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Non-constructible Complexes and the Bridge Index

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We show that if a three-dimensional polytopal complex has a knot in its 1-skeleton, where the bridge index of the knot is larger than the number of edges of the knot, then the complex is not constructible, and hence, not shellable. As an application we settle a conjecture of Hetyei concerning the shellability of cubical barycentric subdivisions of 3-spheres. We also obtain similar bounds concluding that a 3-sphere or 3-ball is non-shellable or not vertex decomposable. These two last bounds are sharp.

1. INTRODUCTION

In the history of the study of shellability, many examples of non-shellable triangulations of balls and spheres have been constructed. A review can be found in the paper by Ziegler [22]. There are two other important properties that a simplicial complex can satisfy, namely constructibility and vertex decomposability. These properties satisfy the following hierarchy.

$$\text{vertex decomposable} \quad \Longrightarrow \quad \text{shellable} \quad \Longrightarrow \quad \text{constructible.}$$

By considering the contrapositive implications; that is,

$$\text{not vertex decomposable} \quad \Longleftarrow \quad \text{non-shellable} \quad \Longleftarrow \quad \text{non-constructible,}$$

we have that non-shellability is implied by non-constructibility.

Among the examples of non-shellable triangulations, Furch's 3-ball [6] (also shown in Bing's article [2]) and Lickorish's 3-sphere [13] involve a special knot embedded as a 1-dimensional complex of small size. Both of these examples are treated in the paper of Hachimori and Ziegler [9] and were extended to the following theorem.

THEOREM 1.1 (HACHIMORI AND ZIEGLER). *A* 3*-ball with a knotted spanning arc consisting of*

$$\begin{cases} \textit{at most } 2 & \textit{edges is not constructible,} \\ 3 & \textit{edges can be shellable, but not vertex decomposable,} \\ 4 & \textit{edges can be vertex decomposable,} \end{cases}$$

and a triangulated 3*-sphere or* 3*-ball with a knot consisting of*

$$\begin{cases} \textit{at most } 3 & \textit{edges is not constructible,} \\ 4 \textit{ or } 5 & \textit{edges can be shellable, but not vertex decomposable,} \\ 6 & \textit{edges can be vertex decomposable.} \end{cases}$$

Shellability and constructibility naturally extend to polytopal complexes, whereas vertex decomposability only applies to simplicial complexes. We note that the proof of the parts of Theorem 1.1 involving shellable and constructible triangulated manifolds is valid for polytopal complexes and the result naturally extends to polytopal decompositions.

In this paper we extend the Hachimori–Ziegler result for knots of larger size. In Theorem 4.2 we show that if the bridge index of a knot is larger than the number of edges of the knot, then

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the complex is not constructible. Similar bounds hold for concluding that a simplicial complex is not shellable or vertex decomposable.

The present work was inspired by the results of Armentrout [1]. He considered simple cell partitionings that contain a knot through its 2- and 3-cells. If the knot has a bridge index larger than the number of spanning arcs it is partitioned into by the cell partitioning, then he proved that the cell partitioning is not shellable [1, Theorem 3]. Thus one can view Theorem 4.2 as the dual to Armentrout's Theorem 3. Moreover, in Section 6 we extend his result to prove non-constructibility for general polytopal complexes that contain a weakly compatible knot in their two- and three-dimensional faces.

Our proofs rely on extending the bridge index, a knot invariant, to tangles. A tangle is a disjoint collection of paths and knots inside a 3-ball such that the endpoints of the paths are on the boundary of the 3-ball. The bridge index of a knot can be defined using the notion of bridge positions, and we define the bridge index for tangles in the same manner as that for knots. The essential property of the bridge index of tangles is Proposition 3.4 which states that the bridge index is subadditive. This should be compared with the fact that the bridge index for knots is additive under knot addition; see [18].

Our theorem has several applications. One is the existence of triangulations of a PL-d-sphere and a PL-d-ball whose n-fold barycentric subdivision is not constructible for given integers $n \geq 0$ and $d \geq 3$. Another important application is a conjecture by Hetyei on the existence of triangulated 3-spheres whose cubical barycentric subdivisions are non-shellable [10, 11]. Our result solves this conjecture affirmatively.

In the concluding remarks (Section 7) we discuss the sharpness of our bounds. Moreover, further questions for research are presented.

2. Preliminaries

In this section we give the basic definitions related to polytopal complexes, constructibility and vertex decomposability. For other basic material on polytopal complexes, we recommend the book by Ziegler [21].

A *polytopal complex* C is a finite set of (convex) polytopes such that (i) if $P \in C$ then all the faces of P are contained in C, and (ii) if $P, Q \in C$ then $P \cap Q$ is a face of both P and Q. In particular, the empty set $\emptyset$ is always contained in C. The members of C are *faces* of C. The zero-dimensional faces are called *vertices* and the one-dimensional faces are called *edges*. The empty set $\emptyset$ is a (-1)-dimensional face. The *dimension* of a polytopal complex is the largest dimension of its faces. The *face poset* of a polytopal complex is the partially ordered set consisting of all the faces ordered by inclusion. Observe that the face poset is a meet-semilattice. The inclusion-maximal faces are *facets*. If all the facets are of the same dimension then the complex C is *pure*. If all the faces are simplices then it is called a *simplicial complex*, whereas if they are all combinatorially equivalent to cubes then it is called a *cubical complex*. A d-dimensional polytopal complex C is called *simple* if whenever m of its facets have a non-empty intersection its dimension is $d - m + 1$. The *k-skeleton* of a polytopal complex is the collection of all faces of dimension k or less. In particular, the 1-skeleton consists of all vertices and edges. The *link* of a face P in a polytopal complex C, $\mathrm{link}_C(P)$, is the polytopal complex which is combinatorially equivalent to the face figure of P in C, namely, a polytopal complex whose face poset is the same as the upper ideal of all elements containing the face P in the face poset of C. For a simplicial complex C let $u * C = C \cup \{u * \sigma : \sigma \in C\}$ be the *cone* of C, where u is a vertex not belonging to C and $u * \sigma$ is a simplex spanned by u and σ. In addition, let $\Sigma(C) = u * C \cup v * C$ be the *suspension* of C, where u and v are two distinct vertices not in C.

For a polytopal complex C, the union $|C|$ of all the polytopes of C is the *underlying space* of C and C is called a *polytopal decomposition* of $|C|$. A polytopal (simplicial, cubical) manifold is a polytopal (simplicial, cubical, respectively) complex whose underlying space is homeomorphic to the manifold.

DEFINITION 2.1. A d-dimensional pure polytopal complex C is *constructible* if:

(i) C is a d-dimensional polytope, or
(ii) there exist polytopal complexes C_1 and C_2 such that $C = C_1 \cup C_2$, the complexes C_1 and C_2 are d-dimensional pure constructible polytopal complexes and the complex $C_1 \cap C_2$ is a $(d-1)$-dimensional pure constructible polytopal complex.

The idea of constructibility can be seen in combinatorial topology, for instance in Zeeman's book [20]. The first explicit definition of this term is likely due to Hochster [12].

DEFINITION 2.2. A d-dimensional pure polytopal complex C is *shellable* if:

(i) C is a d-dimensional polytope, or
(ii) there exist polytopal complexes C_1 and C_2 such that $C = C_1 \cup C_2$, the complex C_1 is a d-dimensional pure shellable polytopal complex, C_2 is a d-dimensional polytope and the complex $C_1 \cap C_2$ is a $(d-1)$-dimensional pure shellable polytopal complex.

This definition of shellability is a reformulation of the classical definition. The classical definition is that there exists an ordering of the facets $F_1, F_2, \ldots, F_n$ (i.e., a shelling) such that for all $2 \leq j \leq n$ the complex $(\overline{F_1} \cup \cdots \cup \overline{F_{j-1}}) \cap \overline{F_j}$ is $(d-1)$-dimensional and shellable. This definition of shellability is equivalent to the definition used in the paper of Bruggesser and Mani [4], but weaker than the usual definition; see [21]. However, for simplicial complexes and cubical complexes both definitions are equivalent because d-simplices and d-cubes are extendably shellable. By comparing the condition on the complex C_2 in Definitions 2.1 and 2.2, we observe that constructibility is a natural relaxation of shellability.

Let C be a polytopal complex and v a vertex of C. Observe that if C is a constructible complex then the link of the complex C at the vertex v, $\mathrm{link}_C(v)$, is also constructible; see [3, p. 1855]. This allows us to lift a non-constructible object from one dimension to the next. Namely the contrapositive statement is

$$\mathrm{link}_C(v) \text{ is non-constructible} \quad \Longrightarrow \quad C \text{ is non-constructible.}$$

This property will be used in the proof of Propositions 4.4 and 4.5.

We remark that if C is a d-dimensional constructible complex whose $(d-1)$-dimensional faces belong to at most two facets then $|C|$ is a PL-ball or a PL-sphere. (See Björner [3, Theorem 11.4] and Zeeman [20, Chapter 3].) Thus if C is a constructible polytopal ball or a sphere then C_1 and C_2 in Definition 2.1 are constructible polytopal balls.

For C a simplicial complex and v a vertex of the complex C, let $C - v$ denote the simplicial complex consisting of all the faces F in C that do not contain the vertex v.

DEFINITION 2.3. A d-dimensional pure simplicial complex C is *vertex decomposable* if:

(i) C is a d-dimensional simplex, or
(ii) there exists a vertex v in C such that $C - v$ is a pure d-dimensional vertex decomposable simplicial complex and $\mathrm{link}_C(v)$ is a $(d-1)$-dimensional pure vertex decomposable simplicial complex.

The vertex v in part (ii) is called a *shedding vertex* of the simplicial complex C. The definition of vertex decomposability is due to Provan and Billera, who showed that vertex decomposability implies shellability [16].

3. Tangles and the Bridge Index

We now introduce knots, tangles and the bridge index and prove the subadditivity for the bridge index. For references on knot theory, we suggest the books by Lickorish [14] and Livingston [15].

A *knot* is a simple closed arc contained in a three-dimensional space. The three-dimensional spaces we consider are 3-balls and 3-spheres. A *link* is the disjoint union of knots. A *spanning arc* is a simple arc contained in a 3-ball whose endpoints are on the boundary of the ball. A *tangle* is a set of mutually disjoint spanning arcs and knots in a 3-ball or 3-sphere. Observe that a tangle in a 3-sphere is necessarily a link since the 3-sphere has no boundary to which the spanning arcs can be attached. A *semispanning disc* D is a disc contained in a 3-ball C such that $\partial D = \alpha \cup \beta$, where α is some spanning arc of C and β is some simple arc contained in the boundary ∂C of C. A spanning arc α is *straight* if there is a semispanning disc D such that $\alpha \subseteq \partial D$. A set of spanning arcs are *simultaneously straight* if they are mutually disjoint and they have mutually disjoint semispanning discs. Moreover, if these semispanning discs can be taken such that they avoid the interior of a disc B on ∂C, then we say that the spanning arcs are *simultaneously straight with respect to the disc B*.

As in the classical treatment of knots, the knots and tangles in this paper are considered to be piecewise linear. The usual treatment of knots and tangles requires the arcs, except for their endpoints, to be in the interior of the 3-space. However, on this point we will differ. We just require that the whole tangles are in the 3-space, allowing the relative interior of the arcs to intersect with the boundary of the 3-space. For instance, this was done in [9]. To make the equivalence relation precise, we give here a definition of tangle equivalence used in this paper.

Two tangles T_1 and T_2 are related by an *elementary deformation* if they only differ locally by one of the following two cases.

(i) The segments $[p, q]$ and $[q, r]$ are in the tangle T_1, the segment $[p, r]$ is in T_2 and the disc spanned by $[p, q, r]$ intersects T_2 only in the segment $[p, r]$.

(ii) The segment $[p, r]$ is in the tangle T_1, the point p is an endpoint of T_1, the segment $[q, r]$ is in T_2, the point q is an endpoint of T_2 and the disc spanned by $[p, q, r]$ intersects T_2 only in the segment $[q, r]$.

The first case is the classical elementary deformation in knot theory; see [15, Chapter 2.3]. The second case allows us to move the endpoints of spanning arcs. Observe that the endpoints must remain on the boundary of the 3-ball. We say that two tangles T_1 and T_2 are *equivalent* if there is a sequence $T_1 = \tau_0, \tau_1, \ldots, \tau_t = T_2$ of tangles in which τ_i is derived from τ_{i-1} by an elementary deformation.

The bridge index is a classical knot invariant [15, 18]. We now extend this invariant to tangles. Our definition is a generalization of the one given by Armentrout [1]. For a different view of this invariant, see Proposition 3.3 and the paragraph preceding it.

DEFINITION 3.1. Let T be a tangle in a 3-ball C. The tangle T is in an *m-bridge position* if T is composed of m mutually disjoint spanning arcs α_i in C which are simultaneously straight and whose relative interiors are contained in the interior of C, and some other simple arcs β_j contained in the boundary of C. Moreover, every connected component of T is required to contain at least one α_i.

In the following we talk of α_i in the definition as 'α-arcs' and β_i as 'β-arcs'.

If T is in an m-bridge position, then every connected component of the tangle T is composed of alternating α-arcs and β-arcs.

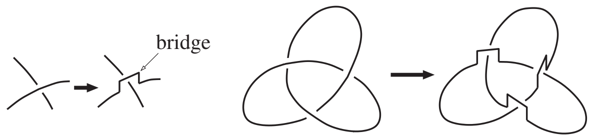

FIGURE 1. Making bridges on a projected trefoil.

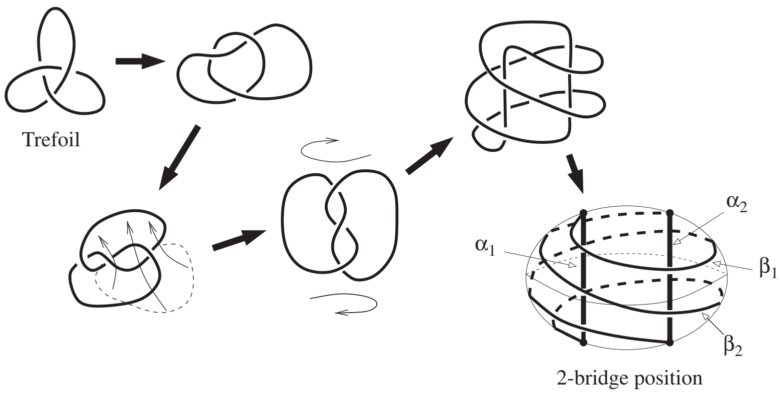

FIGURE 2. The trefoil knot and its 2-bridge position.

We claim that every tangle T is equivalent to a tangle τ which is in an m-bridge position for some positive integer m. Consider the tangle T in a 3-ball. By elementary deformations of the tangle T we can move all the endpoints of the spanning arcs to be on the equator of the ball, obtaining tangle T'. Now we project the tangle straight down on the southern hemisphere of the ball. When drawing this projection on the hemisphere one has to draw which strands of the tangle cross over which strands. This is the same as when drawing the diagram of a knot; see the discussion in [15, Chapter 2.4]. Now each overpass in the diagram can be replaced with a small bridge; see Figure 1. (Hence the name bridge index.) If there is a component which has no overpasses (either the component is a straight arc or the unknot) then we make a small bridge in the middle of this component. Each bridge is a straight spanning arc. More importantly, this collection of bridges is simultaneously straight. The tangle τ obtained this way is in an m-bridge position, where m is the number of overpasses. Moreover, the τ only differ from the T' by elementary deformations. In fact, Schubert's [18] original definition of the bridge index was the smallest number of 'bridges' needed to realize a knot in this manner.

Observe that the typical drawing of the trefoil has three overpasses; see Figure 1. Hence the trefoil is equivalent to a knot in a 3-bridge position. But we can do better. In Figure 2 the trefoil is moved by elementary transformations into a 2-bridge position.

DEFINITION 3.2. For a tangle T we define the *bridge index* $b(T)$ as the minimum positive integer m such that there is a tangle τ in an m-bridge position and τ is equivalent to T. If a tangle T is in a 3-sphere C (in this case, T is a link) then we take a 3-ball C' in C containing T and define its bridge index with respect to C'.

A few examples of the bridge index are:

- An unknot has bridge index 1. In fact, the unknot is the only knot having bridge index 1.
- The trefoil knot has bridge index 2; see Figure 2.
- A straight spanning arc has bridge index 1.

- A tangle consisting of n simultaneously straight spanning arcs has bridge index n.
- The bridge index for knots satisfies $b(K_1 \# K_2) = b(K_1) + b(K_2) - 1$, where # denotes knot addition [14, 15, 18]. Hence for every positive integer n there is a knot with bridge index n, for example, the $(n-1)$-fold knot sum $K^{\#(n-1)}$, where K denotes the trefoil knot. Since the trefoil K has bridge index 2, we conclude that $K^{\#(n-1)}$ has bridge index n.

The bridge index of a knot can be viewed as the minimum number of local maxima over all knot diagrams of the knot; see [15, Chapter 7.3]. The following proposition shows this is also the case for our definition of the bridge index for tangles. A *height function* h on the closed 3-ball C is a continuous function from the ball C onto an interval $[a, b]$ such that the inverse image $h^{-1}(x)$ is a closed disc for $x \in (a, b)$ and $h^{-1}(a)$ and $h^{-1}(b)$ are both points. A point p on a tangle T is a *local maximum* if p is not an endpoint of the tangle T and there is a positive number ε such that $h(p) \geq h(x)$ for all $x \in T$ with $|x - p| < \varepsilon$.

PROPOSITION 3.3. *The bridge index of the tangle T in the 3-ball C is given by*

$$b(T) = \min\{\textit{number of local maxima of } T \textit{ with respect to } h\} + \textit{number of paths of } T,$$

where h ranges over all possible height functions of the ball C.

Since this proposition is not needed for the later sections of the paper, we omit the proof. This kind of equivalence is well known for knots [7] (in this case the number of paths is zero), and the proof for tangles is almost the same as that for knots.

The next proposition and Proposition 3.6 are the keys to the theorems in the following sections.

PROPOSITION 3.4. *Let C be a 3-ball (3-sphere) and C_1 and C_2 be 3-balls such that $C = C_1 \cup C_2$ and $C_1 \cap C_2$ is a 2-ball (2-sphere). Let T be a tangle of C. Set T_1 to be the intersection $T \cap C_1$ and let T_2 be the topological closure of $T - T_1$. (Hence T_1 and T_2 are tangles of C_1 and C_2, respectively.) Then we have*

$$b(T) \leq b(T_1) + b(T_2).$$

PROOF. Consider first the case when C is a 3-sphere. It is possible to choose a 3-ball $C' \subseteq C$ such that T is contained in C', $C'_i = C' \cap C_i$ is a 3-ball for $i = 1, 2$, the tangle T_i is contained in C'_i for $i = 1, 2$ and $C'_1 \cap C'_2$ is a 2-ball in $C_1 \cap C_2$. Now when replacing C, C_1, C_2 by C', C'_1, C'_2 the bridge indices of T, T_1 and T_2 do not change. Hence we can assume that C is a 3-ball.

We will construct a tangle τ which is equivalent to the tangle T and is in a $(b(T_1) + b(T_2))$-bridge position. This will prove that $b(T) = b(\tau) \leq b(T_1) + b(T_2)$ which is the claim of the proposition.

The intersection $T_1 \cap T_2$ is a set P of points $\{p_1, p_2, \ldots, p_t\}$ in $C_1 \cap C_2$. Using some elementary deformations, we can assume that all the points of P lie on the boundary of the disc $C_1 \cap C_2$.

Let τ_i be a tangle which is equivalent to T_i and in a $b(T_i)$-bridge position in C_i, $i = 1, 2$. Without loss of generality, we can assume that the endpoints in τ_i do not lie in $C_1 \cap C_2$. Let p'_{ij} be the endpoint of τ_i corresponding to p_j of T_i. Then we connect p_j and p'_{ij} by an arc on the boundary of C_i ($i = 1, 2$) such that $\tau = \tau_1 \cup \tau_2 \cup \{p'_{1j} p_j p'_{2j}\}$ is equivalent to T. The fact that such a connection is possible can be checked step by step according to the elementary deformations from T_i to τ_i.

Observe that τ is now a tangle in a $(b(T_1)+b(T_2))$-bridge position. Moreover τ is equivalent to T, thus proving the desired inequality. □

We remark that the requirement in Definition 3.2 that every connected component must have at least one α-arc is necessary in the proof of Proposition 3.4. The following case could otherwise occur: a spanning arc τ_i consists of exactly one β-arc on the boundary of C and an arc $p'_{ij} p_j$ would have to cross this arc. This situation would make the construction in the proof fail.

Proposition 3.4 gives a bound for $b(T)$ in terms of $b(T_1)$ and $b(T_2)$. In the case when T_2 is restricted to a collection of simultaneously straight spanning arcs, Lemma 3.5 and Proposition 3.6 improve the bound for $b(T)$. These two results will be useful in Sections 5 and 6.

LEMMA 3.5. *Let C be a* 3*-ball* (3*-sphere*) *and C_1 and C_2 be* 3*-balls such that $C = C_1 \cup C_2$ and $C_1 \cap C_2$ is a* 2*-ball* (2*-sphere*)*. Let T be a tangle of C. Set T_1 to be the intersection $T \cap C_1$ and let T_2 be the topological closure of $T - T_1$. Assume that T_2 is a straight spanning arc.*

(i) If $T_1 \cap T_2$ consists of two points then $b(T) \leq b(T_1)$.
(ii) If $T_1 \cap T_2$ is one point then $b(T) = b(T_1)$.
(iii) If $T_1 \cap T_2 = \emptyset$ then $b(T) = b(T_1) + 1$.

The proof of this lemma is straightforward and hence omitted. The next proposition generalizes the previous lemma.

PROPOSITION 3.6. *Let C be a* 3*-ball* (3*-sphere*) *and C_1 and C_2 be* 3*-balls such that $C = C_1 \cup C_2$ and $C_1 \cap C_2$ is a* 2*-ball* (2*-sphere*)*. Let T be a tangle of C. Set T_1 to be the intersection $T \cap C_1$ and let T_2 be the topological closure of $T - T_1$. Assume that T_2 consists of*

- *a straight spanning arcs each of which intersects with T_1 in two points,*
- *b straight spanning arcs each of which intersects with T_1 in one point, and*
- *c straight spanning arcs each of which intersects with T_1 in zero points.*

If T_2 is simultaneously straight with respect to $C_1 \cap C_2$ then we have

$$b(T) \leq b(T_1) + c.$$

PROOF. Similar to Proposition 3.4, the case when C is a 3-sphere reduces to the case when C is a 3-ball. Hence we may assume that C is a 3-ball. Because T_2 is simultaneously straight with respect to $C_1 \cap C_2$, the arcs of T_2 have mutually disjoint semispanning discs avoiding the interior of $C_1 \cap C_2$. Along these semispanning discs, we can move the arcs onto $\partial C_2 - C_1$ by elementary moves. Thus we can assume without loss of generality that the arcs of T_2 are all on the boundary of C.

Now take a tubular neighborhood $N(k_i)$ for each arc k_i of T_2. If we take the neighborhoods small enough then they are mutually disjoint and also disjoint from the arcs of T_1. Let C° be the 3-ball $\overline{C - \bigcup N(k_i)}$. Now add each tube $N(k_i)$ one by one to C°. We observe that each step satisfies the condition of Lemma 3.5, and the inequality follows. $\square$

The condition that T_2 is simultaneously straight *with respect to* $C_1 \cap C_2$ is necessary for Proposition 3.6. If this condition is dropped, it is straightforward to construct counterexamples.

4. CONSTRUCTIBLE COMPLEXES

In this section we show that tangles embedded in the 1-skeleton of a three-dimensional constructible complex must contain at least the bridge index number of edges. For such a tangle T, let $e(T)$ denote the number of edges that the tangle contains.

THEOREM 4.1. *Let C be a three-dimensional polytopal ball or sphere which is constructible. Let T be a tangle contained in the* 1*-skeleton of the polytopal complex C. Then we have the inequality*

$$b(T) \leq e(T).$$

PROOF. The proof is by induction on the number of facets of C. The induction basis is when C is a three-dimensional polytope. In this case T is a disjoint union of path and cycles (unknots). Let k be the number of components of T. Then $b(T) = k \leq e(T)$, and the induction basis is complete.

The induction step is as follows. By condition (ii) of Definition 2.1, we have two 3-dimensional complexes C_1 and C_2 which are constructible 3-balls and $C = C_1 \cup C_2$. Let $T_1 = T \cap C_1$ and $T_2 = \overline{T - T_1}$. By Proposition 3.4 and the induction hypothesis, we obtain

$$b(T) \leq b(T_1) + b(T_2) \leq e(T_1) + e(T_2) = e(T).$$

This completes the induction. □

Theorem 4.1 implies the following result.

THEOREM 4.2. *Let C be a three-dimensional polytopal ball or sphere. Assume that the* 1*-skeleton of the complex C contains a knot K such that*

$$e(K) \leq b(K) - 1.$$

Then the polytopal complex C is non-constructible.

Thus our theorem proves the existence of non-constructible triangulations of a 3-sphere or a 3-ball, if we can embed a knot with large bridge index using a small number of edges. The following well-known proposition states that such an embedding is possible. In fact, it says that any knot can be embedded into a triangulated 3-sphere or a 3-ball using e edges, where e is any integer greater than or equal to 3. For references see Lickorish [13, Lemma 3] or Ziegler [22, Section 3.2].

PROPOSITION 4.3. *Given any knot K and an integer $e \geq 3$, there exists a triangulation of a* 3*-sphere or a* 3*-ball which embeds K as a subcomplex consisting of e edges* (*and hence e vertices*).

We now present two applications of Theorem 4.2. For a simplicial complex C, denote by $\Delta(C)$ the barycentric subdivision of C.

PROPOSITION 4.4. *Let d be greater than or equal to* 3, *and n be any non-negative integer. Then there exists a triangulation $C_d(n)$ of the d-dimensional sphere* (*or ball*) *which is piecewise linear* (*PL*), *such that the n-fold barycentric subdivision $\Delta^n(C_d(n))$ is non-constructible.*

PROOF. The proof is by induction on dimension d. First we consider the case $d = 3$. Choose a knot K with bridge index larger than or equal to $3 \cdot 2^n + 1$. Let $C_3(n)$ be a triangulation of the three-dimensional sphere (or ball) that contains K on three edges. Such a triangulation is guaranteed by Proposition 4.3. Observe that when taking the barycentric subdivision each edge is divided into two edges. Hence the knot K contained in $\Delta^n(C_3(n))$ consists of $3 \cdot 2^n$ edges. From Theorem 4.2, it now follows that the complex $\Delta^n(C_3(n))$ is non-constructible. Finally, observe that all triangulations of three-dimensional spheres (and balls) are piecewise linear.

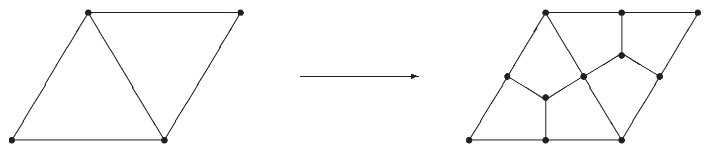

FIGURE 3. Two triangles joined at an edge and the cubical barycentric subdivision.

Assume now that $d \geq 4$. In the case of spheres, let $C_d(n)$ be the suspension of $C_{d-1}(n)$; that is, $C_d(n) = \Sigma(C_{d-1}(n)) = u * C_{d-1}(n) \cup v * C_{d-1}(n)$, where u and v are newly introduced vertices. This is a triangulation of the PL-d-sphere since $C_{d-1}(n)$ is a PL-$(d-1)$-sphere. In the case of balls, we consider the cone over $C_{d-1}(n)$ instead of the suspension; that is, we let $C_d(n) = v * C_{d-1}(n)$, where v is a newly introduced vertex. This yields a PL-d-ball since $C_{d-1}(n)$ is a PL-$(d-1)$-ball. In both cases, observe that

$$\operatorname{link}_{C_d(n)}(v) = C_{d-1}(n).$$

For a simplicial complex D and a vertex v of D we have

$$\operatorname{link}_{\Delta^n(D)}(v) \cong \Delta^n(\operatorname{link}_D(v)),$$

where $\cong$ denotes combinatorial equivalence. Using these relations, we have

$$\operatorname{link}_{\Delta^n(C_d(n))}(v) \cong \Delta^n(\operatorname{link}_{C_d(n)}(v)) = \Delta^n(C_{d-1}(n)).$$

Since $\operatorname{link}_{\Delta^n(C_d(n))}(v)$ is not constructible we conclude that $\Delta^n(C_d(n))$ is not constructible either, from what has been stated in Section 2. $\square$

Given a simplicial complex C, the *cubical barycentric subdivision* of the complex C is the abstract cubical complex $\square(C)$ such that:

(i) the set of vertices of $\square(C)$ is the set of non-empty faces of C, and
(ii) a face of the cubical complex $\square(C)$ is an interval of the face poset of C.

It is straightforward to see that the cubical barycentric subdivision $\square(C)$ is a cubical complex and that $\square(C)$ is a subdivision of the simplicial complex C. Hence the simplicial complex C and its cubical barycentric subdivision $\square(C)$ have the same geometrical realization. See Figure 3 for an example of cubical barycentric subdivision.

PROPOSITION 4.5. *Let d be greater than or equal to 3. Then there exists a d-dimensional simplicial PL-sphere C_d such that the cubical barycentric subdivision $\square(C_d)$ is non-constructible.*

PROOF. Consider first the case when d is equal to 3. Choose a knot K with bridge index larger than or equal to 7 and let C_3 be a simplicial complex that contains the knot K on three edges. Observe that the complex C_3 is non-constructible. By the same argument as in Proposition 4.4, the cubical complex $\square(C_3)$ is non-constructible.

The remaining part of the proof is by induction on dimension. Let C_d be the suspension of C_{d-1}; that is, $C_d = \Sigma(C_{d-1}) = u * C_{d-1} \cup v * C_{d-1}$, where u and v are newly introduced vertices. Then we have that $\operatorname{link}_{C_d}(v) = C_{d-1}$, and hence C_d is non-constructible. Observe that $\operatorname{link}_{\square(C_d)}(v) = C_{d-1}$, and hence $\square(C_d)$ is also non-constructible. $\square$

Proposition 4.5 settles a conjecture of Hetyei [10, 11] on the existence of a triangulation C of the d-dimensional sphere such that $\square(C)$ is not shellable. For dimensions d greater than or equal to 4 this was settled by Readdy (unpublished). The second half of our proof is essentially her argument.

In the light of Propositions 4.4 and 4.5 we have the following proposition. Its proof follows the lines of the two previous proofs.

PROPOSITION 4.6. *Let d be greater than or equal to* 3 *and n be any non-negative integer. Then there exists a d-dimensional simplicial PL-sphere $C_d(n)$ such that the cubical complex $\square(\Delta^n(C_d(n)))$ is non-constructible.*

5. SHELLABLE AND VERTEX DECOMPOSABLE SIMPLICIAL COMPLEXES

In this section we improve the results of Section 4 for shellable and vertex decomposable complexes.

THEOREM 5.1. *Let C be a three-dimensional polytopal ball or sphere which is shellable. Let K be a knot contained in the* 1*-skeleton of the simplicial complex C. Then we have the inequality*

$$2 \cdot b(K) \leq e(K).$$

PROOF. We may assume that K is not the unknot. Since C is shellable there is an ordering of the facets $F_1, F_2, \ldots, F_n$ such that $(\overline{F}_1 \cup \cdots \cup \overline{F}_{j-1}) \cap \overline{F}_j$ is a shellable 2-complex.

Let $C_1^{(n+1)} = C$, $C_1^{(i)} = \overline{F_1 \cup \cdots \cup F_{i-1}}$, and $C_2^{(i)} = F_i$. Let $T_1^{(n+1)} = K$, $T_1^{(i)} = T_1^{(i+1)} \cap C_1^{(i)}$, and $T_2^{(i)} = \overline{T_1^{(i+1)} - T_1^{(i)}}$. ($T_1^{(1)} = \emptyset$.) Note that $C_1^{(i+1)} = C_1^{(i)} \cup C_2^{(i)}$ and $T_1^{(i+1)} = T_1^{(i)} \cup T_2^{(i)}$ are decompositions of the type described in Proposition 3.4.

Observe that $T_2^{(i)}$ is in $\partial C_2^{(i)} - C_1^{(i)}$. This assures that $T_2^{(i)}$ is simultaneously straight with respect to $C_1^{(i)} \cap C_2^{(i)}$; that is, the condition of Proposition 3.6 is satisfied for each i. Let a_i, b_i and c_i be the number of arcs of $T_2^{(i)}$ described in Proposition 3.6. Then we have

$$b(T_1^{(i+1)}) \leq b(T_1^{(i)}) + c_i. \tag{5.1}$$

Moreover, by considering the Euler characteristic of the tangle $T_1^{(i+1)}$ we have that

$$\chi(T_1^{(i+1)}) = \chi(T_1^{(i)}) - a_i + c_i. \tag{5.2}$$

Adding all the inequalities in (5.1) and separately adding all the equalities in (5.2), using the fact that $T_1^{(1)} = \emptyset$, $T_1^{(n+1)} = K$, $b(\emptyset) = 0$, $\chi(\emptyset) = 0$ and $\chi(K) = 0$, we obtain the following inequality and equality

$$b(K) \leq \sum_{i=1}^{n} c_i \qquad \text{and} \qquad \sum_{i=1}^{n} c_i = \sum_{i=1}^{n} a_i.$$

Hence we have

$$e(K) \geq \sum_{i=1}^{n}(a_i + b_i + c_i) \geq \sum_{i=1}^{n}(a_i + c_i) = 2 \cdot \sum_{i=1}^{n} c_i \geq 2 \cdot b(K). \qquad \square$$

For vertex decomposability we obtain an even better bound.

THEOREM 5.2. *Let C be a three-dimensional simplicial ball or sphere which is vertex decomposable. Let K be a knot contained in the* 1*-skeleton of the simplicial complex C. Then we have the inequality*

$$3 \cdot b(K) \leq e(K).$$

PROOF. If C is vertex decomposable, Definition 2.3 shows that there is a sequence of vertices $x_n, x_{n-1}, \ldots, x_1$ of C such that x_i is a shedding vertex of $(\cdots((C - x_n) - x_{n-1}) \cdots - x_{i+1})$. Let $C_1^{(n+1)} = C$, $C_1^{(i)} = C_1^{(i+1)} - x_i$, and $C_2^{(i)} = x_i * \mathrm{link}_{C_1^{(i+1)}}(x_i)$. Let $T_1^{(n+1)} = K$, $T_1^{(i)} = T_1^{(i+1)} \cap C_1^{(i)}$, and $T_2^{(i)} = \overline{T_1^{(i+1)} - T_1^{(i)}}$. ($T_1^{(1)} = \emptyset$.) Observe that $C_1^{(i+1)} = C_1^{(i)} \cup C_2^{(i)}$ and $T_1^{(i+1)} = T_1^{(i)} \cup T_2^{(i)}$ are the decompositions described in Proposition 3.4.

The proof follows the same lines as the proof of Theorem 5.1. Similarly we obtain $b(K) \leq \sum_{i=1}^{n} c_i$ and $\sum_{i=1}^{n} c_i = \sum_{i=1}^{n} a_i$. The major difference in this proof is that a spanning arc in $T_2^{(i)}$ counted by a_i consists of at least two edges. Hence we have the inequality

$$e(T_1^{(i+1)}) \geq e(T_1^{(i)}) + 2 \cdot a_i + b_i + c_i.$$

Thus we have

$$e(K) \geq \sum_{i=1}^{n} (2 \cdot a_i + b_i + c_i) \geq \sum_{i=1}^{n} (2 \cdot a_i + c_i) = 3 \cdot \sum_{i=1}^{n} c_i \geq 3 \cdot b(K).$$

It is important to note that this proof only depends on Lemma 3.5 and not on the more general Proposition 3.6. □

6. COMPATIBLE AND WEAKLY COMPATIBLE TANGLES

Theorems 5.1 and 4.2 can be viewed, respectively, as a dual result to Armentrout's Theorems 1 and 3 in [1]. In this section we generalize his result to hold for polytopal 3-balls and 3-spheres. Again our conclusions from the inequalities are non-constructibility and non-shellability.

Let C be a three-dimensional polytopal ball or sphere. A tangle T is *compatible* with the complex C if T and the 2-skeleton of C are in relative general position and for all facets F of C the intersection $F \cap T$ is empty or a straight spanning arc in the facet F. Similarly, T is *weakly compatible* with the complex C if $F \cap C$ is a set of simultaneously straight spanning arcs in the facet F. The tangle T is naturally partitioned by the complex C. Let $p(T)$ denote the number of arcs in this partition. For such weakly compatible knots contained in C, we show the following analogue of Theorem 4.1.

THEOREM 6.1. *If C is a constructible three-dimensional polytopal ball or sphere and C contains a tangle T which is weakly compatible with C then*

$$b(T) \leq p(T).$$

PROOF. The proof is by induction on the number of facets of C. If C is a three-dimensional polytope then T is a set of simultaneously straight spanning arcs. In this case $b(T)$ and $p(T)$ are both equal to the number of spanning arcs of T. Hence the induction basis is complete.

The induction step is the same as Theorem 4.1. By condition (ii) of Definition 2.1 we have two three-dimensional complexes C_1 and C_2 which are constructible 3-balls and $C = C_1 \cup C_2$. Let $T_1 = T \cap C_1$ and $T_2 = \overline{T - T_1}$. By Proposition 3.4 and the induction hypothesis, we obtain

$$b(T) \leq b(T_1) + b(T_2) \leq p(T_1) + p(T_2) = p(T).$$

This completes the induction. □

Hence we conclude with the following theorem.

THEOREM 6.2. *Let C be a three-dimensional polytopal ball or sphere. If there is a knot K which is weakly compatible with C such that*

$$p(K) \leq b(K) - 1,$$

then C is non-constructible.

Armentrout's theorem [1, Theorem 3] states that if a weakly compatible knot K in a cell partitioning has $p(K) \leq b(K) - 1$ then the partitioning is non-shellable. This theorem was shown to be a consequence of the fact that if a compatible knot K in a cell partitioning satisfies $p(K) \leq 2 \cdot b(K) - 1$ then the partitioning is non-shellable [1, Theorem 1]. This theorem can also be re-proved by a very simple proof similar to that for Theorem 5.1. Observe that Armentrout's results are about simple polytopal spheres, whereas our proofs extend to non-simple polytopal balls or spheres.

The next result is a strengthening of Armentrout's theorem [1, Theorem 1].

THEOREM 6.3. *If C is a shellable three-dimensional polytopal ball or sphere and C contains a knot K which is compatible with C then*

$$2 \cdot b(K) \leq p(K).$$

PROOF. As in the proof of Theorem 5.1, we define $C_1^{(i)}$, $C_2^{(i)}$, $T_1^{(i)}$ and $T_2^{(i)}$. There are now four possible cases of $T_2^{(i)}$ in $C_2^{(i)}$.

(1) $T_2^{(i)}$ in $C_2^{(i)}$ is an arc and $T_1^{(i)} \cap T_2^{(i)}$ consists of two points.
(2) $T_2^{(i)}$ in $C_2^{(i)}$ is an arc and $T_1^{(i)} \cap T_2^{(i)}$ is one point.
(3) $T_2^{(i)}$ in $C_2^{(i)}$ is an arc and $T_1^{(i)} \cap T_2^{(i)}$ is empty.
(4) $T_2^{(i)}$ in $C_2^{(i)}$ is empty.

Let m_j denote the number of cases of type (j). Again by studying how the bridge index (using Lemma 3.5) respectively the Euler characteristic change, we obtain the inequality $b(K) \leq m_3$ and the equality $m_1 = m_3$. Hence we have

$$\begin{aligned} p(K) &= m_1 + m_2 + m_3 \\ &\geq 2 \cdot m_3 \\ &\geq 2 \cdot b(K). \end{aligned}$$ □

It is desirable to improve Theorem 6.3 by replacing the compatible condition with weakly compatible. However one cannot prove this stronger statement by the same technique used in the proof of Theorem 5.1 since Proposition 3.6 does not apply.

7. CONCLUDING REMARKS

In discussions with Ziegler we conjecture the following strengthening of the results in Theorem 4.2.

CONJECTURE 7.1. † *Let C be a three-dimensional polytopal ball or sphere and let K be a knot contained in the* 1*-skeleton of the complex C. If*

$$e(K) \leq 2 \cdot b(K) - 1,$$

then the polytopal complex C is non-constructible.

The bound in Conjecture 7.1 is sharp. Namely, by the same construction as [9, Examples 2], it is straightforward to produce examples of shellable simplicial 3-spheres (and 3-balls) which have a knot K consisting of $2 \cdot b(K)$ edges. From this observation, one can see that Theorem 5.1 achieves the sharp bound and that the conjecture is at least true in the case of shellable complexes.

Consider a 3-sphere containing the trefoil knot on three edges. By Theorem 1.1 this sphere is shown to be non-constructible. But the trefoil knot has bridge index 2. Hence observe that the non-constructibility of this sphere does not follow from Theorem 4.2, but it would follow from Conjecture 7.1.

Analogously, by the same construction as [9, Example 4], we can build examples of vertex decomposable 3-spheres (balls) which have a knot K consisting of $3 \cdot b(K)$ edges. This shows that Theorem 5.2 achieves the sharp bound.

In Proposition 4.4 it is shown that there are triangulated 3-spheres or 3-balls whose n-fold barycentric subdivisions are not constructible for any given n. Such a result for non-shellability was already known as a consequence of Lickorish's theorem [13]. On the other hand, the barycentric subdivision of a constructible complex is always constructible and the same is true for shellability. This leads one to conjecture that for a given 3-sphere or a 3-ball C there is a non-negative integer n_C such that n_C-fold barycentric subdivision is constructible. For dimensions greater than or equal to 5, non-PL-spheres are counterexamples to this problem (because constructible spheres are piecewise linear), but for the cases of dimensions 3 and 4, and that of PL-spheres, the problem is open.

Some non-shellable examples of triangulated 3-balls are constructible. For example, Rudin's 3-ball [17], Grünbaum's 3-ball (unpublished; a description can be found in [5] and [8]) and Ziegler's 3-ball [22] are known to be constructible; see [8, 16]. Is Vince's non-shellable 3-sphere [19] constructible? Is there a large class of objects which are constructible but not shellable?

Finally, our bounds in Theorems 4.2, 5.1, 5.2 and 6.2 are all in terms of the bridge index of the knot. Could there be similar results in terms of other knot invariants? It seems plausible that knot invariants which are additive or subadditive such as the genus and the braid index could play a role in future results.

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†Very recently, Koya Shimokawa and the second author announced that Conjecture 7.1 was settled affirmatively.

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