1+r^2}\right)$$

$$+\frac{(n+1)}{2n}\,C(n)\cos^2\theta\, r^2(1+r^2)^{-n-3/2}\,F\left(1/2, n+3/2, n/2+3/2, \frac{r^2}{1+r^2}\right) \tag{4.17}$$

$$= G_{11}(r) + G_{12}(r)\cos^2\theta$$

and

$$\frac{\partial^2}{\partial z_1^2}F_2(r) = B(n+1)(1+r^2)^{-n-3/2}\,F\left(1/2, n+3/2, n/2+1/2, \frac{r^2}{1+r^2}\right)$$

$$+\ C(n+1)\cos^2\theta\, r^2\,(1+r^2)^{-n-5/2}\,F\left(1/2, n+5/2, n/2+3/2, \frac{r^2}{1+r^2}\right) \tag{4.18}$$

$$= G_{21}(r) + G_{22}(r)\cos^2\theta.$$

From (4.16), (4.17), (4.18) we see that $\frac{\partial g_{\epsilon\epsilon}}{\partial t}$ is calculated in terms of the hypergeometric function.

Next we write $[|\bar{\nabla}\psi|^2 - \psi_t^2](x,0)$ in polar coordinates. If $r = |x|, x_1 = r\cos\theta$, then

$$|\bar{\nabla}\psi|^2 - \psi_t^2(x,0) = -\frac{2n^2(n-2)^2 r^2\cos^2\theta}{(r^2+1)^{(n+2)}}$$

$$+\frac{n(n-2)^3 r^2\cos^2\theta}{(r^2+1)^{(n+1)}} + \frac{(n-2)^2}{(r^2+1)^n} \tag{4.19}$$

$$= g_1(r)\cos^2\theta + g_2(r)\cos^2\theta + g_3(r).$$

In order to find the integrals with integrand involving $g_i\frac{\partial}{\partial t}g_{\epsilon\epsilon},\ 1 \le i \le 3$, we first consider integrals of the form:

$$J(b,h,k,l) = \int_0^\infty r^l\,(1+r^2)^{-h-k}\,F\left(1/2, k, b, \frac{r^2}{1+r^2}\right)dr$$

when either $l = n-2, n, n+2$ and $h, k > 0$ with $h+k \ge 2n+1/2$. Interchanging the integration and summation signs in the above integral (permissible by the Tonelli theorem), as well as using the substitution $r = \tan\phi$, we get

$$J(b,h,k,l) = \sum_{m=0}^\infty \frac{(1/2)_m(k)_m}{(b)_m\,m!}\int_0^\infty \frac{r^{2m+l}}{(1+r^2)^{m+h+k}}\,dr$$

$$= \sum_{m=0}^\infty \frac{(1/2)_m(k)_m}{m!(b)_m}\int_0^{\pi/2}(\cos\phi)^{(2h+2k-2-l)}\,(\sin\phi)^{(2m+l)}d\phi \tag{4.20}$$

$$= \frac{\Gamma(h+k-l/2-1/2)\Gamma(l/2+1/2)}{2\Gamma(h+k)}\ \sum_{m=0}^\infty \frac{(l/2+1/2)_m\,(1/2)_m\,(k)_m}{(h+k)_m\,(b)_m\,m!}.$$

We first use (4.19) with $l = n-2, b = n/2 - 1/2$, to get

$$J(n/2-1/2, h, k, n-2) = \frac{\Gamma(h+k-n/2+1/2)\Gamma(n/2-1/2)}{2\Gamma(h+k))} \sum_{m=0}^{\infty} \frac{(1/2)_m (k)_m}{(k+h)_m\, m!}$$

$$\frac{\Gamma(h+k-n/2+1/2)\Gamma(n/2-1/2)}{2\Gamma(h+k))} F(1/2, k, h+k, 1) \tag{4.21}$$

$$= \frac{\Gamma(n/2-1/2)\Gamma(h+k-n/2+1/2))\Gamma(h-1/2)}{2\Gamma(h)\Gamma(h+k-1/2)}.$$

In the above display we have used the fact that if $\alpha, \beta, \gamma > 0$ and $\gamma - \alpha - \beta > 0$, then

$$F(\alpha, \beta, \gamma, 1) = \frac{\Gamma(\gamma-\alpha-\beta)\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}. \tag{4.22}$$

From (4.21) with $h = n, k = n + 1/2$, (4.16) for F_3, and (4.4) we get

$$H_1 = \int_0^\infty \int_0^\pi \sin^{n-3}\theta\, F_3(r)\, g_3(r)\, r^{(n-2)}\, dr$$

$$= -(n-2)^2(n+3)A(n+1)\frac{\Gamma(n/2-1)\Gamma(1/2)}{\Gamma(n/2-1/2)} J(n/2-1/2, n, n+1/2, n-2) \tag{4.23}$$

$$= -8(n+3)(n-2)^2 n^2 D.$$

Here

$$D = \frac{\pi\Gamma(n/2-1)^2\Gamma(3n/2+1)\Gamma(n-1/2)}{2^{(4+2n)}\Gamma(n/2-1/2)\Gamma(n+1)^3}. \tag{4.24}$$

Similarly from (4.21) with $h = n, k = n + 3/2$, (4.16) for F_4, and (4.4) we get

$$H_2 = \int_0^\infty \int_0^\pi \sin^{n-3}\theta\, F_4(r)\, g_3(r)\, r^{(n-2)}\, d\theta dr = 2(3n+2)(2n+1)n(n-2)^2 D. \tag{4.25}$$

Next we put $b = n/2 - 1/2, l = n$ in the sum on the last line of (4.20). We deduce

$$\sum_{m=0}^{\infty} \frac{(n/2+1/2)_m (1/2)_m (k)_m}{(h+k)_m (n/2-1/2)_m m!} = F(1/2, k, h+k) + \frac{k}{(h+k)(n-1)} F(3/2, k+1, h+k+1).$$

Using this equality, (4.22), (4.20) and simplifying we find that

$$J(n/2-1/2, h, k, n) = \frac{\Gamma(h+k-n/2-1/2)\Gamma(n/2+1/2)\Gamma(h-3/2)}{4(n-1)\Gamma(h+k-1/2)\Gamma(h)} [(n-1)(2h-3)+2k]. \tag{4.26}$$

Using (4.26) with $b = n/2 - 1/2, h = n+2, k = n+1/2, l = n$, (4.16) for F_3, and (4.4) we get

$$\begin{aligned} H_3 &= \int_0^\infty \int_0^{\pi/2} \sin^{n-3}\theta\, \cos^2\theta\, F_3(r)\, g_1(r)\, r^{(n-2)}\, dr \\ &= 2n^2(n-2)^2(n+3)A(n+1)\frac{\Gamma(n/2-1)\Gamma(1/2)}{(n-1)\Gamma(n/2-1/2)} J(n/2-1/2, n+2, n+1/2, n) \qquad (4.27) \\ &= \frac{(2n-1)(3n+2)(n-2)^2 n^3 (n+3)}{2(n^2-1)} D. \end{aligned}$$

Arguing as above with $b = n/2 - 1/2, h = n+2, k = n+3/2, l = n$, we also obtain,

$$\begin{aligned} H_4 &= \int_0^\infty \int_0^{\pi/2} \sin^{n-3}\theta\, \cos^2\theta\, F_4(r)\, g_1(r)\, r^{(n-2)}\, dr \\ &= -\frac{n^2(n-2)^2(2n-1)(2n^2+n+2)(3n+2)(3n+4)}{8(n+1)^2(n-1)} D. \end{aligned} \tag{4.28}$$

Next with b, l as above, $h = n+1, k = n+1/2$, we find

$$\begin{aligned} H_5 &= \int_0^\infty \int_0^{\pi/2} \sin^{n-3}\theta\, \cos^2\theta\, F_3(r)\, g_2(r)\, r^n\, dr \\ &= -\frac{(2n^2-n+2)n(n-2)^3(n+3)}{n-1} D. \end{aligned} \tag{4.29}$$

Also for b, l as above and $h = n+1, k = n+3/2$,

$$\begin{aligned} H_6 &= \int_0^\infty \int_0^{\pi/2} \sin^{n-3}\theta\, \cos^2\theta\, F_4(r)\, g_2(r)\, r^{(n-2)}\, dr \\ &= \frac{n(n-2)^3(3n+2)(2n^2-n+4)}{2(n-1)} D. \end{aligned} \tag{4.30}$$

To continue, we note that if $b = n/2 + 1/2, l = n$ in the sum in (4.20), one deduces as in (4.21) that

$$J(n/2+1/2, h, k, n) = \frac{\Gamma(h+k-n/2-1/2)\Gamma(n/2+1/2)\Gamma(h-1/2)}{2\Gamma(h)\Gamma(h+k-1/2)}. \tag{4.31}$$

In view of (4.31) with $b = n/2+1/2, h = n+2, k = n+1/2, l = n$, we find from (4.17), (4.19) that

$$\begin{aligned} H_7 &= \int_0^\infty \int_0^{\pi/2} \sin^{n-3}\theta\, \cos^2\theta\, G_{11}(r)\, g_1(r)\, r^{(n-2)}\, dr \\ &= -\frac{(3n+2)(2n-1)(n-2)^3 n^2}{2(n-1)} D. \end{aligned} \tag{4.32}$$

With b, h, l as in (4.32) and $k = n + 3/2$ we deduce as above,

$$\begin{aligned} H_8 &= \int_0^\infty \int_0^{\pi/2} \sin^{n-3}\theta \, \cos^2\theta \, G_{21}(r) \, g_1(r) \, r^{(n-2)} \, dr \\ &= \frac{n^2(n-2)^3(3n+2)(3n+4)(4n^2-1)}{8(n-1)(n+1)^2} D. \end{aligned} \tag{4.33}$$

Next we repeat our program for the integrals in (4.32), (4.33) with $h = n+1$ and all other parameters as in (4.32), (4.33) respectively. We get

$$\begin{aligned} H_9 &= \int_0^\infty \int_0^{\pi/2} \sin^{n-3}\theta \, \cos^2\theta \, G_{11}(r) \, g_2(r) \, r^{(n-2)} \, dr \\ &= \frac{(n-2)^4(n+1)(2n-1)n}{n-1} D \end{aligned} \tag{4.34}$$

and

$$\begin{aligned} H_{10} &= \int_0^\infty \int_0^{\pi/2} \sin^{n-3}\theta \, \cos^2\theta \, G_{21}(r) \, g_2(r) \, r^{(n-1)} \, dr \\ &= -\frac{n(n-2)^4(3n+2)(2n-1)}{2(n-1)} D. \end{aligned} \tag{4.35}$$

As in (4.31) we see that if $l = n+2, b = n/2 + 3/2$, then

$$J(n/2 + 3/2, h, k, n+2) = \frac{\Gamma(h + k - n/2 - 3/2)\Gamma(n/2 + 3/2)\Gamma(h - 1/2)}{2\Gamma(h)\Gamma(h + k - 1/2)}. \tag{4.36}$$

Using (4.36) with b, l as in (4.31) and $h = n+2, k = n + 3/2$ we find

$$\begin{aligned} H_{11} &= \int_0^\infty \int_0^{\pi/2} \sin^{n-3}\theta \, \cos^4\theta \, G_{12}(r) \, g_1(r) \, r^{(n-2)} \, dr \\ &= \frac{3(3n+2)(n-2)^3 n^3(4n^2-1)}{8(n+1)^2(n-1)} D. \end{aligned} \tag{4.37}$$

Also for $b, l,$ as in (4.36) we have with $h = n+2, k = n + 5/2$,

$$\begin{aligned} H_{12} &= \int_0^\infty \int_0^{\pi/2} \sin^{n-3}\theta \, \cos^4\theta \, G_{22}(r) \, g_1(r) \, r^{(n-2)} \, dr \\ &= -\frac{3(3n+2)(3n+4)(n-2)^3 n^3(4n^2-1)}{16(n^2-1)(n+1)^2} D. \end{aligned} \tag{4.38}$$

Next we repeat the argument that gave us (4.35), (4.36) with b, l, k as in these displays and $h = n+1$. Doing this we get

$$\begin{aligned} H_{13} &= \int_0^\infty \int_0^{\pi/2} \sin^{n-3}\theta \, \cos^4\theta \, G_{12}(r) \, g_2(r) \, r^{(n-2)} \, dr \\ &= -\frac{3(n-2)^4 n^2(2n-1)}{2(n-1)} D. \end{aligned} \tag{4.39}$$

$$
\begin{aligned}
H_{14} = \int_0^\infty \int_0^{\pi/2} & \sin^{n-3}\theta\, \cos^4\theta\, G_{22}(r)\, g_2(r)\, r^{(n-2)}\, dr \\
& = \frac{3(2n+3)(3n+2)(n-2)^4 n^2 (2n-1)}{8(n^2-1)(n+1)} D.
\end{aligned} \tag{4.40}
$$

Next we note from integrating by parts and radial symmetry that

$$
\begin{aligned}
&\int_0^\infty \int_0^{\pi/2} \sin^{n-3}\theta\, g_3(r) \frac{\partial^2}{\partial x_1^2}(F_1+F_2)(r) r^{n-2} d\theta dr \\
&\quad = \int_0^\infty \int_0^{\pi/2} \sin^{n-3}\theta\, \frac{\partial^2 g_3}{\partial x_1^2}(r)\, (F_1+F_2)(r) r^{n-2} d\theta dr \\
&\quad = -2n(n-2)^2 \int_0^\infty \int_0^{\pi/2} \sin^{n-3}\theta\, (F_1+F_2)(r) \frac{r^{n-2}}{(r^2+1)^{(n+1)}} d\theta \\
&\quad + 4n(n+1)(n-2)^2 \int_0^\infty \int_0^{\pi/2} \sin^{n-3}\theta\, \cos^2\theta (F_1+F_2)(r) \frac{r^n}{(r^2+1)^{(n+2)}} d\theta \\
&\qquad = H_{15} + H_{16}.
\end{aligned} \tag{4.41}
$$

To find H_{15} we first write this integral as a sum of two integrals involving F_1, F_2. We then use $b = n/2 - 1/2, h = n+1, k = n - 1/2, l = n-2$ in (4.21) to evaluate the integral involving F_1 and then $b = n/2 - 1/2, h = n+1, k = n+1/2, l = n-2$ in (4.21) to evaluate the integral involving F_2. Altogether we get

$$
H_{15} = [8n^2(n-2)^2(n+1) - 2n(n-2)^2(2n-1)(3n+2)]D. \tag{4.42}
$$

Next we use (4.26) and argue as above to find

$$
H_{16} = \left[-\frac{2(n+1)(n-2)^2 n(2n^2+n-2)}{n-1} + \frac{(3n+2)(n-2)^2 n^2 (2n-1)}{n-1} \right] D. \tag{4.43}
$$

Combining the integrals in (4.23)-(4.43) and simplifying we find from (4.8) that

$$
\begin{aligned}
&\int_{\mathbb{R}^{n-1}} 6 \frac{\partial g_{\epsilon\epsilon}}{\partial t} (|\bar{\nabla}\psi|^2 - \psi_t^2) dx \;=\; \alpha_n \int_0^{\pi/2} \int_0^\infty \sin^{n-3}\theta\, \frac{\partial g_{\epsilon\epsilon}}{\partial t}(r,\theta)\, r^{n-2} dr d\theta \\
&\quad = 12n(n-2)^2 \sum_{i=1}^{16} H_i = -\alpha_n \frac{3n^3(n-2)^4(16+16n-72n^2-44n^3-11n^4+12n^6)}{4(n-1)(n+1)^3} D
\end{aligned} \tag{4.44}
$$

where $\alpha_n = \frac{2\pi^{(n/2-1)}}{\Gamma(n/2-1)}$.

5. Proof of Theorem 2. In this section we first calculate the integral involving the rest of the integrand in $I''''(0)$ and then use Stirling's formula to finally get Theorem 2. To begin we note as before that if $x \in \mathbb{R}^{n-1}$ and $r = |x|$, then

$$|\bar{\nabla}\psi|^2(x,0) = \frac{(n-2)^2}{(r^2+1)^n} - \frac{(n-2)^2 n^2 x_1^2}{(r^2+1)^{(n+2)}} + \frac{(n-2)^3 n x_1^2}{(r^2+1)^{(n+1)}}. \tag{5.1}$$

and

$$\begin{aligned} |\bar{\nabla}\psi|^4(x,0) = &-\frac{2(n-2)^4 n^2 x_1^2}{(r^2+1)^{2n+2}} + \frac{2(n-2)^5 n x_1^2}{(r^2+1)^{(2n+1)}} - \frac{2(n-2)^5 n^3 x_1^4}{(r^2+1)^{(2n+3)}} \\ &+ \frac{(n-2)^4}{(r^2+1)^{2n}} + \frac{(n-2)^4 n^4 x_1^4}{(r^2+1)^{(2n+4)}} + \frac{(n-2)^6 n^2 x_1^4}{(r^2+1)^{(2n+2)}}. \end{aligned} \tag{5.2}$$

Introducing spherical coordinates $x_1 = r\cos\theta$, in (5.2) and integrating the resulting expression term by term, using once again the substitution $r = \tan\phi$ as well as (4.4), we obtain that

$$\int_{\mathbb{R}^{n-1}} 6|\bar{\nabla}\psi|^4 dx \;=\; \alpha_n \frac{3(n-2)^3(3n^4+8n^3+133n^2+392n+336)}{16(n+1)(2n+3)(2n+1)} E \tag{5.3}$$

where α_n is defined after (4.44) and

$$E = \frac{\sqrt{\pi}\Gamma(1+n/2)\Gamma(3n/2+1/2)}{\Gamma(2n)} = \frac{\pi\Gamma(1+n/2)\Gamma(3n/2+1/2)}{\Gamma(n)\Gamma(n+1/2)2^{2n-1}} \tag{5.4}$$

by Legendre's duplication formula for the Γ function. We note that

$$\psi_t^2(x,0) = \frac{(n-2)^2 n^2 x^2}{(r^2+1)^{(n+2)}}. \tag{5.5}$$

Multiplying (5.5) by (5.1) and then integrating we find that

$$\int_{\mathbb{R}^{n-1}} 12\psi_t^2|\bar{\nabla}\psi|^2\, dx \;=\; \alpha_n \frac{3(n-2)^3(3n^3+n^2+4n+24)(3n+1)}{8(2n+1)(n+1)(2n+3)} E. \tag{5.6}$$

Also squaring (5.4) and integrating we find

$$\int_{\mathbb{R}^{n-1}} 2\psi_t^4\, dx \;=\; \alpha_n \frac{9(n-2)^3 n^2(3n+1)}{16(2n+1)(2n+3)} E. \tag{5.7}$$

Next we note that

$$\bar{\Delta}\psi(x,0) = -\frac{n(n-2)x}{(r^2+1)^{n/2+1}} + \frac{n(n^2-4)x}{(r^2+1)^{n/2+2}}. \tag{5.8}$$

Multiplying together (5.8), (4.1), (5.1) and then integrating we get

$$\int_{\mathbb{R}^{n-1}} 12\psi|\bar{\nabla}\psi|^2\bar{\Delta}\psi\, dx \;=\; -\alpha_n \frac{9(n-2)^3(3n^4+4n^3+n^2+32n+48)}{8(2n+1)(2n+3)(n+1)} E. \tag{5.9}$$

To handle the last term in the integral observe that

$$\frac{\partial|\bar{\nabla}\psi|^2}{\partial t}(x,0) = -\frac{2n(n-2)^2}{(r^2+1)^{n+1}} + \frac{2n^2(n-2)^2(n+2)x^2}{(r^2+1)^{n+3}} - \frac{2n(n-2)^2(n^2-2)x^2}{(r^2+1)^{n+2}}. \tag{5.10}$$

Multiplying together (5.10), the positive square root of (5.5), (4.1) and integrating using (4.2) we obtain

$$\int_{\mathbb{R}^{n-1}} 24\psi\psi_t\bar{\nabla}\psi\cdot\bar{\nabla}\psi_t\,dx \;=\; \alpha_n\frac{3(n-2)^3(3n^3+7n^2-2n-12)(3n+1)}{4(n+1)(2n+1)(2n+3)}\;E. \tag{5.11}$$

Adding together, (5.3), (5.6), (5.7), (5.9), and (5.11) we conclude that

$$\begin{aligned}&\int_{\mathbb{R}^{n-1}} \left(12\psi|\bar{\nabla}\psi|^2\bar{\Delta}\psi + 24\psi\psi_t\langle\bar{\nabla}\psi,\bar{\nabla}\psi_t\rangle\; + 2\psi_t^4 + 12|\bar{\nabla}\psi|^2\psi_t^2 + 6|\bar{\nabla}\psi|^4\right)\,dx\\ &\qquad = \;\alpha_n\frac{3(3n^3+2n^2+7n+2)(n-2)^3}{2(n+1)(2n+1)}E.\end{aligned} \tag{5.12}$$

Combining (5.12) with (4.44) we get

$$\begin{aligned}I''''(0) &= -\alpha_n\frac{3n^3(n-2)^4(16+16n-72n^2-44n^3-11n^4+12n^6)}{4(n-1)(n+1)^3}D\\ &+\alpha_n\frac{3(3n^3+2n^2+7n+2)(n-2)^3}{2(n+1)(2n+1)}E\\ &= A\left\{\frac{-1}{16}\frac{\Gamma(n+\frac{1}{2})\Gamma(n-\frac{1}{2})\Gamma(\frac{n}{2}-1)\Gamma(\frac{3n}{2}+1)}{\Gamma(n+1)^2\Gamma(\frac{n}{2}-\frac{1}{2})\Gamma(\frac{3n}{2}+\frac{1}{2})}\times\frac{n(16+16n-72n^2-44n^3-11n^4+12n^6)}{(n+1)^2(n-1)}\right.\\ &\qquad\qquad\left.+\frac{3n^3+2n^2+7n+2}{2n+1}\right\}\end{aligned} \tag{5.13}$$

where

$$A=\frac{3n\alpha_n\pi(n-2)^4\Gamma(n/2-1)\Gamma(3n/2+1/2)2^{-2n-2}}{\Gamma(n)\Gamma(n+1/2)(n+1)}.$$

Let L be the ratio of Gamma functions in (5.13). That is,

$$L=\frac{\Gamma(n+\frac{1}{2})\Gamma(n-\frac{1}{2})\Gamma(\frac{n}{2}-1)\Gamma(\frac{3n}{2}+1)}{\Gamma(n+1)\Gamma(n+1)\Gamma(\frac{n}{2}-\frac{1}{2})\Gamma(\frac{3n}{2}+\frac{1}{2})}$$

In order to show that (5.13) is positive we need to estimate L from above. Substituting $\Gamma(\frac{3n}{2}+1)=\frac{3n}{2}\Gamma(\frac{3n}{2}+1)$ and $\Gamma(n+1)=n\Gamma(n)$, for one of the $\Gamma(n+1)$ factors gives

$$L=\frac{3}{2}\frac{\Gamma(n+\frac{1}{2})\Gamma(n-\frac{1}{2})\Gamma(\frac{n}{2}-1)\Gamma(\frac{3n}{2}+1)}{\Gamma(n+1)\Gamma(n)\Gamma(\frac{n}{2}-\frac{1}{2})\Gamma(\frac{3n}{2}+\frac{1}{2})}.$$

This gives four ratios of Gamma functions, in the order they appear, of the form $\dfrac{\Gamma(z)}{\Gamma(z+\frac{1}{2})}$ Using Stirling's formula (see [A, page 203]) we see that

$$\frac{\Gamma(z)}{\Gamma(z+\frac{1}{2})} \;=\; \left(\frac{z}{z+\frac{1}{2}}\right)^z\frac{e^{\frac{1}{2}}}{z^{\frac{1}{2}}}e^{J(z)-J(z+\frac{1}{2})}$$

where
$$J(z)=\frac{1}{\pi}\int_0^\infty \frac{z}{\eta^2+z^2}\log\frac{1}{1-e^{-2\pi\eta}}\,d\eta$$
for $z=n+\frac{1}{2},n-\frac{1}{2},\frac{n}{2}-1,\frac{3n}{2}+1$.

Next observe that
$$\left(\frac{z}{z+\frac{1}{2}}\right)^z=e^{z\ln(1-\frac{1}{2z+1})}\le e^{z\frac{-1}{2z+1}}\le e^{-\frac{n-2}{2(n-1)}}$$
for all z occurring above. Therefore,
$$L\le(3/2)\frac{e^{-2\frac{n-2}{(n-1)}}e^2}{\sqrt{(n+\frac{1}{2})(n-\frac{1}{2})(\frac{n}{2}-1)(\frac{3n}{2}+1)}}S\le\frac{\sqrt{3}\,e^{\frac{2}{(n-2)}}}{n^2\sqrt{(1-\frac{1}{2n})(1-\frac{2}{n})}}S \tag{5.14}$$
for $n\ge 3$ where
$$S=\exp(J(n+\tfrac{1}{2})-J(n+1)+J(n-\tfrac{1}{2})-J(n)$$
$$+J(\tfrac{n}{2}-1)-J(\tfrac{n}{2}-\tfrac{1}{2})+J(\tfrac{3n}{2})-J(\tfrac{3n}{2}+\tfrac{1}{2})).$$
We estimate $J(z)-J(z+\frac{1}{2})$ using the formula above, and the mean value theorem ($z\le\xi\le z+\frac{1}{2}$) :
$$|J(z)-J(z+\tfrac{1}{2})|=\frac{1}{2\pi}\left|\int_0^\infty\frac{\eta^2-\xi^2}{(\eta^2+\xi^2)^2}\log(\frac{1}{1-e^{-2\pi\eta}})\,d\eta\right|$$
$$\le\frac{1}{2\pi\xi^2}\int_0^\infty\log(\frac{1}{1-e^{-2\pi\eta}})\,d\eta\le\frac{1}{24\xi^2}\le\frac{1}{24z^2}$$
where we have also used the fact that the last integral is $\pi/12$ (see [A, page 205]). Again using that the least z above is $z=\frac{n-2}{2}$ we see that
$$S\le\left(e^{\frac{1}{24(\frac{n}{2}-1)^2}}\right)^4=e^{\frac{2}{3(n-2)^2}}.$$
Using this inequality in (5.14) we have
$$L\le\frac{\sqrt{3}\exp\left[\frac{2}{n-2}+\frac{2}{3(n-2)^2}\right]}{n^2\sqrt{(1-\frac{1}{2n})(1-\frac{2}{n})}}. \tag{5.15}$$
We also have for $n\ge 3$
$$\frac{n(16+16n-72n^2-44n^3-11n^4+12n^6)}{(n+1)^2(n-1)}\le\frac{12n^7}{n^3}=12n^4. \tag{5.16}$$
Combining (5.16) and (5.15) we deduce that
$$\frac{1}{16}\frac{\Gamma(n+\frac{1}{2})\Gamma(n-\frac{1}{2})\Gamma(\frac{n}{2}-1)\Gamma(\frac{3n}{2}+1)}{\Gamma(n+1)^2\Gamma(\frac{n}{2}-\frac{1}{2})\Gamma(\frac{3n}{2}+\frac{1}{2})}\times\frac{n(16+16n-72n^2-44n^3-11n^4+12n^6)}{(n+1)^2(n-1)}$$
$$\le\frac{3n^2\sqrt{3}\exp\left[\frac{2}{n-2}+\frac{1}{6(n-2)^2}\right]}{4\sqrt{(1-\frac{1}{2n})(1-\frac{2}{n})}}\le\frac{3n^3+2n^2+7n+2}{2n+1} \tag{5.17}$$

for $n \geq 100$ as is easily seen. From (5.17) and (5.13) we conclude that $I''''(0) \geq 0$ for $n \geq 100$. For $3 \leq n \leq 100$ one can check (5.13) directly to see that $I''''(0) \geq 0$. This is easily done using a computer. In fact it turns out that (5.17) is valid for $n \geq 23$ so that one need only verify directly that (5.13) is positive for $3 \leq n \leq 23$. The proof of Theorem 2 is now complete. □

6. Proof of Theorem 3. Let ψ be as in Theorem 2 and put $\theta = \psi\psi_t$. We note from (3.5) that $g_{\epsilon^2} \equiv 0$ so $g_{\epsilon^2 t} \equiv 0$ and hence the so called "bad term" in the last calculation, now vanishes. Also note that

$$-24\int_{\mathbb{R}^{n-1}} \langle\bar{\nabla}\psi, \bar{\nabla}(\psi\psi_t)\rangle\psi_t\, dx = -24\int_{\mathbb{R}^{n-1}} |\bar{\nabla}\psi|^2\, \psi_t^2 dx \;-24\int_{\mathbb{R}^{n-1}} \langle\bar{\nabla}\psi_t, \bar{\nabla}\psi\rangle\psi\psi_t dx. \tag{6.1}$$

Putting (6.1) into (3.20) and collecting terms we see that

$$I''''(0) \;= \int_{\mathbb{R}^{n-1}} \left[12\psi|\bar{\nabla}\psi|^2\bar{\Delta}\psi + 2\psi_t^4 - 12|\bar{\nabla}\psi|^2\psi_t^2 + 6|\bar{\nabla}\psi|^4\right] dx. \tag{6.2}$$

Subtracting (5.6) from the sum of (5.3), (5.7), (5.9), we conclude that

$$\begin{aligned} I''''(0) \;&= -3\alpha_n \frac{(n-2)^3 n(3n^3+2n^2-13n-6)}{2(n+1)(2n+3)(2n+1)}\; E \\ &= -\frac{3\,\pi^{n/2}\,\Gamma(\frac{3n+1}{2})(n-2)^4\, n^2\,(3n^3+2n^2-13n-6)}{\Gamma(n+\frac{1}{2})\Gamma(n)\, 2^{2n+1}\,(n+1)(2n+3)(2n+1)} < 0. \end{aligned} \tag{6.3}$$

The proof of Theorem 3 is now complete. □

Remark. We note that (3.20) can be written as

$$\begin{aligned} I''''(0) = \textstyle\int_{\mathbb{R}^{n-1}} \big\{&12\psi|\bar{\nabla}\psi|^2\bar{\Delta}\psi + 24\psi\psi_t\langle\bar{\nabla}\psi, \bar{\nabla}\psi_t\rangle \;+\; 12(|\bar{\nabla}\psi|^2 - \psi_t^2)\,\mathcal{P}(\psi\psi_t)_t \\ &+ 2\psi_t^4 + 12|\bar{\nabla}\psi|^2\psi_t^2 + 6|\bar{\nabla}\psi|^4 \;-12\theta_t\,(|\bar{\nabla}\psi|^2 - \psi_t^2) \;-24\langle\bar{\nabla}\psi, \bar{\nabla}\theta\rangle\psi_t\big\}\; dx. \end{aligned} \tag{6.4}$$

Note that θ occurs linearly in this integral and only in the last two terms. Thus since we can always multiply θ by a large positive or negative constant it would suffice to prove the following conjecture in order to obtain examples where $I''''(0)$ has both signs.

Conjecture 1. Given $0 \not\equiv \psi \in C_0^\infty(\mathbb{R}^{n-1}), n \geq 3$, with $I'''(0) = 0$, there exists $\theta \in C_0^\infty(\mathbb{R}^{n-1})$ with

$$\begin{aligned} 0 \neq &\int_{\mathbb{R}^{n-1}} \left[\,12\,\theta_t\,(|\bar{\nabla}\psi|^2 - \psi_t^2) \;+24\langle\bar{\nabla}\psi, \bar{\nabla}\theta\rangle\psi_t\right]\; dx \\ &= \int_{\mathbb{R}^{n-1}} \left[12\,\theta_t\,(|\bar{\nabla}\psi|^2 - \psi_t^2) \;-\; 12\,\theta\,(|\bar{\nabla}\psi|^2 - \psi_t^2)_t\right] dx = \int_{D(0)} 12\theta\bar{\Delta}[|\bar{\nabla}\psi|^2 - \psi_t^2] dx. \end{aligned}$$

Let $\theta(x,t) = \frac{\Gamma(n/2)}{\pi^{n/2}(2-n)}\,(|x-y|^2+t^2)^{1-n/2}$ for fixed $y \in \mathbb{R}^{n-1}$. Using this θ in Conjecture 1 and letting $t \to 0$ through positive values we see from well known facts about the Poisson integral that Conjecture 1 is equivalent to the following inequality.

$$(|\bar{\nabla}\psi|^2 - \psi_t^2)(y,0) \not\equiv 2\sum_{i=1}^{n-1} R_i(\psi_{x_i}\psi_t)(y,0)$$

when $y \in \mathbb{R}^{n-1}$, where $R_i(x) = \frac{\Gamma(n/2)}{\pi^{n/2}}\frac{x_i}{|x|}, 1 \le i \le n-1$, are Riesz transforms and $R_i f$ denotes convolution of R_i with f. A less likely conjecture is

Conjecture 2. Given $\psi \in C_0^\infty(\mathbb{R}^{n-1})$ with $I'''(0) = 0$ and $\theta \equiv 0$, it is always true that the integral in (3.20) is positive.

The following examples, all with $n = 4$, lend some credence to the above conjectures. The first example involves

$$\psi = \frac{1}{r^2+(1+t)^2} - \frac{a^2}{r^2+(t+a)^2}$$

where $0 < a < 1$ and $r^2 = x^2+y^2+z^2$. For this ψ we find, using residues, that $I'''(0) = 0$ for all a. The remaining calculations in this section are all from Maple. The fourth derivatives are

$$I''''_{\theta=0}(0) = \frac{1}{8}\,\frac{\pi^2\left(3\,a^2+22\,a+3\right)(a-1)^4}{a\,(a+1)^5} \qquad I''''_{\theta=\psi\psi_t}(0) = \frac{1}{4}\,\frac{\left(3\,a^2+26\,a+3\right)(a-1)^4\pi^2}{a\,(a+1)^5}$$

These derivatives are positive and unequal, so support both the first and second conjectures. To calculate $I''''_{\theta=0}(0)$ for this example we needed to calculate $g_{\epsilon\epsilon}$ which we did by first using partial fractions on $2\psi\psi_t$ and second by solving a system of equations based on the observation that $g_{\epsilon\epsilon}$ can be realized as a linear combination of t derivatives of $u = \frac{1}{r^2+(t+1)^2}$ and $v = \frac{1}{r^2+(t+a)^2}$.

One obtains more evidence for Conjectures 1 and 2 from the following examples which also imply, along with the above example, that $I''''_{\theta=\psi\psi_t}(0)$ may have either sign. All the examples involve linear combinations of derivatives of $u = \frac{1}{r^2+(t+1)^2}$. Again, this makes finding $g_{\epsilon\epsilon}$ possible when $\theta = 0$ using derivatives of u. We always use an odd number of x derivatives so that $I'''(0) = 0$.

Here is a table of some derivatives.

ψ	$I''''_{\theta=0}(0)$	$I''''_{\theta=\psi\psi_t}(0)$
$u_x = -2xu^2$	$\frac{141\pi^2}{320}$	$-\frac{83}{40}\,\pi^2$
$u_{xt} = 8x(t+1)u^3$	$\frac{103899\pi^2}{1600}$	$-\frac{48051}{160}\,\pi^2$
$u_{xxx} = -48x^3u^4+24xu^3$	$\frac{302804163}{125440}\,\pi^2$	$-\frac{6567399}{640}\,\pi^2$
$u_{ttx} = -48\,u^4\,(t+1)^2\,x+8\,u^3x$	$\frac{462648861}{17920}\,\pi^2$	$-\frac{75385143}{640}\,\pi^2$
$u_{xxxxx} = -3840\,u^6x^5+3840\,u^5x^3-720\,u^4x$	$\frac{4739410625625}{5632}\,\pi^2$	$-\frac{423800960625}{128}\,\pi^2$

Here are some linear combinations.

$$\begin{aligned} \psi &= u_x + a u_{xt} \\ I''''_{\theta=0}(0) &= \frac{1}{1600}\,\pi^2\left(-9165\,a - 112284\,a^3 + 47111\,a^2 + 705 + 103899\,a^4\right) \\ I''''_{\theta=\psi\psi_t}(0) &= -\frac{1}{160}\,\pi^2\left(-4316\,a - 52530\,a^3 + 22153\,a^2 + 332 + 48051\,a^4\right) \end{aligned}$$

$$
\begin{aligned}
\psi \quad &= u_x + a u_{xxx} \\
I''''_{\theta=0}(0) \quad &= \tfrac{3}{627200}\,\pi^2\,(-2190496\,a - 187227180\,a^3 + 28823592\,a^2 + 504673605\,a^4 + 92120) \\
I''''_{\theta=\psi\psi_t}(0) \quad &= -\tfrac{1}{4480}\,\pi^2\,(45971793\,a^4 + 9296 - 214200\,a + 2819700\,a^2 - 17892900\,a^3)
\end{aligned}
$$

$$
\begin{aligned}
\psi \quad &= u_{xt} + a u_{xxx} \\
I''''_{\theta=0}(0) \quad &= \tfrac{3}{627200}\,\pi^2\,(120281560\,a + 760731228\,a^3 + 444523278\,a^2 + 504673605\,a^4 + 13576136) \\
I''''_{\theta=\psi\psi_t}(0) \quad &= -\tfrac{3}{4480}\,\pi^2\,(3890212\,a + 23718213\,a^3 + 14150139\,a^2 + 448476 + 15323931\,a^4)
\end{aligned}
$$

$$
\begin{aligned}
\psi \quad &= u_x + a u_{ttx} \\
I''''_{\theta=0}(0) \quad &= \tfrac{1}{89600}\,\pi^2\,(490793508\,a^3 + 2313244305\,a^4 + 39480 + 1908928\,a + 42920808\,a^2) \\
I''''_{\theta=\psi\psi_t}(0) \quad &= -\tfrac{1}{640}\,\pi^2\,(75385143\,a^4 + 1457916\,a^2 + 16579332\,a^3 + 64472\,a + 1328)
\end{aligned}
$$

$$
\begin{aligned}
\psi \quad &= u_{tx} + a u_{ttx} \\
I''''_{\theta=0}(0) \quad &= \tfrac{3}{89600}\,\pi^2\,(218197002\,a^2 - 32970616\,a - 661467012\,a^3 + 1939448 + 771081435\,a^4) \\
I''''_{\theta=\psi\psi_t}(0) \quad &= -\tfrac{9}{640}\,\pi^2\,(-7246011\,a^3 + 2399675\,a^2 - 363052\,a + 8376127\,a^4 + 21356)
\end{aligned}
$$

$$
\begin{aligned}
\psi \quad &= u_{xxx} + a u_{ttx} \\
I''''_{\theta=0}(0) \quad &= \tfrac{27}{125440}\,\pi^2\,(213382734\,a^2 + 119946001\,a^4 - 79253748\,a - 258873252\,a^3 + 11214969) \\
I''''_{\theta=\psi\psi_t}(0) \quad &= -\tfrac{9}{4480}\,\pi^2\,(-124781964\,a^3 + 58632889\,a^4 + 101236758\,a^2 - 36908748\,a + 5107977)
\end{aligned}
$$

To reiterate in all these examples $I''''_{\theta=0}(0) > 0$, supporting conjecture 2, while $I''''_{\theta=\psi\psi_t}(0) < 0$, supporting conjecture 1, and showing along with example 1 that $I''''_{\theta=\psi\psi_t}(0)$ may have either sign.

Another way to generate examples is to use separation of variables in (r,t). Then for any $\lambda > 0$ the functions $e^{-\lambda t}\sin(\lambda r)/r$ are harmonic in $\mathbb{R}^4_+$ and, with appropriate integrability,

$$
g(r,t) = \sqrt{2/\pi}\int_0^\infty \frac{e^{-\lambda t}\sin(\lambda r)}{r} f(\lambda)\,d\lambda
$$

represents the harmonic function with boundary values

$$
g(r,0) = \sqrt{2/\pi}\int_0^\infty \frac{\sin(\lambda r)}{r} f(\lambda)\,d\lambda
$$

So one can calculate examples in the variables (r,t) using fourier sine transforms when $n = 4$, or by using Hankel transforms in higher dimensions. The examples we worked out did not lead to anything new.

Finally let $\mu, \tilde{\mu}$ denote (as in section 1) harmonic measure defined with respect to the above $D, \mathbb{R}^n \setminus \bar{D}$. Since both these measures are doubling measures it follows easily from a monotonicity formula of [ACF] (see also [FH]) that if $x \in \partial D, 0 \le r \le 1$, then

$$
\mu(B(x,r)) \cdot \tilde{\mu}(B(x,r)) \;\le\; c r^{2n-2}
$$

where c depends only on n. Now for μ, D as in Corollary 1 and almost every $x \in \operatorname{supp} \mu$ it follows from Wolff's argument and Theorem 2 that

$$\lim_{r \to 0} \frac{\ln \mu(B(x,r))}{\ln r} < n - 1,$$

A similar statement holds for $\tilde{\mu}$. Using these equalities in the monotonicity formula we reach a contradiction unless μ and $\tilde{\mu}$ are mutually singular. On the other hand there is no contradiction to the above inequality if $\mu, \tilde{\mu}, D$ are as in Corollary 2. Thus we conjecture:

Conjecture 3. There exists a Wolff snowflake for which $\mu, \tilde{\mu}$ are not mutually singular.

In two dimensions (see [BCGJ]) a necessary and sufficient condition for these measures to be mutually singular in a Jordan domain is that the set of boundary points with tangents has H^1 measure zero. Clearly Wolff snowflakes have this property so $\mu, \tilde{\mu}$ are always mutually singular in $\mathbb{R}^2$.

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