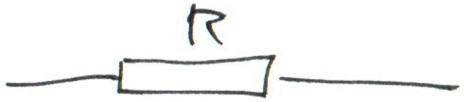
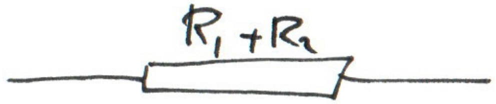
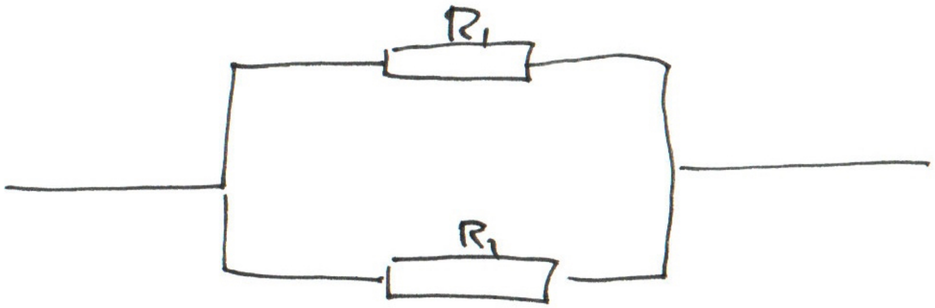
Electrical Networks



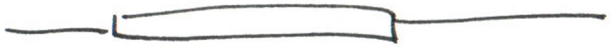
$\cong$





$\cong$

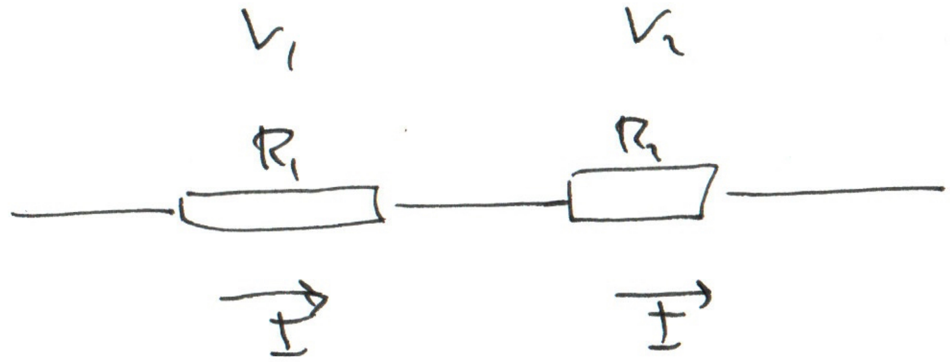
$$\frac{1}{\frac{1}{R_1} + \frac{1}{R_2}} = \frac{R_1 \cdot R_2}{R_1 + R_2}$$



Ohm's Law $\quad R = \dfrac{V}{I}$

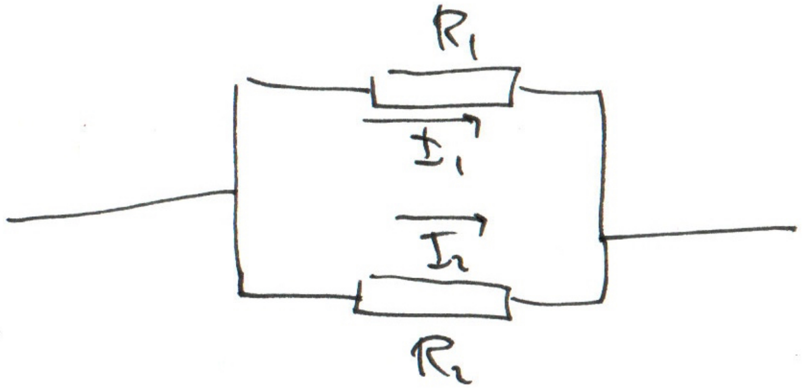
V = potential difference or voltage

I = current.



$$V = V_1 + V_2$$

$$R = \frac{V}{I} = \frac{V_1 + V_2}{I} = \frac{V_1}{I} + \frac{V_2}{I} = R_1 + R_2$$



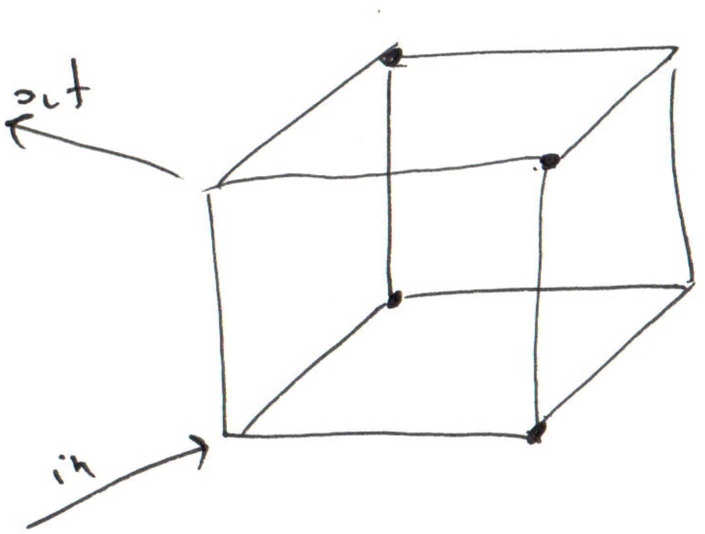
$$\frac{1}{R} = \frac{I}{V} = \frac{I_1 + I_2}{V} =$$

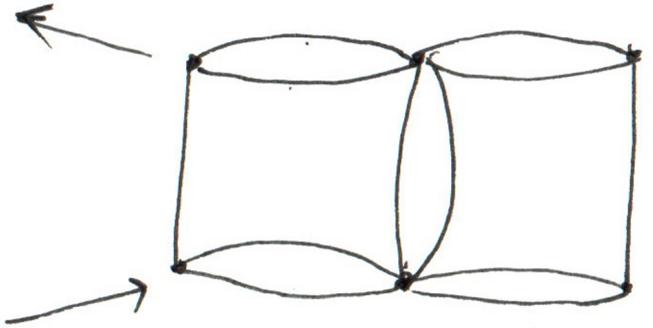
$$= \frac{I_1}{V} + \frac{I_2}{V} = \frac{1}{R_1} + \frac{1}{R_2}$$

$$R = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}}$$

III

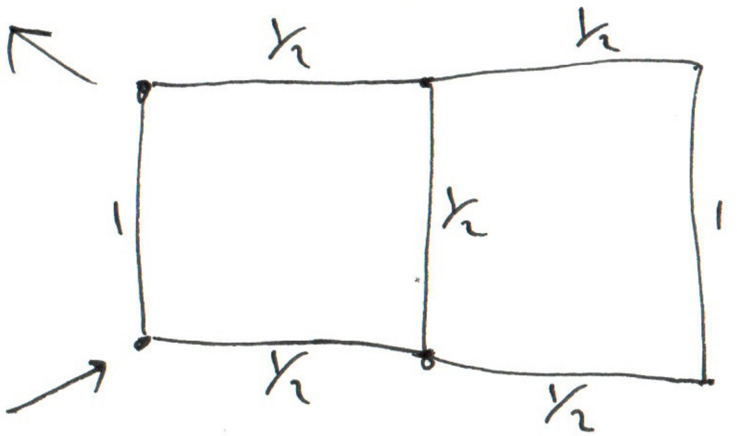
Each edge is
a unit resistor.



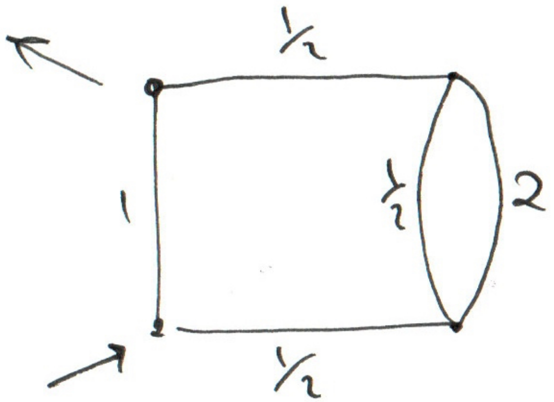


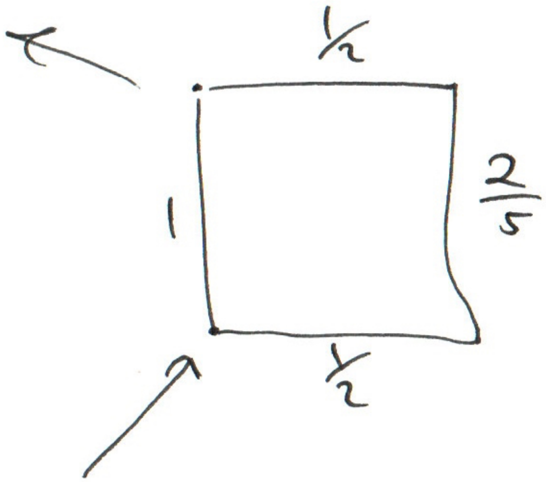
IV

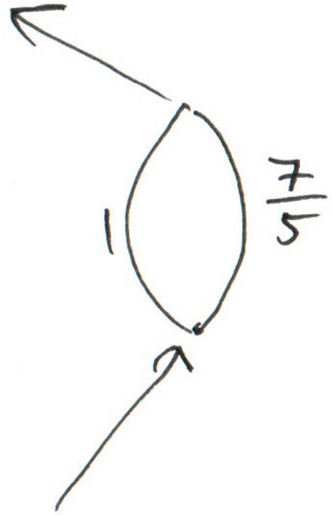
$$\frac{1}{\frac{1}{1/2}+\frac{1}{2}} = \frac{1}{2+\frac{1}{2}} = \frac{1}{\frac{5}{2}} = \frac{2}{5}$$



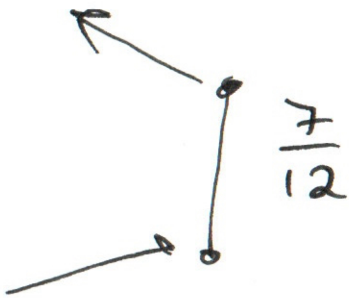
$$\frac{1}{2} + \frac{2}{5} + \frac{1}{2} = \frac{7}{5}$$

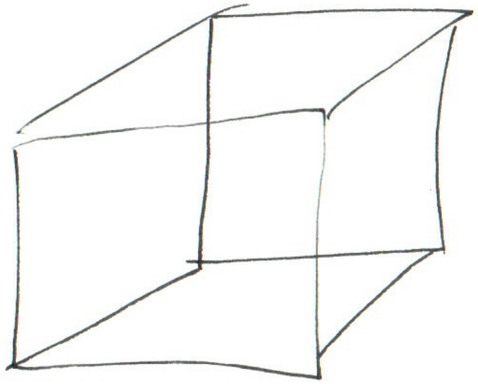






$$\frac{1}{1+\frac{1}{7/5}} = \frac{1}{1+\frac{5}{7}} = \frac{1}{\frac{12}{7}} = \frac{7}{12}$$



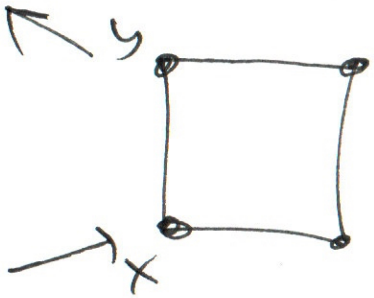


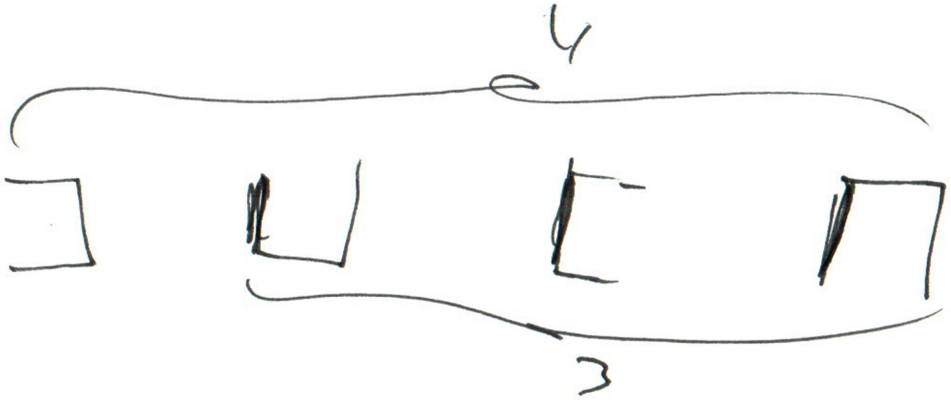
$f_1 = 12$

$f_0 - 1 = 7 =$ # edges in a spanning tree

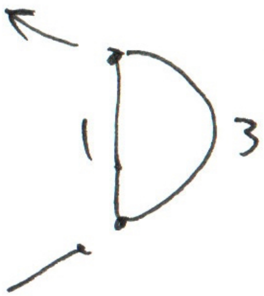
Theorem Let G be a graph and xy an edge in the graph. Replace each edge with a unit resistor. Then the total resistance between nodes x and y is given by

$$\frac{\#\ \text{spanning trees in } T \text{ containing the edge } xy}{\#\ \text{spanning trees.}}$$

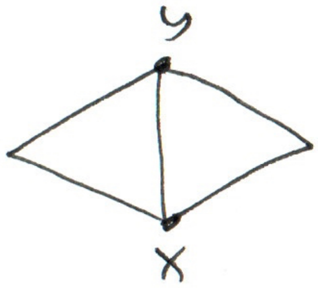




$3/4$



$$\frac{1}{\frac{1}{1}+\frac{1}{3}} = \frac{1}{\frac{4}{3}} = \frac{3}{4}$$



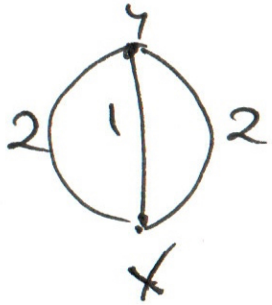
spanning trees containing xy

$2 \times 2 = 4$

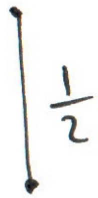
$R_{xy} = \frac{4}{8} = \frac{1}{2}$

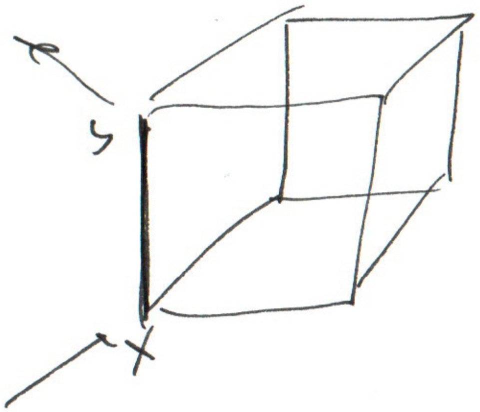
4

spanning trees $= 4 + 4 = 8$



$$\frac{1}{\frac{1}{2} + \frac{1}{1} + \frac{1}{2}} = \frac{1}{2}$$





$\#$ spanning trees $=$ big number.

$N = \#$ spanning trees

$N_e = \#$ spanning trees containing a given edge e

Count pairs (T, e) where T is a spanning tree, e ~~edge~~ an edge and $e \in T$.

$$N \cdot \underset{\substack{\uparrow \\ \text{pick the edge}}}{7} = \underset{\substack{\uparrow \\ \text{pick the edge}}}{12} \cdot N_e$$

N ↑ pick tree

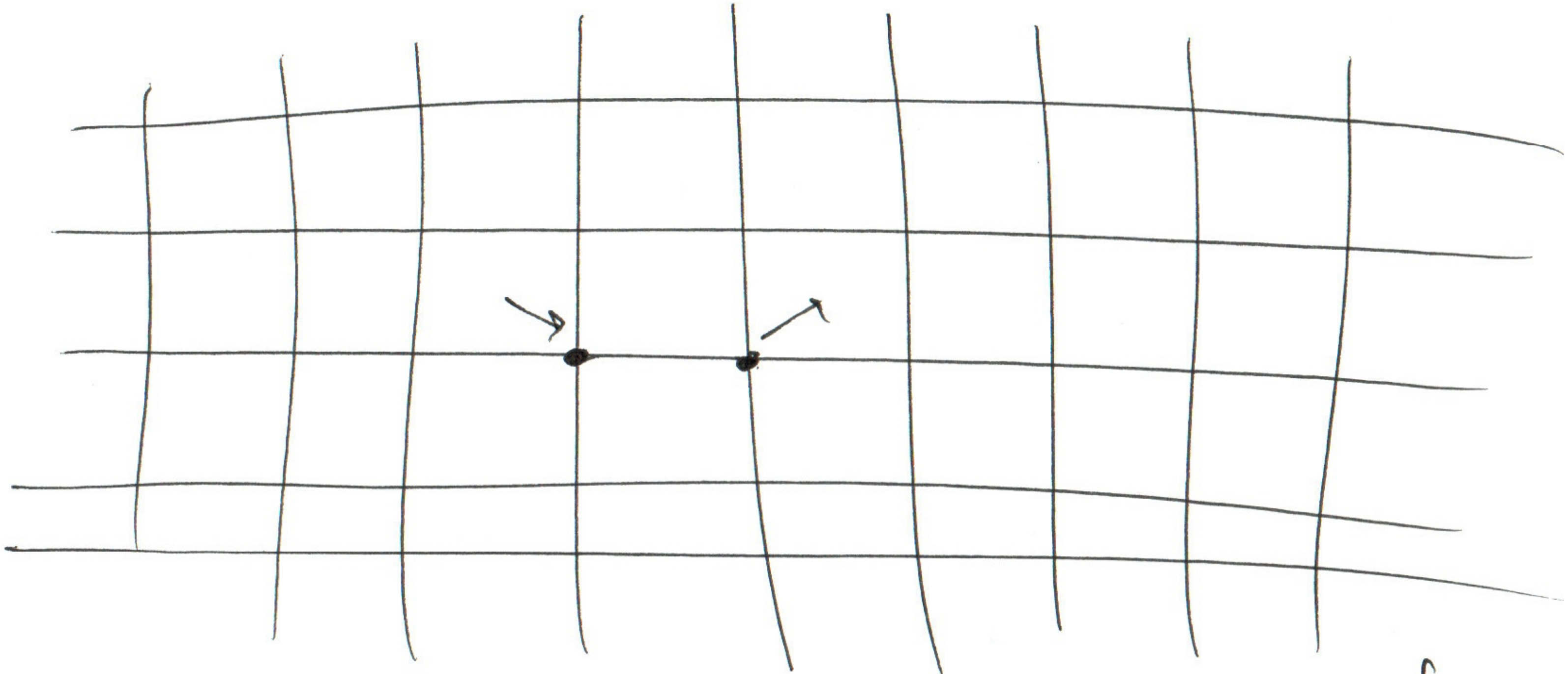
$$\frac{7}{12} = \frac{N_e}{N} = \text{total resistance between } x \,\&\, y.$$

Icosahedron

Resistance between xy (xy is an edge)

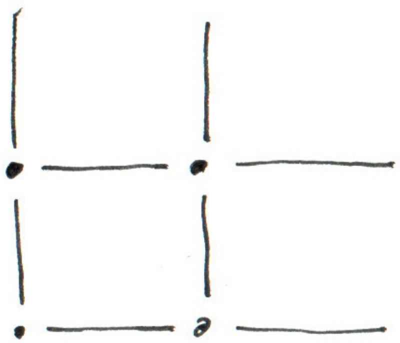
$$= \frac{f_0 - 1}{f_1} = \frac{12-1}{30} = \frac{11}{30}$$

Infinite grid



$$\frac{f_{\infty - 1}}{f_1}$$

$$\frac{1}{2}$$



$2 \cdot f_\infty = f_1$

$$\frac{f_\infty}{f_1} = \frac{1}{2}$$

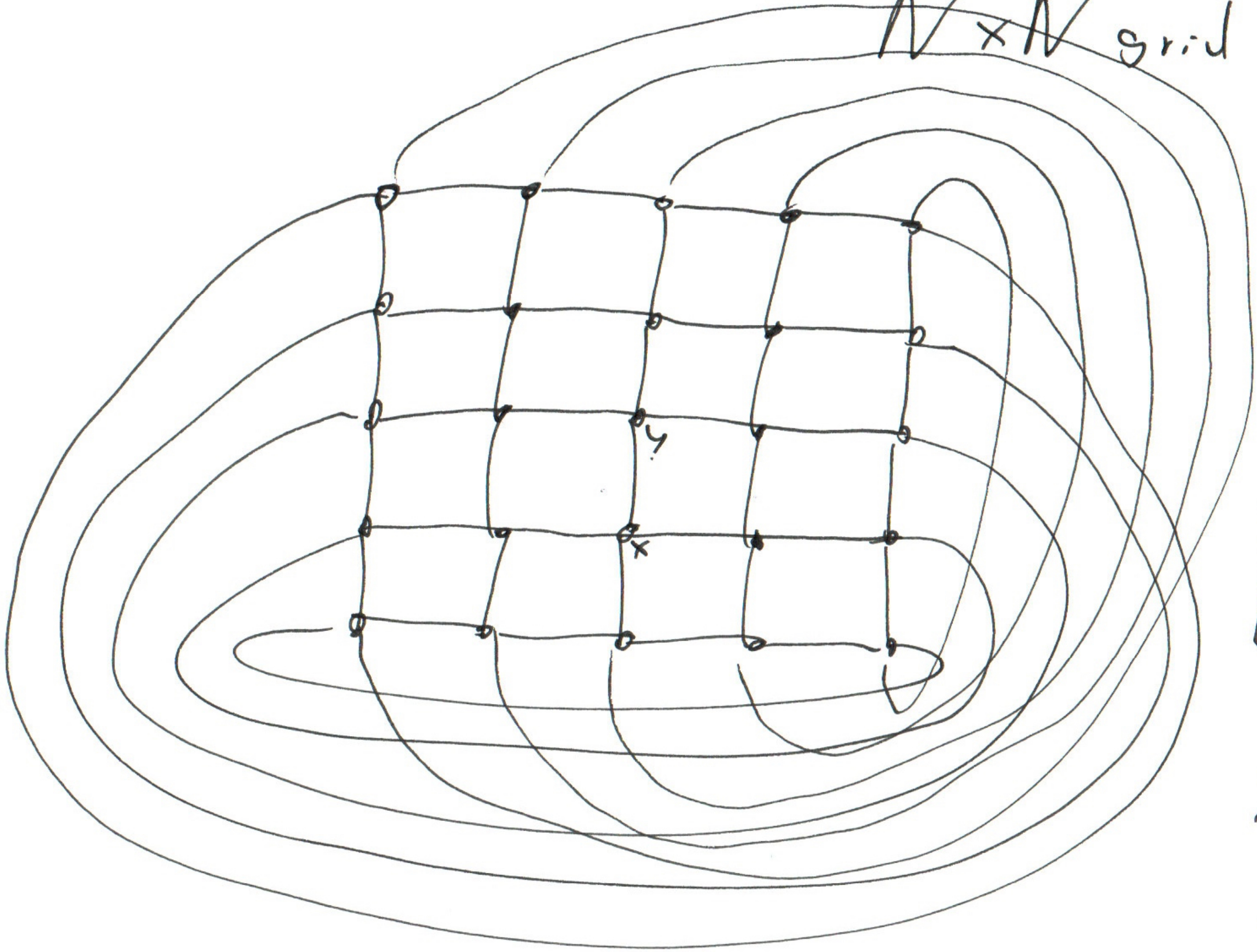
But

$f_{\infty-1} \approx f_\infty$

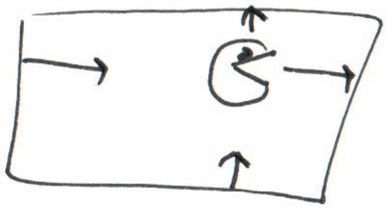
since

$f_\infty = \infty$

$N \times N$ grid



torus

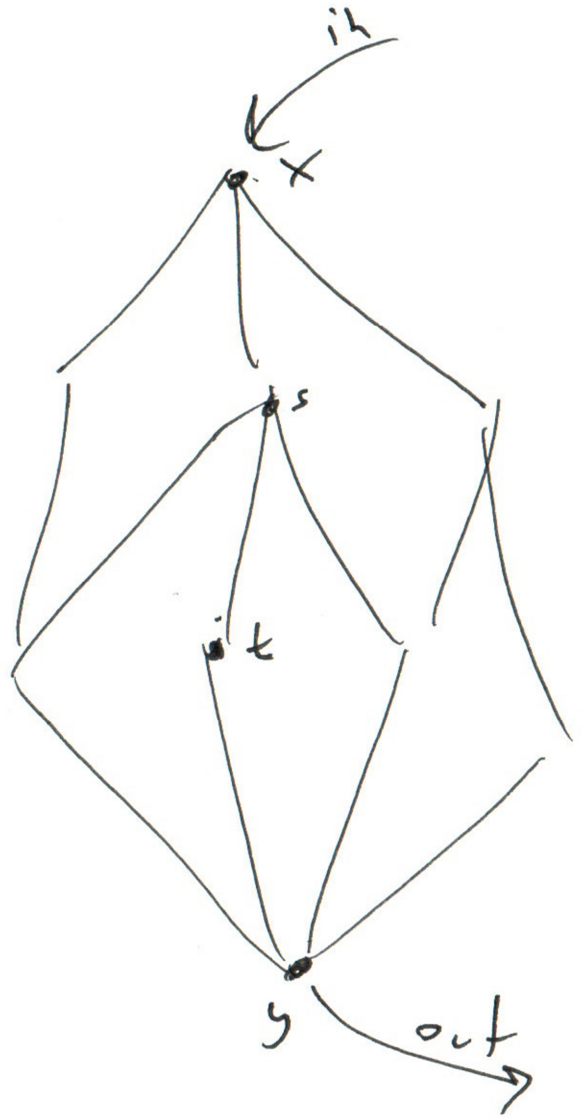


$$f_0 = N^2$$

$$f_1 = 2N^2$$

$$R_{xy} = \frac{f_0 - 1}{f_1} = \frac{N^2 - 1}{2N^2} \xrightarrow[N \to +\infty]{} \frac{1}{2}$$

XIV



Each edge is
a unit resistor.

$$I_{st} = V_{st} = \text{Then...}$$

$$= \frac{N(x,s,t,y) - N(x,t,s,y)}{N}$$

N = total number of dimers

$N(x,s,t,y) =$

$N(x,t,s,y) =$