Discrete Comput Geom 23:261–271 (2000)
DOI: 10.1007/s004549910017

Discrete & Computational
Geometry

Cutting Polytopes and Flag f-Vectors*

R. Ehrenborg,[1] D. Johnston,[2] R. Rajagopalan,[3] and M. Readdy[1]

[1]School of Mathematics, Institute for Advanced Study,
Olden Lane, Princeton, NJ 08540, USA
{jrge,readdy}@math.ias.edu

[2]Department of Mathematics,
Washington University in St. Louis,
St. Louis, MO 63130-4899, USA
dbjohnst@artsci.wustl.edu

[3]Department of Mathematics,
Malott Hall, Cornell University,
Ithaca, NY 14853-4201, USA
mohan@math.cornell.edu

Abstract. We show how the flag f-vector of a polytope changes when cutting off any face, generalizing work of Lee for simple polytopes. The result is in terms of explicit linear operators on **cd**-polynomials. Also, we obtain the change in the flag f-vector when contracting any face of the polytope.

1. Introduction

The flag f-vector records the face incidence information of a polytope. For an n-dimensional polytope there are linear dependencies among the 2^n entries of the flag f-vector. These dependencies are given by the generalized Dehn–Sommerville relations [2] which determine a subspace of dimension the nth Fibonacci number. Many bases exist for this subspace, but the one given by the **cd**-index [4] has been particularly fruitful for exploring and answering questions about flag vectors and revealing their underlying algebraic structure.

For a general polytope determining the **cd**-index is as difficult as determining its flag f-vector. The groundbreaking result which has enabled the **cd**-index to be used as a tool

* This research was supported by National Science Foundation DMS-9619681 as part of an REU project directed by Ehrenborg and Readdy at Cornell University.

to understand the combinatorics of polytopes was found by Ehrenborg and Readdy [10]. They showed the **cd**-index is a coalgebra homomorphism and applied their coproduct techniques to determine how the **cd**-index of a polytope, and, more generally, of an Eulerian poset, changes under geometric operations, such as taking the pyramid or the prism. Both these operations are expressed as derivations on **cd**-polynomials. Billera et al. [8] used these coalgebra techniques to show the flag f-vector of zonotopes satisfy precisely the same linear relations as those of all polytopes. Additionally, the coproduct formulations of the pyramid and prism operations enabled them to give a very compact proof of Bayer and Billera's result [2] that the flag f-vectors of all polytopes span the linear space determined by the generalized Dehn–Sommerville equations [8]. As a consequence, they prove the **cd**-index of zonotopes is coefficientwise minimized on the cube of the same dimension [7]. Other work in this vein includes how to compute the **cd**-index of products of polytopes, due to Ehrenborg and Fox [9], and how to compute the toric h-vector of posets, due to Bayer and Ehrenborg [3].

Recently Billera and Ehrenborg [6] succeeded in proving a long-outstanding conjecture of Stanley, namely, that the **cd**-index of polytopes is minimized on the simplex. The proof of this result and that of the cubical analogue for zonotopes relied heavily on the ability to compute **cd**-indices. In the first case, Billera et al. gave an explicit expression for the **cd**-index of a zonotope in terms of the corresponding intersection lattice [7]. Such a correspondence was known to exist by Bayer and Sturmfels [5] but had not before been made so concrete. For the second result, Billera and Ehrenborg determined how the **cd**-index of a polytope changes under an S-shelling. Both of these results point to the future role the **cd**-index will have in proving inequalities for flag vectors. Hence, it is a fundamental question to understand how changes in a polytope affect the **cd**-index.

In his dissertation Lee studied how the h-vector changes under operations applied to a simplicial complex. For example, one of his results (dualized) is that the h-polynomial of a simple n-dimensional polytope P with a k-dimensional face F cut off is given by

$$h(P - F) = h(P) + h(F) \cdot \left(x + \cdots + x^{n-k-1}\right).$$

See Proposition 2.10.1(iv) of [11]. For simple polytopes the h-vector determines the flag f-vector and the **cd**-index of the polytope [13, Theorem 3.1].

Generalizing Lee's result, in this paper we consider the impact on the flag f-vector after cutting off any face F from a polytope. The technique we use is to contract the face F into a vertex and then cut off this vertex. Although the resulting object after contracting the face may not be a polytope, it is a regular cell complex and results about the **cd**-index extend to this case.

When contracting a face F, the change in the **cd**-index is a linear combination of the **cd**-indices of the face figures of all the subfaces of the face F. The coefficients in this linear combination are **cd**-polynomials. The previously known result [10] for cutting off a vertex v expresses the change in the **cd**-index in terms of a derivation of the **cd**-index of the vertex figure of v. In our generalization to cutting off any face F, the change depends on a family of explicit linear operators applied to the same face figures as in the contraction case.

The problem of determining all the linear inequalities for flag f-vectors of polytopes is settled in three dimensions [14] and is still open in higher dimensions. See Bayer's paper [1] for the best-known results for 4-polytopes. One application of knowing how flag f-vectors behave under the cutting and contraction operations would be to construct sequences of polytopes whose flag f-vectors approach the extreme rays of the cone generated by all flag f-vectors of polytopes. This would give a method to prove that a given linear inequality on flag f-vectors is the best possible and cannot be improved.

2. Definitions and Notation

We define the basic terminology used throughout this paper. The statements of the results will be phrased in geometric language, while the proofs will be in terms of the partially ordered sets (posets) corresponding to these geometric objects. For a standard reference on polytopes, see [15].

Given a convex n-dimensional polytope P and $0 \leq i \leq n-1$, let f_i be the number of i-dimensional faces of the polytope P. The vector $(f_0, \ldots, f_{n-1})$ is called the *f-vector* of P. A classic result is that the f-vector satisfies the Euler–Poincaré relation $\sum_{i=0}^{n-1}(-1)^i f_i = 1-(-1)^n$. The f-vector has a natural extension by counting chains of faces in the polytope. For a subset $S \subseteq \{0, \ldots, n-1\}$, we denote by f_S the number of chains of faces (*flags*) in P, $F_1 \subseteq \cdots \subseteq F_k$, with $S = \{\dim F_1 < \cdots < \dim F_k\}$. The vector consisting of all the numbers f_S, $S \subseteq \{0, \ldots, n-1\}$, is called the *flag f-vector* of P. Observe that $f_{\{i\}} = f_i$ and $f_\emptyset = 1$. The linear span of the flag f-vectors of all polytopes, and more generally, of all Eulerian posets, is described by a system of linear equations known as the generalized Dehn–Sommerville equations [2].

For any $S \subseteq \{0, \ldots, n-1\}$, we set $h_S = \sum_{T \subseteq S}(-1)^{|S-T|} \cdot f_T$, and we call the vector of all such numbers the *flag h-vector* of P. Define a polynomial in the noncommuting variables **a** and **b**, called the **ab**-*index*, by

$$\Psi(P) = \sum_S h_S \cdot u_S,$$

where $u_S = u_0 \cdots u_{n-1}$, $u_i = \mathbf{b}$ if $i \in S$ and $u_i = \mathbf{a}$ if $i \notin S$. A result conjectured by Fine and proved by Bayer and Klapper [4] is that the **ab**-index of a polytope can be written as a polynomial in the variables $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a}$. This polynomial in terms of the variables **c** and **d** is called the **cd**-*index*. It gives an implicit encoding of the generalized Dehn–Sommerville equations [2].

Let F be a nonempty face of the polytope P. There exists a linear functional ℓ and a real number c such that for all points $\mathbf{x} \in P$ we have that $\ell(\mathbf{x}) \geq c$, but $\mathbf{x} \in P$ and $\ell(\mathbf{x}) = c$ implies that the point $\mathbf{x}$ belongs to the face F. That is, the hyperplane $\ell(\mathbf{x}) = c$ is a *supporting hyperplane* of the face F. We define the polytope $P - F$, that is, the *polytope P with the face F cut off*, by

$$P - F = \{\mathbf{x} \in P \colon \ \ell(\mathbf{x}) \geq c + \delta\},$$

where δ is an arbitrary small positive real number. Observe that the polytope $P - F$ depends on the choice of ℓ, c, and δ, but the combinatorial type of $P - F$ is independent of these variables.

Let v be a vertex of the n-dimensional polytope P and let $\ell(\mathbf{x}) = c$ be a supporting hyperplane of the vertex v. The *vertex figure* of v is the $(n-1)$-dimensional polytope P

defined by

$$P/v = \{\mathbf{x} \in P\colon\ \ell(\mathbf{x}) = c + \delta\},$$

where δ is an arbitrary small positive real number. As before, the combinatorial type of P/v is well-defined. Observe that every face of the polytope P/v corresponds to a face of P that contains v. For a k-dimensional face F of the polytope P the *face figure* P/F is an $(n-k-1)$-dimensional polytope with the property that every face of P/F corresponds to a face of P that contains F. More formally, choose a maximal chain of faces $F_0 \subseteq F_1 \subseteq \cdots \subseteq F_k = F$ such that $\dim(F_i) = i$. Then the face figure P/F is the iterated quotient

$$P/F = (\cdots((P/F_0)/F_1)\cdots)/F_k.$$

Observe that the face F_i corresponds to a vertex in the quotient $(\cdots(P/F_0)\cdots)/F_{i-1}$ and hence the iterated expression is well-defined.

Let $\mathbb{Z}\langle\mathbf{a},\mathbf{b}\rangle$ be the polynomial ring in the noncommutative variables $\mathbf{a}$ and $\mathbf{b}$. Let $\mathbb{Z}\langle\mathbf{c},\mathbf{d}\rangle$ be the subring generated by $\mathbf{c} = \mathbf{a}+\mathbf{b}$ and $\mathbf{d} = \mathbf{ab}+\mathbf{ba}$. The ring $\mathbb{Z}\langle\mathbf{a},\mathbf{b}\rangle$ has a grading by letting the degree of $\mathbf{a}$ and $\mathbf{b}$ be 1. The ring $\mathbb{Z}\langle\mathbf{c},\mathbf{d}\rangle$ inherits this grading; thus $\mathbf{c}$ has degree 1 and $\mathbf{d}$ has degree 2.

Recall that a *derivation* f on a ring R is a linear map which satisfies the Leibniz rule (or product rule) $f(x\cdot y) = f(x)\cdot y + x\cdot f(y)$. To determine a derivation, it is enough to specify it on the generators. Let E be a derivation on $\mathbb{Z}\langle\mathbf{a},\mathbf{b}\rangle$ by letting $E(\mathbf{a}) = \mathbf{ab}$ and $E(\mathbf{b}) = \mathbf{ba}$. This derivation restricts to a derivation on $\mathbb{Z}\langle\mathbf{c},\mathbf{d}\rangle$. To see this, it is enough to verify $E(\mathbf{c}) = \mathbf{d}$ and $E(\mathbf{d}) = \mathbf{dc}$. Observe that this derivation increases the degree by one.

In [10] the authors gave a formula for the **cd**-index of a polytope with a vertex cut off in terms of the derivation E.

Proposition 2.1 [10]. *Let P be a convex polytope and let v be a vertex of P. Then the* **cd**-*index of the polytope $P - v$, that is, the polytope P with the vertex v cut off, is given by*

$$\Psi(P - v) = \Psi(P) + E(\Psi(P/v)).$$

We now introduce some poset terminology. A standard reference for basic concepts is Chapter 3 of [12]. A graded poset Q is a poset with minimal element $\hat{0}$, maximal element $\hat{1}$, and a rank function ρ such that $\rho(\hat{0}) = 0$ and $\rho(x) = \rho(y) - 1$ for y covering x. The rank of Q is defined to be the rank of the maximal element $\hat{1}$, denoted by $\rho(Q)$. For $x, y \in Q$ and $x \le y$, the *interval* $[x, y]$ is the set $\{z\colon\ x \le z \le y\}$. Observe the interval $[x, y]$ is also a graded poset of rank $\rho(x, y) = \rho(y) - \rho(x)$.

The notion of the flag f-vector can be extended to graded posets. We present a different approach to define the **ab**-index by counting chains in a poset. For a chain $c = \{\hat{0} = x_0 < x_1 < \cdots < x_{k+1} = \hat{1}\}$ in a graded poset Q of rank $n+1$, define the *weight* of the chain c to be the product $\mathrm{wt}(c) = w_1 \cdots w_n$, where

$$w_i = \begin{cases} \mathbf{b} & \text{if}\quad i \in \{\rho(x_1), \ldots, \rho(x_k)\}, \\ \mathbf{a}-\mathbf{b} & \text{otherwise.} \end{cases}$$

The **ab**-index is then given by

$$\Psi(Q) = \sum_{c} \mathrm{wt}(c), \tag{2.1}$$

where the sum is over all chains c in the poset Q.

For a polytope P, let $L(P)$ denote the set of all faces of P, including the polytope itself and the empty face $\emptyset$, where the elements of $L(P)$ are ordered by inclusion. Observe $L(P)$ is a graded poset, and, in fact, is a lattice. The rank function is given by $\rho(x) = \dim(x) + 1$, the minimal element is the empty face and the maximal element is the polytope P. The intervals of the face lattice also have a geometric interpretation. For F and G two faces of P such that $F \subseteq G$, the face lattice of the face figure G/F is the interval $[F, G]$. Notice that the **ab**-index of a polytope and the **ab**-index of its face lattice are the same.

The *Möbius function* on a poset Q is defined as $\mu(x, x) = 1$ and $\mu(x, y) = -\sum_{x \leq z < y} \mu(x, z)$ for $x < y$. A poset Q is called *Eulerian* if its Möbius function on any interval $[x, y]$ in Q is given by $\mu(x, y) = (-1)^{\rho(x,y)}$. An important fact which follows from the Euler–Poincaré formula is that face lattices of convex polytopes are Eulerian. The result by Bayer and Klapper of the existence of the **cd**-index extends to Eulerian posets, that is, every Eulerian poset has a **cd**-index.

Let $\mathbb{B}^k$ denote the k-dimensional open unit ball $\{\mathbf{x} \in \mathbb{R}^k\colon\ \|\mathbf{x}\| < 1\}$ and similarly let $\mathbb{S}^{k-1}$ denote the $(k-1)$-dimensional sphere $\{\mathbf{x} \in \mathbb{R}^k\colon\ \|\mathbf{x}\| = 1\}$. A finite *regular cell complex* Γ is a finite collection of nonempty pairwise disjoint open cells $\sigma_i \subseteq \mathbb{R}^n$ so that $(\sigma_i, \partial\sigma_i)$ is homeomorphic to $(\mathbb{B}^k, \mathbb{S}^{k-1})$, where $k = \dim \sigma_i$ and $\partial\sigma_i$ can be expressed as a union of σ_j's. Again, we refer to Chapter 3 of [12] as a reference on regular cell complexes. The geometric realization of Γ, denoted $|\Gamma|$, is $|\Gamma| = \bigcup \sigma_i$. We only consider cell complexes where the geometric realization is an $(n-1)$-dimensional sphere. Form the *face poset* $P(\Gamma)$ from Γ by defining a partial order on the cells by $\sigma_i \leq \sigma_j$ if $\overline{\sigma_i} \subseteq \overline{\sigma_j}$ and adjoining a minimal element $\hat{0}$ and a maximal element $\hat{1}$.

The essential property we will need is that the face poset of a finite regular cell complex is Eulerian; see Proposition 3.8.9 of [12]. Observe that every polytope is a regular cell complex and that the notion of a face figure extends to regular cell complexes. Thus, Proposition 2.1 generalizes to regular cell complexes as follows.

Proposition 2.2. *Let Γ be a finite regular cell complex and let v be a vertex of Γ. Then the* **cd***-index of the regular cell complex $\Gamma - v$, that is, the complex Γ with the vertex v cut off, is given by*

$$\Psi(\Gamma - v) = \Psi(\Gamma) + E(\Psi(\Gamma/v)).$$

The motivation for needing the generality of regular cell complexes is that when we contract a face of a polytope the result may not be a polytope. For example, contracting an edge of a triangle gives a 2-gon. However, after contracting a face of a polytope, the result is a regular cell complex Γ whose face poset is Eulerian and hence has a **cd**-index.

3. The Facet Obtained by Cutting a Face

Let P be a polytope and let F be a face of the polytope P. When cutting the face F off P we create a new facet T of the polytope. This facet is described by

$$T = \{\mathbf{x} \in P\colon\ \ell(\mathbf{x}) = c + \delta\}.$$

When the face F is a vertex v, T is the vertex figure of v. Observe that every nonempty face of T corresponds to a face of one dimension higher from the polytope P. Moreover, these faces of P are not faces of F, but instead they strictly contain a subface of F. Hence let K denote the subposet of the face lattice of the polytope P:

$$K = \{x \in L(P)\colon \text{ there exists } y \in (\hat{0}, F] \text{ such that } y \le x\}.$$

Thus we have that the face lattice of the facet T is isomorphic to

$$L(T) \cong (K - (\hat{0}, F]) \cup \{\hat{0}\}.$$

For $k \ge 0$ define the **cd**-polynomials

$$\tau_{2k} = (\mathbf{c}^2 - 2\mathbf{d})^k \quad \text{and} \quad \tau_{2k+1} = -(\mathbf{c}^2 - 2\mathbf{d})^k \cdot \mathbf{c}.$$

As **ab**-polynomials the τ_n satisfy the identity

$$(\mathbf{a} - \mathbf{b}) \cdot \tau_n = (\mathbf{a} - \mathbf{b})^n \cdot ((-1)^n \cdot \mathbf{a} - \mathbf{b}).$$

Theorem 3.1. *Let P be a polytope with nonempty face F. Let T be the facet created by cutting off the face F from the polytope P. Then the* **cd***-index of the facet T is given by*

$$\Psi(T) = \sum_{X} \tau_{\dim(X)} \cdot \Psi(P/X),$$

where X ranges over all nonempty subfaces of the face F.

Proof. Let K be the subposet defined in the previous discussion. Consider $(K - (\hat{0}, F]) \cup \{\hat{0}\}$ as a subposet of the face lattice of the polytope P. That is, the rank of an element x in K is the rank of x in the original face lattice $L(P)$. Since the subposet $(K-(\hat{0}, F])\cup\{\hat{0}\}$ does not have any elements of rank 1, we have that $\Psi((K-(\hat{0}, F])\cup\{\hat{0}\})$ is equal to $(\mathbf{a} - \mathbf{b}) \cdot \Psi(T)$ by the chain definition (2.1) of the **ab**-index.

For a chain $c = \{x_1 < x_2 < \cdots < x_{k+1} = \hat{1}\}$ in K, let $m(c)$ denote the smallest element of the chain c, that is, $m(c) = x_1$. Moreover, define

$$r(c) = \max\{y\colon\ \hat{0} < y \le F \text{ and } y \le m(c)\}.$$

Note $r(c) = F \wedge m(c)$, where $\wedge$ denotes the meet operation of the face lattice $L(P)$.

For $x \in (\hat{0}, F]$ define

$$C_x = \sum_{c} \mathrm{wt}(c),$$

where the sum is over all chains c in K satisfying $r(c) = x$ and $m(c) > x$. Hence we have

$$\sum_{\hat{0}<x\leq F} C_x = \Psi\big((K - (\hat{0}, F]) \cup \{\hat{0}\}\big) = (\mathbf{a} - \mathbf{b}) \cdot \Psi(T). \tag{3.1}$$

We claim the following identity holds:

$$(\mathbf{a}-\mathbf{b})^{\rho(x)-1} \cdot ((\mathbf{a}-\mathbf{b})+\mathbf{b}) \cdot \Psi([x, \hat{1}]) = \sum_{x\leq y\leq F} \big(C_y + (\mathbf{a}-\mathbf{b})^{\rho(y)-1} \cdot \mathbf{b} \cdot \Psi([y, \hat{1}])\big), \tag{3.2}$$

where $\hat{0} < x \leq F$. To see this, consider all the chains c in K such that $m(c) \geq x$. The weight of these chains is counted by the left-hand side of (3.2). To count the right-hand side of (3.2), let $y = r(c)$ so that $x \leq y \leq F$. Two cases occur. First, if $y < m(c)$, then all such chains are counted by C_y. If $y = m(c)$, then these chains are counted by $(\mathbf{a} - \mathbf{b})^{\rho(y)-1} \cdot \mathbf{b} \cdot \Psi([y, \hat{1}])$. Thus the identity in (3.2) holds.

By applying the Möbius inversion theorem to (3.2), we have

$$C_x + (\mathbf{a} - \mathbf{b})^{\rho(x)-1} \cdot \mathbf{b} \cdot \Psi([x, \hat{1}]) = \sum_{x\leq y\leq F} (-1)^{\rho(x,y)} \cdot (\mathbf{a} - \mathbf{b})^{\rho(y)-1} \cdot \mathbf{a} \cdot \Psi([y, \hat{1}]).$$

Summing over all x satisfying $\hat{0} < x \leq F$ gives

$$\begin{aligned}\sum_{\hat{0}<x\leq F} C_x \;=\; &-\sum_{\hat{0}<x\leq F} (\mathbf{a} - \mathbf{b})^{\rho(x)-1} \cdot \mathbf{b} \cdot \Psi([x, \hat{1}]) \\ &+ \sum_{\hat{0}<x\leq F} \sum_{x\leq y\leq F} (-1)^{\rho(x,y)} \cdot (\mathbf{a} - \mathbf{b})^{\rho(y)-1} \cdot \mathbf{a} \cdot \Psi([y, \hat{1}]).\end{aligned}$$

Change the order of summation in the second term of the left-hand side of this equation and use that the interval $[\hat{0}, y]$ is Eulerian. After combining the two terms into one sum, we have

$$\sum_{\hat{0}<x\leq F} C_x = \sum_{\hat{0}<x\leq F} (\mathbf{a} - \mathbf{b})^{\rho(x)-1} \cdot \big((-1)^{\rho(x)-1} \cdot \mathbf{a} - \mathbf{b}\big) \cdot \Psi([x, \hat{1}]).$$

Factoring out an $(\mathbf{a}-\mathbf{b})$ and rewriting using the definition of the **cd**-polynomials τ_j gives

$$\sum_{\hat{0}<x\leq F} C_x = (\mathbf{a} - \mathbf{b}) \cdot \sum_{\hat{0}<x\leq F} \tau_{\rho(x)-1} \cdot \Psi([x, \hat{1}]).$$

This and (3.1), after dividing by $(\mathbf{a} - \mathbf{b})$, yield the desired result. □

As an example, let P be the polytope obtained from the four-dimensional cross-polytope by cutting off a vertex. The **cd**-index of P is $\Psi(P) = \mathbf{c}^4 + 11 \cdot \mathbf{dc}^2 + 23 \cdot \mathbf{cdc} + 15 \cdot \mathbf{c}^2\mathbf{d} + 30 \cdot \mathbf{d}^2$. (This can be computed from Proposition 2.1 and the expression for the bipyramid operation appearing in Corollary 4.7 of [10].) Let $e = uv$ be an edge in P such that u is a vertex from the four-dimensional cross-polytope and v is a vertex created in the cut. The face figure of e is a square, the vertex figure of u is an octahedron, and the vertex figure of v is a pyramid with square base. We have the following **cd**-indices: $\Psi(P/e) = \mathbf{c}^2 + 2 \cdot \mathbf{d}$, $\Psi(P/u) = \mathbf{c}^3 + 4 \cdot \mathbf{dc} + 6 \cdot \mathbf{cd}$, and $\Psi(P/v) = \mathbf{c}^3 + 3 \cdot \mathbf{dc} + 3 \cdot \mathbf{cd}$. The **cd**-index of the facet T created when cutting off the edge e from the polytope P is given by $\Psi(T) = -\mathbf{c} \cdot \Psi(P/e) + \Psi(P/u) + \Psi(P/v) = \mathbf{c}^3 + 7 \cdot \mathbf{dc} + 7 \cdot \mathbf{cd}$.

4. Contracting a Face

Lemma 4.1. *The polynomials*

$$\gamma_n = \mathbf{a} \cdot (\mathbf{a} - \mathbf{b})^{n-1} \cdot \mathbf{b} + (-1)^{n-1} \cdot \mathbf{b} \cdot (\mathbf{a} - \mathbf{b})^{n-1} \cdot \mathbf{a}$$

for $n \geq 1$ with $\gamma_0 = 0$ are **cd**-*polynomials. Furthermore, for $n \geq 0$,*

$$\gamma_n = (\mathbf{a} - \mathbf{b})^n \cdot \mathbf{b} - \mathbf{b} \cdot \tau_n.$$

Proof. Define a linear map $\lambda \colon \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle \longrightarrow \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ by $\lambda(\mathbf{a} \cdot w) = \mathbf{a} \cdot (\mathbf{a} - \mathbf{b}) \cdot w$, $\lambda(\mathbf{b} \cdot w) = -\mathbf{b} \cdot (\mathbf{a} - \mathbf{b}) \cdot w$, and $\lambda(1) = 0$. Observe that $\lambda(\mathbf{c} \cdot w) = (\mathbf{c}^2 - 2\mathbf{d}) \cdot w$ and $\lambda(\mathbf{d} \cdot w) = (\mathbf{cd} - \mathbf{dc}) \cdot w$. Hence λ restricts to a linear map on $\mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle$. Finally observe that $\lambda(\gamma_n) = \gamma_{n+1}$ for $n \geq 1$ and $\gamma_1 = \mathbf{d}$. To obtain the second identity, note that it holds for $n = 0$. For $n \geq 1$, it is straightforward to check. □

Theorem 4.2. *Let P be a convex polytope and let F be a nonempty face of that polytope. Let P_* be the regular cell complex created by contracting F in P. Then*

$$\Psi(P_*) = \Psi(P) - \sum_{X} \gamma_{\dim(X)} \cdot \Psi(P/X),$$

where X ranges over all nonempty subfaces of the face F.

Proof. Let T be the facet created by cutting off the face F from the polytope P. We first give a chain argument to show

$$\Psi(P_*) + \sum_{\hat{0} < x \leq F} (\mathbf{a} - \mathbf{b})^{\rho(x)-1} \cdot \mathbf{b} \cdot \Psi([x, \hat{1}]) = \Psi(P) + \mathbf{b} \cdot \Psi(T).$$

To do this, observe first that there are two types of chains in the contracted cell complex P_*: those which use the new vertex v created from contracting the face F and those which do not use the new vertex v. The weight of those chains in P_* which use the vertex v is given by the term $\mathbf{b} \cdot \Psi(T)$ and those which do not use the vertex v contribute to the weight $\Psi(P)$. The weight of the remaining chains in $\Psi(P)$ corresponds to the weight of those chains in P which include at least one nonempty subface of the face F. For such a chain c if we let x denote the first nonempty subface of F appearing in c, then the weight of all such chains is $\sum_{\hat{0} < x \leq F} (\mathbf{a} - \mathbf{b})^{\rho(x)-1} \cdot \mathbf{b} \cdot \Psi([x, \hat{1}])$. Hence the identity holds.

We thus have

$$\begin{aligned} \Psi(P_*) &= \Psi(P) + \mathbf{b} \cdot \Psi(T) - \sum_{\hat{0} < x \leq F} (\mathbf{a} - \mathbf{b})^{\rho(x)-1} \cdot \mathbf{b} \cdot \Psi([x, \hat{1}]) \\ &= \Psi(P) - \sum_{\hat{0} < x \leq F} \left((\mathbf{a} - \mathbf{b})^{\rho(x)-1} \cdot \mathbf{b} - \mathbf{b} \cdot \tau_{\rho(x)-1} \right) \cdot \Psi([x, \hat{1}]) \\ &= \Psi(P) - \sum_{\hat{0} < x \leq F} \gamma_{\rho(x)-1} \cdot \Psi([x, \hat{1}]). \end{aligned}$$

□

Recall the four-dimensional polytope P given as an example in the previous section. The **cd**-index of the polytope P_* obtained by contracting the edge e is given by $\Psi(P_*) = \Psi(P) - \mathbf{d} \cdot \Psi(P/e) = \mathbf{c}^4 + 10 \cdot \mathbf{dc}^2 + 23 \cdot \mathbf{cdc} + 15 \cdot \mathbf{c}^2\mathbf{d} + 28 \cdot \mathbf{d}^2$.

5. Cutting Off a Face

In this section we obtain an expression for cutting off any dimensional face F from a polytope P. Geometrically this follows from realizing this operation is equivalent to first contracting the face F into a point v and then cutting off the point v.

Define the family of linear operators E_n on $\mathbb{Z}\langle \mathbf{c}, \mathbf{d}\rangle$ for $n \geq 0$ by

$$E_n(w) = \begin{cases} (\mathbf{c}^2 - 2\mathbf{d})^{(n-1)/2} \cdot [-2\mathbf{d} \cdot w - \mathbf{c} \cdot E(w)] & \text{for } n \text{ odd,} \\ (\mathbf{c}^2 - 2\mathbf{d})^{n/2} \cdot E(w) & \text{for } n \text{ even.} \end{cases}$$

Note that $E_0 = E$ is a derivation defined on $\mathbb{Z}\langle \mathbf{c}, \mathbf{d}\rangle$ by $E(\mathbf{c}) = \mathbf{d}$ and $E(\mathbf{d}) = \mathbf{d} \cdot \mathbf{c}$.

Theorem 5.1. *Let P be a polytope and let F be a nonempty face of P. Then*

$$\Psi(P - F) = \Psi(P) + \sum_X E_{\dim(X)}(\Psi(P/X)),$$

where X ranges over all nonempty subfaces of the face F.

In order to prove this theorem, we need the following lemma.

Lemma 5.2. *For all nonnegative integers n we have*

$$E(\tau_n) = \gamma_n + \begin{cases} -(\mathbf{c}^2 - 2\mathbf{d})^{(n-1)/2} \cdot 2\mathbf{d} & \text{if } n \text{ odd,} \\ 0 & \text{if } n \text{ even.} \end{cases}$$

Proof. Let ε_n denote the expression given in the bracelet. We prove the statement by induction on n. The base cases $n = 0$ and $n = 1$ are straightforward to check. Assume now that the identity holds for n and we prove it for $n + 2$. We have

$$E(\tau_{n+2}) = E(\mathbf{c}^2 - 2\mathbf{d}) \cdot \tau_n + (\mathbf{c}^2 - 2\mathbf{d}) \cdot E(\tau_n).$$

Applying the induction hypothesis gives

$$E(\tau_{n+2}) = (\mathbf{cd} - \mathbf{dc}) \cdot \tau_n + (\mathbf{c}^2 - 2\mathbf{d}) \cdot \gamma_n + (\mathbf{c}^2 - 2\mathbf{d}) \cdot \varepsilon_n.$$

Expanding γ_n with Lemma 4.1 and observing $\varepsilon_{n+2} = (\mathbf{c}^2 - 2\mathbf{d}) \cdot \varepsilon_n$ gives

$$E(\tau_{n+2}) = (\mathbf{cd} - \mathbf{dc}) \cdot \tau_n + (\mathbf{a} - \mathbf{b})^2 \cdot ((\mathbf{a} - \mathbf{b})^n \cdot \mathbf{b} - \mathbf{b} \cdot \tau_n) + \varepsilon_{n+2}.$$

After simplifying and using Lemma 4.1 again, we have

$$E(\tau_{n+2}) = (\mathbf{a} - \mathbf{b})^{n+2} \cdot \mathbf{b} - \mathbf{b} \cdot \tau_{n+2} + \varepsilon_{n+2} = \gamma_{n+2} + \varepsilon_{n+2}. \qquad \square$$

Observe that Lemma 5.2 gives a different proof that the polynomials γ_n are **cd**-polynomials.

Proof of Theorem 5.1. We first contract the face F to a point v to form the cell complex P_* and then cut v from P_* to form the polytope $P - F$. Observe that $E_n(w) = \varepsilon_n \cdot w + \tau_n \cdot E(w)$. Now by Theorems 3.1 and 4.2 we have

$$\begin{aligned}
\Psi(P-F) &= \Psi(P_*) + E(\Psi(T)) \\
&= \Psi(P) - \sum_X \gamma_{\dim(X)} \cdot \Psi(P/X) + E\left(\sum_X \tau_{\dim(X)} \cdot \Psi(P/X)\right) \\
&= \Psi(P) + \sum_X (E(\tau_{\dim(X)}) - \gamma_{\dim(X)}) \cdot \Psi(P/X) + \sum_X \tau_{\dim(X)} \cdot E(\Psi(P/X)) \\
&= \Psi(P) + \sum_X \varepsilon_{\dim(X)} \cdot \Psi(P/X) + \sum_X \tau_{\dim(X)} \cdot E(\Psi(P/X)) \\
&= \Psi(P) + \sum_X E_{\dim(X)}(\Psi(P/X)),
\end{aligned}$$

where X ranges over all nonempty subfaces of the face F. □

Finally in our example, cutting off the edge e from the polytope P we have $\Psi(P-e) = \Psi(P) + E_1(\Psi(P/e)) + E_0(\Psi(P/u)) + E_0(\Psi(P/v)) = \mathbf{c}^4 + 18 \cdot \mathbf{dc}^2 + 31 \cdot \mathbf{cdc} + 16 \cdot \mathbf{c}^2\mathbf{d} + 42 \cdot \mathbf{d}^2$.

Acknowledgments

The authors would like to thank Louis Billera and the two referees for helpful comments.

References

1. M. Bayer, The extended f-vectors of 4-polytopes, *J. Combin. Theory Ser. A* **44** (1987), 141–151.
2. M. Bayer and L. Billera, Generalized Dehn–Sommerville relations for polytopes, spheres and Eulerian partially ordered sets, *Invent. Math.* **79** (1985), 143–157.
3. M. Bayer and R. Ehrenborg, On the toric h-vectors of posets, to appear in *Trans. Amer. Math. Soc.*
4. M. Bayer and A. Klapper, A new index for polytopes, *Discrete Comput. Geom.* **6** (1991), 33–47.
5. M. Bayer and B. Sturmfels, Lawrence polytopes, *Canad. J. Math.* **42** (1990), 62–79.
6. L. J. Billera and R. Ehrenborg, Monotonicity properties of the **cd**-index for polytopes, to appear in *Math. Z.*
7. L. J. Billera, R. Ehrenborg, and M. Readdy, The **c**-2**d**-index of oriented matroids, *J. Combin. Theory Ser. A* **80** (1997), 79–105.
8. L. J. Billera, R. Ehrenborg, and M. Readdy, The **cd**-index of zonotopes and arrangements, *Mathematical Essays in Honor of Gian-Carlo Rota* (B. E. Sagan and R. P. Stanley, eds.), Birkhäuser, Boston, 1998, pages 23–40.
9. R. Ehrenborg and H. Fox, Products of polytopes and inequalities, Preprint, 1999.
10. R. Ehrenborg and M. Readdy, Coproducts and the **cd**-index, *J. Algebraic Combin.* **8** (1998), 273–299.

11. C. W. Lee, Counting the faces of simplicial convex polytopes, Doctoral dissertation, Cornell University, 1981.
12. R. P. Stanley, *Enumerative Combinatorics*, Vol. I, Wadsworth and Brooks/Cole, Pacific Grove, CA, 1986.
13. R. P. Stanley, Flag f-vectors and the cd-index, *Math. Z.* **216** (1994), 483–499.
14. E. Steinitz, Über die Eulerischen Polyderrelationen, *Arch. Math. Phys.* **11** (1906), 86–88.
15. G. M. Ziegler, *Lectures on Polytopes*, Graduate Texts in Mathematics 152, Springer-Verlag, New York, 1995.

Received July 13, 1998, *and in revised form April* 14, 1999.