# Manifold arrangements

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**Abstract**

We determine the **cd**-index of the induced subdivision arising from a manifold arrangement. This generalizes earlier results in several directions: (*i*) One can work with manifolds other than the $n$-sphere and $n$-torus, (*ii*) the induced subdivision is a Whitney stratification, and (*iii*) the submanifolds in the arrangement are no longer required to be codimension one.

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## 1 Introduction

In the paper [12] Ehrenborg, Goresky and Readdy extend the theory of flag enumeration in polytopes and regular CW-complexes to Whitney stratified manifolds. Their key insight is to replace flag enumeration with Euler flag enumeration, that is, a chain of strata is weighted by the Euler characteristic of each link in the chain; see Theorem 5.4. The classical results for the generalized Dehn–Sommerville relations and the **cd**-index [1, 2, 31] carry over to this setting.

The **cd**-index of a polytope, and more generally, an Eulerian poset, is a minimal encoding of the flag vector having coalgebraic structure which reflects the geometry of the polytope [13]. It is known that the coefficients of the **cd**-index of polytopes, spherically-shellable posets and Gorenstein* posets are nonnegative [23, 31]. Hence, inequalities among the **cd**-coefficients imply inequalities among the flag vector entries of the original objects. For Whitney stratified manifolds, the coefficients of the **cd**-index are no longer restricted to being nonnegative [12, Example 6.15]. This broadens the research program of understanding flag vectors of polytopes to that of manifolds. See [10] for the best currently known inequalities for flag vectors of polytopes.

One would like to understand the combinatorics of naturally occurring Whitney stratified spaces. One such example is that of manifold arrangements. These arrangements are motivated by subspace arrangements in Euclidean space. Classically Goresky and MacPherson determined the cohomology of the complement of subspace arrangements using intersection homology [21]. The stable homotopy type of the complement was studied by Ziegler and Živaljević [37]. Arrangements of

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submanifolds and subvarieties have been considered in connection with blowups in algebraic geometry; see for instance MacPherson and Procesi's work on conical stratification [25], and Li's work on arrangements of subvarieties [24]. Face enumeration issues were examined by Zaslavsky [36] and Swartz [34]. See also [16] where Forge and Zaslavsky extend the notion of hyperplane to topological hyperplanes.

The case of toric arrangements, that is, a collection of subtori inside an $n$-dimensional torus $T^n$, was studied by Novik, Postnikov and Sturmfels [28] in reference to minimal cellular resolutions of unimodular matroids. Other results for toric arrangements include that of De Concini and Procesi [8], who computed the cohomology of the complement and related this to polytopal lattice point enumeration, and D'Antonio and Delucchi [7], who considered the homotopy type and the fundamental group of the complement.

Billera, Ehrenborg and Readdy studied oriented matroids and the lattice of regions [4]. The **cd**-index of this lattice only depends upon the flag $f$-vector of the intersection lattice, which is a smaller poset. Their work shows how to determine the **cd**-index of induced subdivisions of the sphere $S^n$. Ehrenborg, Readdy and Slone considered toric arrangements that induce regular subdivisions of the torus $T^n$ [15]. Yet again, the associated **cd**-index depends only upon the flag $f$-vector of the intersection poset.

In this paper we consider arrangements of manifolds and the subdivisions they induce. In the manifold setting the computation of the **cd**-index now depends upon the intersection poset and the Euler characteristic of the elements of this quasi-graded poset. This extends the earlier studied spherical and toric arrangements [4, 15].

In order to obtain this generalization, we first review the notion of a quasi-graded poset. See Section 2. This allows us to work with intersection posets that are not necessarily graded. A short discussion of the properties of the Euler characteristic with compact support and its relation to the Euler characteristic is included. We then introduce manifold arrangements in Section 3. The intersection poset of an arrangement is defined. This notion is not unique. However, this gives us the advantage of choosing the most suitable poset for calculations. In Section 4 the Euler characteristic of the complement is computed for these arrangements. This is a manifold analogue of the classical result concerning the number of regions of a hyperplane arrangement [35]. In Section 5 we review the notions of Eulerian quasi-graded posets, the **cd**-index and Whitney stratified spaces.

In the oriented matroid setting the coalgebraic structure of flag vector enumeration was essential in developing the results. In Section 6 we describe the underlying coalgebraic structure in the more general quasi-graded poset setting and summarize the essential operators from [4]. Using these operators, we define the operator $\mathcal{G}$ that will be key to the main result.

In Section 7 we state and prove the main result; see Theorem 7.3. The previous proof techniques for studying subdivisions induced by oriented matroids and toric arrangements depended upon finding a natural map from the face poset of the given subdivision to the intersection poset and studying the inverse image of a chain under this map. See the proofs of [3, Theorem 4.5], [4, Theorem 3.1] and [15, Theorems 3.12 and 4.10]. In the more general setting of manifold arrangements we can now avoid this step by forming another Whitney stratification having the same **cd**-index; see Proposition 7.6. Namely, we can choose each strata to be a submanifold in the intersection poset without those points included in smaller submanifolds. (It is customary to refer to a single stratum by the plural strata.) In the classical case of hyperplane arrangements this gives a stratification into disconnected strata.

Finally in Section 8 we revisit two important cases studied earlier: spherical and toric arrangements. These arrangements have the property that the Euler characteristic of an element of dimension $k$ in the intersection lattice only depends upon $k$, that is, the Euler characteristic is given by $1 + (-1)^k$, respectively the Kronecker delta $\delta_{k,0}$. In both of these cases Theorem 7.3 reduces to a result which only depends on the intersection poset. The original work for spherical and toric arrangements required the induced subdivision to yield a regular subdivision on the sphere $S^n$, respectively, the torus $T^n$ [4, 15]. This regularity condition is no longer necessary in the arena of Whitney stratified spaces.

An illuminating sample of this theory is to consider a complete flag in $n$-dimensional Euclidean space. Intersecting this flag with the $(n-1)$-dimensional unit sphere $S^{n-1}$ gives a (nested) arrangement of spheres, one of each dimension. The intersection poset is a chain of rank $n$. The induced subdivision of the sphere consists of two cells of each dimension $i$, $0 \leq i \leq n-1$. The face poset is the butterfly poset of rank $n+1$. It is straightforward to see that the classical Billera–Ehrenborg–Readdy formula holds in this case; see Example 8.3.

We end with some open questions in the concluding remarks.

## 2 Preliminaries

A *quasi-graded poset* is a triplet $(P, \rho, \overline{\zeta})$ where

(i) $P$ is a finite poset with a minimal element $\widehat{0}$ and maximal element $\widehat{1}$,

(ii) $\rho$ is a function from $P$ to $\mathbb{N}$ such that $\rho(\widehat{0}) = 0$ and $x < y$ implies $\rho(x) < \rho(y)$, and

(iii) $\overline{\zeta}$ is a function in the incidence algebra of $P$ such that for all $x$ in $P$ we have $\overline{\zeta}(x, x) = 1$.

The notion of quasi-graded poset is due to the authors and Goresky. See [12] for further details and see [32] for standard poset terminology. In this paper we will assume $\overline{\zeta}$ is integer-valued, though in general this is not necessary.

Condition (iii) guarantees that $\overline{\zeta}$ is invertible in the incidence algebra of $P$. Let $\overline{\mu}$ denote the inverse of $\overline{\zeta}$. Observe that when the weighted zeta function $\overline{\zeta}$ is the classical zeta function $\zeta$, that is, $\zeta(x, y) = 1$ for all $x \leq y$, the function $\overline{\mu}$ is the Möbius function $\mu$.

Recall that a *subspace arrangement* is a collection $\{V_i\}_{i=1}^m$ of subspaces in $n$-dimensional Euclidean space. We allow a subspace $V_i$ to be a proper subspace of another subspace $V_j$ of the arrangement. However, if a subspace $V_i$ is the intersection of other subspaces in the arrangement, that is, $V_i = \bigcap_{j \in J} V_j$ for $J \subseteq \{1, 2, \ldots, m\} - \{i\}$, the subspace $V_i$ is redundant for our purposes, and can be removed.

The *intersection lattice* of a subspace arrangement forms a quasi-graded poset $(P, \rho, \overline{\zeta})$ where $P$ is the collection of all intersections of subspaces ordered by reverse inclusion. The minimal element is the ambient space $\mathbb{R}^n$ and the maximal element is the intersection $V_1 \cap \cdots \cap V_m$. The rank function $\rho$ is given by the codimension, that is, $\rho(x) = n - \dim(x)$. Finally, we let the weighted zeta function be given by the classical zeta function $\zeta$, where $\zeta(x, y) = 1$ for $x \leq y$.

One of the topological tools we will need is the *Euler characteristic with compact support* $\chi_c$. For a reference, see the article by Gusein-Zade [22]. We review two essential properties. First, the Euler characteristic with compact support is a *valuation*, that is, it satisfies

$$\chi_c(A) + \chi_c(B) = \chi_c(A \cap B) + \chi_c(A \cup B), \tag{2.1}$$

where the two sets $A$ and $B$ are formed by finite intersections, finite unions and complements of locally closed sets. (A *locally closed set* is the intersection of an open set and a closed set.) The second key property relates the usual Euler characteristic $\chi$ with $\chi_c$.

**Proposition 2.1.** *For an $n$-dimensional manifold $M$ that is not necessarily compact, the Euler characteristic $\chi(M)$ and the Euler characteristic with compact support $\chi_c(M)$ satisfy the relation*

$$\chi_c(M) = (-1)^n \cdot \chi(M). \tag{2.2}$$

*Proof.* By Poincaré duality for non-compact manifolds $M$ we have

$$H_c^i(M; \mathbb{Z}_2) \cong H_{n-i}(M; \mathbb{Z}_2).$$

The result follows by taking dimension of this isomorphism, multiplying by the sign $(-1)^i$ and summing over all $i$. □

Observe that by computing the (co)-homology groups over the field of two elements, we also cover the case when the manifold is non-orientable.

## 3 Manifold arrangements

Given an $n$-dimensional compact manifold $M$ without boundary and a collection of submanifolds $\{N_i\}_{i=1}^m$ of $M$ each without boundary, we call this collection a *manifold arrangement* if it satisfies Bott's [6, Section 5] *clean intersection property*, defined as follows. For every point $p$ in the manifold $M$, there exist (i) a neighborhood $U$ of $p$, (ii) a neighborhood $W$ in $\mathbb{R}^n$ of the origin, (iii) a subspace arrangement $\{V_i\}_{i=1}^k$ in $\mathbb{R}^n$, and (iv) a diffeomorphism $\phi : U \longrightarrow W$ such that the point $p$ is mapped to the origin and the collection of manifolds restricted to the neighborhood $U$, that is, $\{N_i \cap U\}_{i=1}^m$ is mapped to the restriction of the subspace arrangement $\{V_i \cap W\}_{i=1}^k$.

Similar to the setup for subspace arrangements we allow a manifold $N_i$ to be a proper submanifold of another manifold $N_j$ in the arrangement; see Examples 8.3 and 8.7. A manifold $N_i$ is redundant if it is the intersection of other manifolds in the arrangement, that is, $N_i = \bigcap_{j \in J} N_j$ for $J \subseteq \{1, 2, \ldots, m\} - \{i\}$.

**Example 3.1.** Let $M$ be the sphere $x^2 + y^2 + z^2 = 2$, and let $\{N_1, N_2\}$ consist of the two circles $x^2+y^2 = 1$, $z = 1$; and $x = 1$, $y^2+z^2 = 1$. Observe that the line $x = z = 1$ is tangent to both circles at their point of intersection $p = (1, 0, 1)$. Hence at the point $p$ the clean intersection property is not satisfied so that $\{N_1, N_2\}$ is not a manifold arrangement.

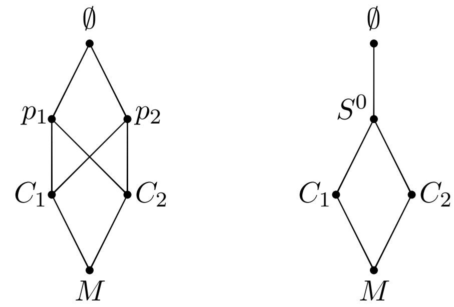


Figure 1: The two possible intersection posets for the arrangement in Example 3.5.

**Proposition 3.2.** *An intersection $\bigcap_{i\in I} N_i$ where $I \subseteq \{1,\ldots,m\}$ in a manifold arrangement $\{N_i\}_{i=1}^m$ of a compact manifold $M$ consists of a disjoint union of a finite number of connected manifolds.*

*Proof.* Assume that the intersection $\bigcap_{i\in I} N_i$ consists of an infinite number of connected components. Pick a point $p_j$ from each connected component. Since $M$ is compact the sequence $\{p_j\}_{j\geq 1}$ has a convergent subsequence. Let $p$ be the limit point of this subsequence. Observe now that the clean intersection property does not hold at $p$, contradicting the assumption of the existence of an infinite number of components.

The clean intersection property implies that the neighborhood of a point in the intersection $\bigcap_{i\in I} N_i$ is a relatively open ball, that is, each connected component is a manifold. □

**Example 3.3.** To illustrate why compactness is a necessary condition, consider the two curves $y = \sin(x)$ and $y = -\sin(x)$ in the plane $\mathbb{R}^2$. They intersect in an infinite number of points.

The connected components in Proposition 3.2 could be manifolds of different dimensions. We illustrate this behavior in Example 3.4.

**Example 3.4.** Let $A$, $B$, $C$ and $D$ be the following four 2-dimensional spheres in $\mathbb{R}^4$:

$$\begin{aligned} A &= \{(x,y,z,w) \,:\, x = 0, (y-1)^2 + z^2 + w^2 = 1\}, \\ B &= \{(x,y,z,w) \,:\, x^2 + y^2 + (z-1)^2 = 1, w = 0\}, \\ C &= \{(x,y,z,w) \,:\, (x-9)^2 + y^2 + z^2 = 2, w = 0\}, \\ D &= \{(x,y,z,w) \,:\, (x-11)^2 + y^2 + z^2 = 2, w = 0\}. \end{aligned}$$

The spheres $A$ and $B$ intersect in two points, the spheres $C$ and $D$ intersect in a circle, and there are no other intersections. Now construct the connected sums $A\#C$ and $B\#D$ by attaching disjoint tubes. We obtain two spheres $A\#C$ and $B\#D$ which intersect in two points and a circle. Finally, take the one-point compactification of $\mathbb{R}^4$ to obtain an arrangement in the four-dimensional sphere $S^4$.

We now introduce the notion of intersection poset. Depending on particular circumstances there could be several suitable intersection posets for a given arrangement.

**Example 3.5.** Let $M$ be the sphere $x^2 + y^2 + z^2 = 1$. Let $\{C_1, C_2\}$ be the arrangement consisting of the two circles $x^2 + y^2 = 1$, $z = 0$; and $x = 0$, $y^2 + z^2 = 1$, which intersect in two points $p_1$

and $p_2$. We can either view these two points as separate elements in an intersection poset or as one zero-dimensional sphere $S^0$. We will see that both views are useful. The two possible intersection posets are displayed in Figure 1.

**Definition 3.6.** *An intersection poset $P$ of a manifold arrangement $\{N_i\}_{i=1}^m$ of a compact manifold $M$ is a poset whose elements are ordered by reverse inclusion that satisfies:*

*(i) The empty set is an element of $P$.*

*(ii) Each non-empty element of $P$ is a disjoint union of connected components, all of the same dimension, of a non-empty intersection $\bigcap_{i \in I} N_i$ where $I \subseteq \{1, \ldots, m\}$.*

*(iii) Given a connected component $C$ of an intersection $\bigcap_{i \in I} N_i$, there exists exactly one element of $P$ having $C$ as one of its connected components.*

*Conditions (ii) and (iii) imply that each intersection can be written uniquely as a disjoint union of non-empty elements of $P$.*

*(iv) Let $I \subseteq J$ be two index sets. Then we have unique subsets $A$ and $B$ of $P - \{\emptyset\}$ such that*

$$\bigsqcup_{x \in A} x = \bigcap_{i \in I} N_i \supseteq \bigcap_{j \in J} N_j = \bigsqcup_{y \in B} y.$$

*If $x \in A$ and $y \in B$ intersect non-trivially then the element $x$ contains the element $y$, that is,*

$$x \cap y \neq \emptyset \Longrightarrow x \supseteq y.$$

The condition that the elements of $P$ consist of manifolds all of the same dimension ensures that the dimension of a non-empty element of $P$ is well-defined. Also note that condition (*iv*) mimics the condition of the frontier for stratified spaces; see equation (5.3).

**Example 3.7.** Let $M$ be a compact manifold of dimension greater than one and let $N_1$ and $N_2$ be two one-dimensional submanifolds of $M$, that is, $N_1$ and $N_2$ are closed curves. Assume that $N_1$ and $N_2$ intersect in $k$ points. Then the number of possible intersection posets of the manifold arrangement $\{N_1, N_2\}$ is given by the $k$th Bell number, that is, the number of set partitions of a $k$-element set.

As an example of an intersection poset, we may take the elements to consist of all connected components of the non-empty intersections. This is the approach taken in the paper [15] when studying toric arrangements. However, this does not work for spherical arrangements since the zero-dimensional sphere consists of two points and hence is disconnected. See Section 8 for further discussion regarding these two special cases.

**Example 3.8.** Let $\{N_i\}_{i=1}^m$ be a manifold arrangement of a manifold $M$ such that each intersection $\bigcap_{i \in I} N_i$ is *pure*, that is, each component of $\bigcap_{i \in I} N_i$ has the same dimension. Then as an intersection poset we may choose

$$L = \left\{ \bigcap_{i \in I} N_i \;:\; I \subseteq \{1, \ldots, m\} \right\} \cup \{\emptyset\}.$$

Observe that this intersection poset is indeed a lattice and hence it is called the *intersection lattice* of the arrangement. However, as Example 3.4 shows there are arrangements which do not have an intersection lattice.

Finally, we define a *quasi-graded intersection poset* $(P, \rho, \overline{\zeta})$ of a manifold arrangement $\{N_i\}_{i=1}^m$ of a manifold $M$ to consist of (i) an intersection poset $P$ of the arrangement, (ii) the rank function $\rho$ given by $\rho(x) = \dim(M) - \dim(x)$ and $\rho(\emptyset) = \dim(M) + 1$, and (iii) the weighted zeta function $\overline{\zeta}$ given by the classical zeta function $\zeta$.

## 4 The complement of a manifold arrangement

We define the *manifold Zaslavsky invariant* of a quasi-graded poset $(P, \rho, \overline{\zeta})$ with respect to a function $f$ defined on $P$ to be

$$Z_M(P, \rho, \overline{\zeta}; f) = \sum_{\widehat{0} \leq x \leq \widehat{1}} (-1)^{\rho(x)} \cdot \overline{\mu}(\widehat{0}, x) \cdot \overline{\zeta}(x, \widehat{1}) \cdot f(x).$$

In our applications the elements of the poset will be geometric objects and the function $f$ will be the Euler characteristic $\chi$. In the case when $f$ is integer-valued the manifold Zaslavsky invariant $Z_M$ is an integer.

**Theorem 4.1.** *Let $\{N_i\}_{i=1}^m$ be a manifold arrangement in a manifold $M$ with quasi-graded intersection poset $(P, \rho, \zeta)$, and where $\chi$ is the Euler characteristic of the elements in the intersection poset. Then the Euler characteristic of the complement is given by*

$$\chi\left(M - \bigcup_{i=1}^{m} N_i\right) = Z_M(P, \rho, \zeta; \chi).$$

*Proof.* For a manifold $x$ in the intersection poset $P$, define $x^\circ$ by

$$x^\circ = x - \bigcup_{y \subsetneq x} y = x - \bigcup_{y > x} y, \tag{4.1}$$

that is, $x^\circ$ consists of all points in $x$ not contained in any submanifold in $P$. Observe that $x^\circ$ is a manifold that is not necessarily compact. Directly for all submanifolds $x$ in the intersection poset we have the following disjoint union:

$$x = \dot{\bigcup_{x \leq y}} y^\circ.$$

Applying the Euler characteristic with compact support, and using the fact $\chi_c$ is additive on disjoint unions, we have

$$\chi_c(x) = \sum_{x \leq y} \chi_c(y^\circ).$$

Möbius inversion yields

$$\chi_c(y^\circ) = \sum_{y \leq x} \mu(y,x) \cdot \chi_c(x).$$

By setting $y$ to be the entire manifold $M$, that is, the minimal element in the intersection poset $P$, using Proposition 2.1 and observing that $(-1)^{\dim(M)} = (-1)^{\rho(x)} \cdot (-1)^{\dim(x)}$, the result follows. □

Theorem 4.1 is an extension of Zaslavsky's classical result on enumerating the number of regions of a hyperplane arrangement [35]. Before stating his result, we define the *Zaslavsky invariant* of a quasi-graded poset $(P, \rho, \overline{\zeta})$ to be

$$Z(P, \rho, \overline{\zeta}) = \sum_{\widehat{0} \leq x \leq \widehat{1}} (-1)^{\rho(x)} \cdot \overline{\mu}(\widehat{0}, x) \cdot \overline{\zeta}(x, \widehat{1}).$$

See [15] for the graded poset case. Zaslavsky's immortal result can now be stated as follows.

**Theorem 4.2.** *Let $\{V_i\}_{i=1}^m$ be a hyperplane arrangement in $\mathbb{R}^n$ with intersection lattice $L$. Then the number of chambers in the complement of the hyperplane arrangement is given by $Z(L, \rho, \zeta)$.*

For our purposes we need to extend Theorem 4.2 to subspace arrangements:

**Theorem 4.3.** *Let $\{V_i\}_{i=1}^m$ be a subspace arrangement in $\mathbb{R}^n$ with quasi-graded intersection poset $(P, \rho, \zeta)$ and let $S^{n-1}$ be an $(n-1)$-dimensional sphere centered at the origin. Then the Euler characteristic of the complement of the arrangement in the sphere $S^{n-1}$ is given by*

$$\chi\left(S^{n-1} - \bigcup_{i=1}^{m} V_i\right) = Z(P, \rho, \zeta).$$

*Proof.* Observe that $\{S^{n-1} \cap V_i\}_{i=1}^m$ is a spherical arrangement on the sphere $S^{n-1}$. Furthermore, the subspace arrangement and the spherical arrangement have the same intersection poset $P$. Hence by Theorem 4.1 the Euler characteristic of the complement is given by

$$\begin{aligned}
\chi\left(S^{n-1} - \bigcup_{i=1}^{m} V_i\right) &= \sum_{x \in P} (-1)^{\rho(x)} \cdot \mu(\widehat{0}, x) \cdot \chi(x) \\
&= \sum_{x \in P} (-1)^{\rho(x)} \cdot \mu(\widehat{0}, x) \cdot \left((-1)^{\dim(x)} + 1\right) \\
&= \sum_{x \in P} \left((-1)^n + (-1)^{\rho(x)}\right) \cdot \mu(\widehat{0}, x) \\
&= \sum_{x \in P} (-1)^{\rho(x)} \cdot \mu(\widehat{0}, x),
\end{aligned}$$

where in the third step we use $\rho(x) + \dim(x) = n$ and in the fourth step we use $\sum_{x \in P} \mu(\widehat{0}, x) = 0$. □

We should be mindful that $Z(P, \rho, \overline{\zeta})$ only depends on the quasi-graded poset structure, whereas $Z_M(P, \rho, \overline{\zeta}; f)$ also depends on the function values $f(x)$ for elements $x$ in $P$.

## 5 The cd-index and Whitney stratifications

In this section we review the theory of the **cd**-index for Eulerian quasi-graded posets and the important subclass of face posets of manifolds which have Whitney stratified boundaries. For more details, see the article [12].

Let **a** and **b** be two non-commutative variables. Given a quasi-graded poset $(P, \rho, \overline{\zeta})$, the *weight* of a chain $c = \{\widehat{0} = x_0 < x_1 < \cdots < x_k = \widehat{1}\}$ is

$$\mathrm{wt}(c) = (\mathbf{a} - \mathbf{b})^{\rho(x_0,x_1)-1} \cdot \mathbf{b} \cdot (\mathbf{a} - \mathbf{b})^{\rho(x_1,x_2)-1} \cdot \mathbf{b} \cdots \mathbf{b} \cdot (\mathbf{a} - \mathbf{b})^{\rho(x_{k-1},x_k)-1}, \tag{5.1}$$

where $\rho(x, y)$ denotes the difference $\rho(y)-\rho(x)$. Note that the weight of a chain is an **ab**-polynomial with integer coefficients homogeneous of degree $\rho(\widehat{0}, \widehat{1}) - 1 = \rho(P) - 1$. Furthermore, the *weighted zeta function* of the chain $c$ is the product

$$\overline{\zeta}(c) = \overline{\zeta}(x_0, x_1) \cdot \overline{\zeta}(x_1, x_2) \cdots \overline{\zeta}(x_{k-1}, x_k).$$

The **ab**-*index* of the quasi-graded poset $(P, \rho, \overline{\zeta})$ is defined to be

$$\Psi(P, \rho, \overline{\zeta}) = \sum_{c} \overline{\zeta}(c) \cdot \mathrm{wt}(c),$$

where the sum ranges over all chains $c$ in the quasi-graded poset $P$. Similarly, the **ab**-index of a quasi-graded poset $(P, \rho, \overline{\zeta})$ is an **ab**-polynomial homogeneous of degree $\rho(P) - 1$ with integer coefficients.

**Remark 5.1.** An alternative definition of the **ab**-index of a quasi-graded poset of rank $n + 1$ is to define the flag $\overline{f}$-vector by $\overline{f}(S) = \sum_c \overline{\zeta}(x_0, x_1) \cdot \overline{\zeta}(x_1, x_2) \cdots \overline{\zeta}(x_{k-1}, x_k)$, where the sum is over all chains $c = \{\widehat{0} = x_0 < x_1 < \cdots < x_k = \widehat{1}\}$ satisfying $S = \{\rho(x_1), \ldots, \rho(x_{k-1})\}$. The flag $\overline{h}$-vector is then given by $\overline{h}(S) = \sum_{T \subseteq S} (-1)^{|S-T|} \cdot \overline{f}(T)$. Finally, the **ab**-index is the sum $\Psi(P, \rho, \overline{\zeta}) = \sum_S \overline{h}(S) \cdot u_S$, where the monomial $u_S = u_1 u_2 \cdots u_n$ is given by $u_i = \mathbf{a}$ if $i \notin S$ and $u_i = \mathbf{b}$ if $i \in S$.

In [12, Section 3] the definition of Eulerian poset is extended to quasi-graded posets. A quasi-graded poset $(P, \rho, \overline{\zeta})$ is *Eulerian* if for all elements $x < z$ in the poset $P$ the following equality holds:

$$\sum_{x \le y \le z} (-1)^{\rho(x,y)} \cdot \overline{\zeta}(x, y) \cdot \overline{\zeta}(y, z) = 0. \tag{5.2}$$

From [12, Theorem 4.2] we have that

**Theorem 5.2** (Ehrenborg–Goresky–Readdy)**.** *The* **ab***-index of an Eulerian quasi-graded poset* $(P, \rho, \overline{\zeta})$ *can be written in terms of* $\mathbf{c} = \mathbf{a} + \mathbf{b}$ *and* $\mathbf{d} = \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a}$.

When the **ab**-index is expressed in terms of **c** and **d**, we call it the **cd**-*index*. Also note that the variable **c** has degree 1 and **d** has degree 2. Like the **ab**-index, the **cd**-index satisfies $\Psi(P, \rho, \overline{\zeta}) \in \mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle$.

Observe that when $P$ is a graded poset and the weighted zeta function $\overline{\zeta}$ is the classical zeta function $\zeta$, condition (5.2) reduces to the classical notion of an Eulerian poset. See [12, Section 3] for a detailed discussion. Furthermore, Theorem 5.2 reduces to the usual notion of the **cd**-index [2, 31].

Define the involution $u \longmapsto u^*$ on **ab**-polynomials by reading each monomial in reverse, that is, $(u_1 u_2 \cdots u_n)^* = u_n \cdots u_2 u_1$ where each $u_i$ is either **a** or **b**. Observe that this involution restricts to **cd**-polynomials as well, since $\mathbf{c}^* = \mathbf{c}$ and $\mathbf{d}^* = \mathbf{d}$. Define the dual of a quasi-graded poset $(P, \rho, \overline{\zeta})$ to be $(P^*, \rho^*, \overline{\zeta}^*)$ where $P^*$ is the dual poset, that is, $x \leq_{P^*} y$ if and only if $y \leq_P x$, $\rho^*(x) = \rho(x, \widehat{1})$ and $\overline{\zeta}^*(x, y) = \overline{\zeta}(y, x)$. Directly it follows

$$\Psi(P^*, \rho^*, \overline{\zeta}^*) = \Psi(P, \rho, \overline{\zeta})^*.$$

We now review the notion of a Whitney stratification.

**Definition 5.3.** *A stratification $\Omega$ of a manifold $M$ is a disjoint union of smaller manifolds, called strata, whose union is $M$. We assume that the strata satisfy the* condition of the frontier, *that is, for two strata $X$ and $Y$ in $\Omega$, we have*

$$X \cap \overline{Y} \neq \emptyset \Longleftrightarrow X \subseteq \overline{Y}. \tag{5.3}$$

*This condition defines a partial order on the set of strata, that is, it defines the face poset $P$ of the stratification by $X \leq_P Y$ if and only if $X \subseteq \overline{Y}$. Furthermore, for the stratification $\Omega$ to be a* Whitney stratification, *each strata has to be a (locally closed, not necessarily connected) smooth submanifold of $M$ and $\Omega$ must satisfy Whitney's conditions $(A)$ and $(B)$:*

> *Let $X <_P Y$ and suppose $y_i \in Y$ is a sequence of points converging to some $x \in X$ and that $x_i \in X$ converges to $x$. Also assume that (with respect to some local coordinate system on the manifold $M$) the secant lines $\ell_i = \overline{x_i y_i}$ converge to some limiting line $\ell$ and the tangent planes $T_{y_i} Y$ converge to some limiting plane $\tau$. Then the following inclusions hold:*

$$\text{(A) } T_x X \subseteq \tau \qquad \text{and} \qquad \text{(B) } \ell \subseteq \tau. \tag{5.4}$$

We refer the reader to [9, 20, 21, 26] for a more detailed discussion. Note that we allow our strata to be disconnected. This will be essential in Section 7.

One important consequence of a Whitney stratification is that the link of a strata in another strata is well-defined. Let $X$ be a $k$-dimensional strata and $p$ a point in $X$. Let $Y$ be another strata such that $X \leq Y$. Let $N_p$ be a normal slice at $p$ to $X$, that is, a submanifold such that $\dim(X) + \dim(N_p) = \dim(M)$ and $X \cap N_p = \{p\}$. Let $B_\epsilon(p)$ be a small ball centered at $x$ of radius $\epsilon > 0$. Then the homeomorphism type of the intersection

$$Y \cap N_p \cap \partial B_\epsilon(p) \tag{5.5}$$

does not depend on the choice of the point $p$ in $X$, the choice of the normal slice $N_p$ or the choice of the radius of the ball $B_\epsilon(p)$ for small enough $\epsilon > 0$. The above intersection (5.5) is defined to be the *link of $X$ in $Y$*, denoted by $\mathrm{link}_Y(X)$. For details, see [12, Section 6]. As an example see Figure 2, where $X$ is a one-dimensional strata and $Y$ is the two-dimensional strata consisting of 4 sheets attached to $X$.

Ehrenborg, Goresky and Readdy provided a geometric source of Eulerian quasi-graded posets, namely, those arising from Whitney stratified manifolds. See [12, Theorem 6.10].

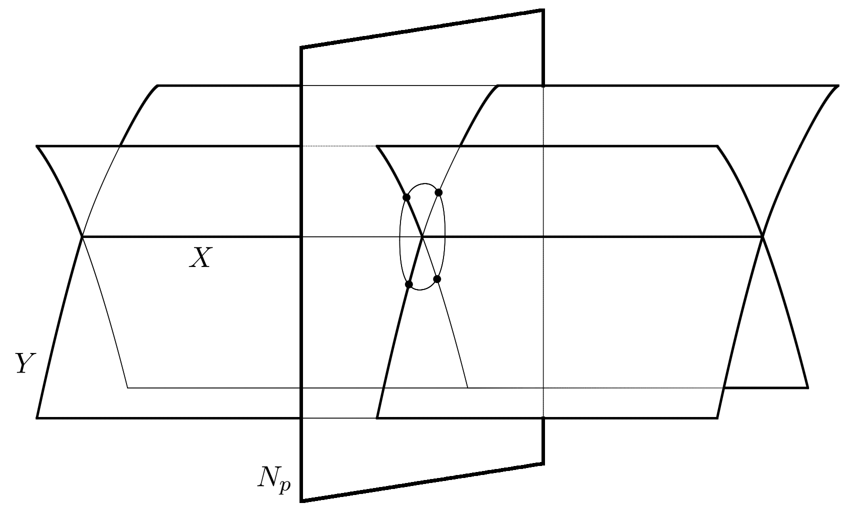


Figure 2: The link of the horizontal line $X$ in the two-dimensional strata $Y$ consists of 4 points.

**Theorem 5.4** (Ehrenborg–Goresky–Readdy)**.** *Let $M$ be a manifold whose boundary has a Whitney stratification. Let $T$ be the face poset of this stratification where the partial order relation is given by* (5.3)*. Let the rank function $\rho$ and the weighted zeta function $\overline{\zeta}$ be*

$$\rho(x) = \begin{cases} \dim(x) + 1 & \text{if } x > \widehat{0}, \\ 0 & \text{if } x = \widehat{0}, \end{cases} \qquad \overline{\zeta}(x, y) = \begin{cases} \chi(\mathrm{link}_y(x)) & \text{if } x > \widehat{0}, \\ \chi(y) & \text{if } x = \widehat{0}. \end{cases}$$

*Then the quasi-graded face poset $(T, \rho, \overline{\zeta})$ is Eulerian.*

We end this section with a result about stratifications.

**Proposition 5.5.** *Let $M$ be a manifold with a Whitney stratification $\Omega$ in its boundary. Assume that there are two strata $X$ and $Y$ of the same dimension satisfying:*

*(i) for all strata $V$ in $\Omega$ the condition $X < V$ is equivalent to $Y < V$, and*

*(ii) for all strata $V$ in $\Omega$ such that $X < V$, the two links $\mathrm{link}_V(X)$ and $\mathrm{link}_V(Y)$ are homeomorphic.*

*Then*

$$\Omega' = \Omega - \{X, Y\} \cup \{X \cup Y\}$$

*is also a Whitney stratification and their* **cd***-indexes are equal:*

$$\Psi(\Omega) = \Psi(\Omega').$$

This result follows from Lemma 5.4 in [12], which shows that we can replace two elements in a quasi-graded poset with their union if their up-sets and their weighted zeta functions agree.

## 6 Coalgebraic techniques and the operator $\mathcal{G}$

Following [13], on the algebra of non-commutative polynomials in the variables $\mathbf{a}$ and $\mathbf{b}$ we define the coproduct $\Delta : \mathbb{Z}\langle \mathbf{a}, \mathbf{b}\rangle \longrightarrow \mathbb{Z}\langle \mathbf{a}, \mathbf{b}\rangle \otimes \mathbb{Z}\langle \mathbf{a}, \mathbf{b}\rangle$ by

$$\Delta(u_1 u_2 \cdots u_k) = \sum_{i=1}^{k} u_1 \cdots u_{i-1} \otimes u_{i+1} \cdots u_k,$$

where $u_1 u_2 \cdots u_k$ is an $\mathbf{ab}$-monomial of length $k$ and extend $\Delta$ by linearity. Note that $\Delta(1) = 0$. Observe that this coproduct satisfies the *Newtonian condition*:

$$\Delta(u \cdot v) = \sum_{u} u_{(1)} \otimes u_{(2)} \cdot v + \sum_{v} u \cdot v_{(1)} \otimes v_{(2)}. \tag{6.1}$$

Next we have the following result; see [12, Theorem 2.5].

**Theorem 6.1** (Ehrenborg–Goresky–Readdy)**.** *For a quasi-graded poset $(P, \rho, \overline{\zeta})$,*

$$\Delta(\Psi(P, \rho, \overline{\zeta})) = \sum_{\widehat{0} < x < \widehat{1}} \Psi([\widehat{0}, x], \rho, \overline{\zeta}) \otimes \Psi([x, \widehat{1}], \rho_x, \overline{\zeta}),$$

*where the rank function $\rho_x$ is given by $\rho_x(y) = \rho(y) - \rho(x)$.*

This result can be stated as the $\mathbf{ab}$-index is a coalgebra map. Namely, let $C$ be the $\mathbb{Z}$-module generated by all isomorphism types of quasi-graded posets and extend the $\mathbf{ab}$-index to be a linear map $\Psi : C \longrightarrow \mathbb{Z}\langle \mathbf{a}, \mathbf{b}\rangle$. Let $C$ be a coalgebra by defining the coproduct by $\Delta(P, \rho, \overline{\zeta}) = \sum_{\widehat{0} < x < \widehat{1}} ([\widehat{0}, x], \rho, \overline{\zeta}) \otimes ([x, \widehat{1}], \rho_x, \overline{\zeta})$. Theorem 6.1 now states $\Delta \circ \Psi = (\Psi \otimes \Psi) \circ \Delta$, that is, $\Psi$ is a coalgebra homomorphism.

We now introduce a number of operators on $\mathbb{Z}\langle \mathbf{a}, \mathbf{b}\rangle$ that will be essential to describe the $\mathbf{cd}$-index of manifold arrangements. The operators $\kappa$, $\eta$, $\varphi$ and $\omega$ were first introduced in [4] when studying flag vectors of oriented matroids.

Define the two algebra maps $\kappa$ and $\overline{\lambda}$ such that $\kappa(1) = \overline{\lambda}(1) = 1$ and

$$\kappa(\mathbf{a}) = \mathbf{a} - \mathbf{b}, \quad \kappa(\mathbf{b}) = 0, \quad \overline{\lambda}(\mathbf{a}) = 0, \quad \text{and} \quad \overline{\lambda}(\mathbf{b}) = \mathbf{a} - \mathbf{b}.$$

We use the notation $\overline{\lambda}$ to be consistent with the notation $\lambda$ in [14].

Define $\eta : \mathbb{Z}\langle \mathbf{a}, \mathbf{b}\rangle \longrightarrow \mathbb{Z}\langle \mathbf{a}, \mathbf{b}\rangle$ by

$$\eta(w) = \begin{cases} 2 \cdot (\mathbf{a} - \mathbf{b})^{m+k} & \text{if } w = \mathbf{b}^m \cdot \mathbf{a}^k, \\ 0 & \text{otherwise.} \end{cases}$$

**Lemma 6.2.** *Let $(P, \rho, \zeta)$ be a quasi-graded poset where the weighted zeta function is the classical zeta function $\zeta$. Then the following identities hold:*

$$\kappa(\Psi(P, \rho, \zeta)) = (\mathbf{a} - \mathbf{b})^{\rho(P)-1}, \tag{6.2}$$

$$\overline{\lambda}(\Psi(P, \rho, \zeta)) = (-1)^{\rho(P)} \cdot \mu(P) \cdot (\mathbf{a} - \mathbf{b})^{\rho(P)-1}, \tag{6.3}$$

$$\eta(\Psi(P, \rho, \zeta)) = Z(P, \rho, \zeta) \cdot (\mathbf{a} - \mathbf{b})^{\rho(P)-1}. \tag{6.4}$$

*Proof.* Equation (6.2) is a direct observation. See also [4, Equation (5)]. Equation (6.3) is a reformulation of Hall's formula for the Möbius function. Finally (6.4) follows from [4, Lemma 5.2 and Equation (6)]. Although this reference only proves this relation for classical graded posets, the proof for quasi-graded posets carries through using the same techniques. ☐

Define the operator $\varphi$ as the sum $\sum_{k \geq 1} \varphi_k$, where $\varphi_k$ is defined by the $k$-ary coproduct

$$\varphi_k(w) = \sum_w \kappa(w_{(1)}) \cdot \mathbf{b} \cdot \eta(w_{(2)}) \cdot \mathbf{b} \cdots \mathbf{b} \cdot \eta(w_{(k)}).$$

From [4, Lemmas 5.6 and 5.7] we have the following lemma for evaluating $\varphi$.

**Lemma 6.3.** *Let $v$ be an* **ab***-polynomial without a constant term. Then $\varphi(v \cdot \mathbf{ab}) = \varphi(v) \cdot 2\mathbf{d}$. Furthermore, let $x$ be either* **a** *or* **b** *and assume that the monomial $v \cdot x$ does not end with* **ab**. *Then $\varphi(v \cdot x) = \varphi(v) \cdot \mathbf{c}$.*

Define the linear map $\omega : \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle \longrightarrow \mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle$ by first defining $\omega$ on a monomial by replacing each occurrence of **ab** by $2\mathbf{d}$ and then replacing the remaining letters by **c**. Extend by linearity to all **ab**-polynomials in $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$. As an example, $\omega(\mathbf{abba}) = 2\mathbf{dc}^2$. The $\omega$ map is equivalent to Stembridge's peak algebra map $\theta$ [33].

From [4, Proposition 5.5] we have:

**Proposition 6.4.** *The two linear operators $\varphi$ and $\omega$ agree on* **ab***-monomials that begin with the letter* **a***, that is, $\varphi(\mathbf{a} \cdot v) = \omega(\mathbf{a} \cdot v)$.*

Now define the operator $\mathcal{G} : \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle \longrightarrow \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ by the relation

$$\mathcal{G}(w) = \varphi(w) \cdot \mathbf{b} + \sum_w \varphi(w_{(1)}) \cdot \mathbf{b} \cdot \overline{\lambda}(w_{(2)}) \cdot (\mathbf{a} - \mathbf{b}).$$

Applying this operator to the **ab**-index of a quasi-graded poset $P$ whose weighted zeta function is the zeta function, we have

$$\mathcal{G}\big(\Psi(P)\big) = \varphi\big(\Psi(P)\big) \cdot \mathbf{b} + \sum_{\widehat{0} < y < \widehat{1}} \varphi\big(\Psi([\widehat{0}, y])\big) \cdot \mathbf{b} \cdot \overline{\lambda}\big(\Psi([y, \widehat{1}])\big) \cdot (\mathbf{a} - \mathbf{b}). \tag{6.5}$$

**Lemma 6.5.** *For an* **ab***-polynomial $w$ we have*

$$\mathcal{G}(w \cdot \mathbf{a}) = \frac{1}{2} \cdot \varphi(w \cdot \mathbf{ab}).$$

*Proof.* By the Newtonian condition (6.1) we have that $\Delta(w \cdot \mathbf{a}) = w \otimes 1 + \sum_w w_{(1)} \otimes w_{(2)} \cdot \mathbf{a}$. Hence we have

$$\begin{aligned} \mathcal{G}(w \cdot \mathbf{a}) &= \varphi(w \cdot \mathbf{a}) \cdot \mathbf{b} + \varphi(w) \cdot \mathbf{b} \cdot \overline{\lambda}(1) \cdot (\mathbf{a} - \mathbf{b}) + \sum_w \varphi(w_{(1)}) \cdot \mathbf{b} \cdot \overline{\lambda}(w_{(2)} \cdot \mathbf{a}) \cdot (\mathbf{a} - \mathbf{b}) \\ &= \varphi(w) \cdot \mathbf{c} \cdot \mathbf{b} + \varphi(w) \cdot \mathbf{b} \cdot (\mathbf{a} - \mathbf{b}) \\ &= \varphi(w) \cdot \mathbf{d} \\ &= 1/2 \cdot \varphi(w \cdot \mathbf{ab}), \end{aligned}$$

where in the second step we used Lemma 6.3, $\overline{\lambda}(1) = 1$ and $\overline{\lambda}(w_{(2)} \cdot \mathbf{a}) = \overline{\lambda}(w_{(2)}) \cdot \overline{\lambda}(\mathbf{a}) = 0$, and in the fourth step Lemma 6.3 was applied again. □

**Proposition 6.6.** *For any* **ab***-polynomial $v$ the operator $\mathcal{G}$ satisfies*

$$\mathcal{G}(1) = \mathbf{b}, \tag{6.6}$$

$$\mathcal{G}(\mathbf{a} \cdot v) = \frac{1}{2} \cdot \omega(\mathbf{a} \cdot v \cdot \mathbf{b}). \tag{6.7}$$

*Proof.* It is a straightforward verification that $\mathcal{G}(1) = \mathbf{b}$. Next we prove statement (6.7) by induction. The induction basis is $v = 1$, which follows from $\mathcal{G}(\mathbf{a}) = \varphi(\mathbf{a}) \cdot \mathbf{b} + \varphi(1) \cdot \mathbf{b} \cdot \overline{\lambda}(1) \cdot (\mathbf{a} - \mathbf{b}) = \mathbf{c} \cdot \mathbf{b} + \mathbf{b} \cdot (\mathbf{a} - \mathbf{b}) = \mathbf{d} = 1/2 \cdot \varphi(\mathbf{ab})$. Assume now that the statement holds for $v$ and let $w = \mathbf{a} \cdot v$. By Lemma 6.5 we know it is true for $v \cdot \mathbf{a}$. The last case to consider is $v \cdot \mathbf{b}$ and again use $w = \mathbf{a} \cdot v$. The Newtonian condition (6.1) implies $\Delta(w \cdot \mathbf{b}) = w \otimes 1 + \sum_w w_{(1)} \otimes w_{(2)} \cdot \mathbf{b}$. Now

$$\begin{aligned}
\mathcal{G}(w \cdot \mathbf{b}) &= \varphi(w \cdot \mathbf{b}) \cdot \mathbf{b} + \varphi(w) \cdot \mathbf{b} \cdot \overline{\lambda}(1) \cdot (\mathbf{a} - \mathbf{b}) + \sum_w \varphi(w_{(1)}) \cdot \mathbf{b} \cdot \overline{\lambda}(w_{(2)} \cdot \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) \\
&= \varphi(w \cdot \mathbf{b}) \cdot \mathbf{b} + \Big(\varphi(w) \cdot \mathbf{b} + \sum_w \varphi(w_{(1)}) \cdot \mathbf{b} \cdot \overline{\lambda}(w_{(2)}) \cdot (\mathbf{a} - \mathbf{b})\Big) \cdot (\mathbf{a} - \mathbf{b}) \\
&= \varphi(w \cdot \mathbf{b}) \cdot \mathbf{b} + \mathcal{G}(w) \cdot (\mathbf{a} - \mathbf{b}) \\
&= \varphi(w \cdot \mathbf{b}) \cdot \mathbf{b} + 1/2 \cdot \varphi(w \cdot \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) \\
&= 1/2 \cdot \varphi(w \cdot \mathbf{b}) \cdot \mathbf{c} \\
&= 1/2 \cdot \varphi(w \cdot \mathbf{bb}),
\end{aligned}$$

where the fourth step is the induction hypothesis and the sixth step is Lemma 6.3, completing the induction. □

# 7 The induced stratification

Let $M$ be an $n$-dimensional manifold and let $\{N_i\}_{i=1}^m$ be a manifold arrangement in the boundary of $M$. The arrangement induces a Whitney stratification of the boundary of $M$ as follows. Recall that for an intersection $x = \bigcap_{i \in I} N_i$ we let $x^\circ$ be all points in $x$ not contained in any submanifold of $x$; see (4.1). Now the induced subdivision $T$ is the collection of all connected components of $\left(\bigcap_{i \in I} N_i\right)^\circ$, where $I$ ranges over all index sets, together with the empty strata $\emptyset$, and the manifold $M$ as the maximal strata. Observe that the empty index set yields the connected components of $(\partial M)^\circ$.

**Proposition 7.1.** *The stratification $T$ is a Whitney stratification.*

*Proof.* Pick two strata $X$ and $Y$ from $T$ where $X <_T Y$ and a point $x$ in $X$. Since the clean intersection property holds at the point $x$ we can choose a local coordinate system around $x$ such that the two strata $X$ and $Y$ are locally straight in a neighborhood $U$ around $x$, that is, for any point $p$ close enough to $x$ the tangent planes $T_pX$, respectively $T_pY$, are independent of the point $p$. Let $y_i \in Y$ be any sequence of points converging to the point $x$. Without loss of generality, we may

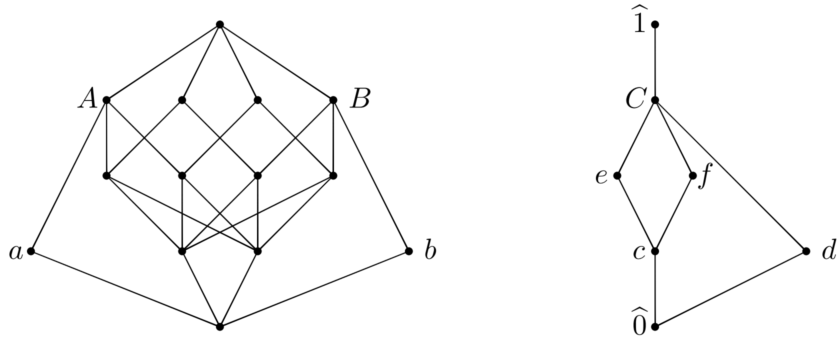


Figure 3: The face poset $T$ of the induced subdivision and the quasi-graded poset $Q$ of Example 7.2.

assume that $y_i$ lies in the neighborhood $U$. Since the tangent planes $T_{y_i}Y$ are all the same, they are in fact equal to the limiting plane $\tau$. Hence Whitney's condition $(A)$ holds: $T_x X \subseteq T_x Y = \tau$. Similarly, let $x_i \in X$ be any sequence of points converging to the point $x$. Again, we may assume that $x_i \in U$. Now all the lines $\ell_i = \overline{x_i y_i}$ lie in the plane $\tau$ so that the limiting line $\ell$ also lies in $\tau$, and thus Whitney's condition $(B)$ holds. $\square$

**Example 7.2.** Consider the subspace arrangement in $\mathbb{R}^3$ consisting of the two planes $x = 0$ and $y = 0$, and the line $x = y = z$. Intersect this arrangement with $S^2$ to obtain a spherical arrangement consisting of two 1-dimensional spheres and one 0-dimensional sphere. The face poset of the induced stratification of the sphere consists of four points $\pm(0, 0, 1)$ and $\pm(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$; four open edges (1-dimensional strata) each emanating from $(0, 0, 1)$ to $(0, 0, -1)$; and finally, four 2-dimensional strata where two of the strata are discs (the $x$ and $y$ coordinates have different signs) and the other two strata are punctured discs (the $x$ and $y$ coordinates have the same sign). See Figure 3 for the face poset $T$.

**Theorem 7.3.** *Let $M$ be an $n$-dimensional manifold. Let $\{N_i\}_{i=1}^m$ be a manifold arrangement in the boundary $\partial M$ with an intersection poset $P$. Let $T$ be the induced Whitney stratification of $M$. Then the reverse of the* **cd***-index of $T$ is given by*

$$\Psi(T)^* = \chi(M) \cdot \begin{cases} (\mathbf{c}^2 - 2\mathbf{d})^{n/2} & \text{if } n \text{ is even,} \\ \mathbf{c} \cdot (\mathbf{c}^2 - 2\mathbf{d})^{(n-1)/2} & \text{if } n \text{ is odd,} \end{cases}$$
$$+ \sum_{\substack{x \in P, x > \widehat{0} \\ \dim(x) \text{ even}}} \frac{1}{2} \cdot \omega\big(\mathbf{a} \cdot \Psi([\widehat{0}, x]) \cdot \mathbf{b}\big) \cdot (\mathbf{c}^2 - 2\mathbf{d})^{\dim(x)/2} \cdot \chi(x).$$

Before proving Theorem 7.3, we first introduce a quasi-graded poset $Q$ that is different than the quasi-graded poset $T$. This new quasi-graded poset will be smaller than $T$. However, it will have the same **cd**-index as $T$. Let $\widehat{P} = P \cup \{\widehat{-1}\}$ be the intersection poset with a new minimal element $\widehat{-1}$ adjoined.

Define $Q$ to be the poset

$$Q = \{x^\circ \ : \ x \in P\} \cup \{M\}$$

where the partial order is $x^\circ \leq y^\circ$ if $x \subseteq y$. Observe that this condition is equivalent to $x^\circ \subseteq \overline{y^\circ}$ since $\overline{y^\circ} = y$. This is the partial order of the face poset of the induced stratification where the strata are of the form $x^\circ$. Furthermore, note that $M$ is the maximal element and it covers the element $(\partial M)^\circ$, which is the unique element of rank $n$.

Define the rank function $\rho_Q$ by

$$\rho_Q(x^\circ) = \begin{cases} 0 & \text{if } x^\circ = \widehat{0}, \\ \dim(x^\circ) + 1 & \text{otherwise,} \end{cases}$$

and the weighted zeta function $\overline{\zeta}_Q$ by

$$\overline{\zeta}_Q(x^\circ, y^\circ) = \begin{cases} \chi(y^\circ) & \text{if } \widehat{0} = x^\circ, \\ \chi(\operatorname{link}_{y^\circ}(x^\circ)) & \text{if } \widehat{0} < x^\circ. \end{cases}$$

As a poset $Q$ is the dual poset of $\widehat{P}$ via the map $\psi(x^\circ) = x$ and $\psi(M) = \widehat{-1}$.

There is a natural order-preserving map $z$ from the quasi-graded face poset $T$ to the poset $Q$. Namely, for an element $x$ in $T$ let $z(x)$ be the rank-wise smallest element in the poset $Q$ containing $x$. Observe the map $z$ also preserves the rank function, that is, $\rho_T(x) = \rho_Q(z(x))$. As a side comment, the reason why this map is called $z$ stems from oriented matroid theory, as it selects the coordinates that are equal to zero in the covectors of the oriented matroid [5, Section 4.6].

The same argument as Proposition 7.1 yields:

**Proposition 7.4.** *The stratification $Q$ is a Whitney stratification.*

Since the links of a Whitney stratification are well-defined, we have the next corollary.

**Corollary 7.5.** *Let $x$ and $y$ be two manifolds in the intersection poset such that $x \subseteq y$. Furthermore, let $p$ and $q$ be two points in $x^\circ$. Let $N_p$ and $N_q$ be the normal slices to $x$ at $p$, respectively at $q$. Then the two spaces*

$$y^\circ \cap N_p \cap \partial B_\epsilon(p) \quad \textit{and} \quad y^\circ \cap N_q \cap \partial B_\epsilon(q) \tag{7.1}$$

*are homeomorphic for small enough $\epsilon > 0$.*

*Proof.* The two spaces in (7.1) are both homeomorphic to $\operatorname{link}_y(x)$. □

**Proposition 7.6.** *The* **cd***-indexes of the two quasi-graded posets $T$ and $Q$ are equal, that is, $\Psi(T, \rho_T, \overline{\zeta}_T) = \Psi(Q, \rho_Q, \overline{\zeta}_Q)$.*

*Proof.* Choose a linear extension of the poset $Q$, that is, $Q = \{x_1, \ldots, x_k\}$ such that $x_i \leq_Q x_j$ implies $i \leq j$. Note that $x_1 = \widehat{0}$ and $x_k = \widehat{1} = M$. Starting with $x_{k-1}$ select two elements $u$ and $v$ such that $\psi(z(u)) = \psi(z(v)) = x_{k-1}$. By Corollary 7.5, the conditions of Proposition 5.5 are satisfied and hence we can replace $u$ and $v$ by their union $u \cup v$ without changing the **cd**-index. Repeat this operation until there are no more such elements. Continue with elements mapping to $x_{k-2}$ and work towards the strata $x_2$. Proposition 5.5 guarantees that at every step the **cd**-index does not change. The end result is the stratification $Q$, proving the proposition. □

**Continuation of Example 7.2.** The weighted zeta function of the quasi-graded poset $T$ takes the value 1 everywhere, except for the four values $\overline{\zeta}_T(\widehat{0}, A) = \overline{\zeta}_T(\widehat{0}, B) = \overline{\zeta}_T(a, A) = \overline{\zeta}_T(b, B) = 0$.

Similarly, the non 1-values of the weighted zeta function for the quasi-graded poset $Q$ are given by $\overline{\zeta}_Q(\widehat{0}, c) = \overline{\zeta}_Q(\widehat{0}, d) = \overline{\zeta}_Q(\widehat{0}, e) = \overline{\zeta}_Q(\widehat{0}, f) = \overline{\zeta}_Q(\widehat{0}, C) = \overline{\zeta}_Q(c, e) = \overline{\zeta}_Q(c, f) = 2$, $\overline{\zeta}_Q(c, C) = 4$, $\overline{\zeta}_Q(d, C) = 0$ and $\overline{\zeta}_Q(e, C) = \overline{\zeta}_Q(f, C) = 2$. Observe that each of the five strata $c$, $d$, $e$, $f$ and $C$ are disconnected.

In both cases we obtain $\Psi(T) = \Psi(Q) = \mathbf{c}^3 + 2 \cdot \mathbf{dc}$.

We now explicitly describe $\overline{\zeta}_Q$ in terms of the invariants $Z$ and $Z_M$.

**Proposition 7.7.** *The weighted zeta function $\overline{\zeta}_Q$ is given by*

$$\begin{aligned} \overline{\zeta}_Q(y^\circ, \widehat{1}) &= 1 && \text{for } \widehat{-1} <_{\widehat{P}} y <_{\widehat{P}} \widehat{1}, \\ \overline{\zeta}_Q(y^\circ, x^\circ) &= Z([x, y], \rho_{\widehat{P}}, \zeta) && \text{for } \widehat{-1} <_{\widehat{P}} x \leq_{\widehat{P}} y <_{\widehat{P}} \widehat{1}, \\ \overline{\zeta}_Q(\widehat{0}, x^\circ) &= Z_M([x, \widehat{1}], \rho_{\widehat{P}}, \zeta; \chi) && \text{for } \widehat{-1} <_{\widehat{P}} x <_{\widehat{P}} \widehat{1}, \\ \overline{\zeta}_Q(\widehat{0}, \widehat{1}) &= \chi(M). \end{aligned}$$

*Proof.* The first and fourth equations are direct. The second equation follows from

$$\overline{\zeta}_Q(y^\circ, x^\circ) = \chi(\mathrm{link}_{x^\circ}(y^\circ)) = Z([x, y], \rho_{\widehat{P}}, \zeta),$$

where the second equality is by Theorem 4.3. Similarly, we have

$$\overline{\zeta}_Q(\widehat{0}, x^\circ) = \chi(x^\circ) = Z_M([x, \widehat{1}], \rho_{\widehat{P}}, \zeta; \chi).$$

□

We are now positioned to prove our main result.

*Proof of Theorem 7.3.* Throughout this proof we suppress the dependency on the rank function $\rho$ and the zeta function $\zeta$ in our notation. By summing over all chains $c$ in the poset $\widehat{P}$ we can compute the **ab**-index of $Q$ and hence of $T$. However, we prefer to compute the reverse **ab**-indexes, that is,

$$\begin{aligned} \Psi(T)^* = \Psi(Q)^* &= \chi(M) \cdot (\mathbf{a} - \mathbf{b})^{n+1} \\ &\quad + \sum_{c} Z([x_1, x_2]) \cdots Z([x_{k-2}, x_{k-1}]) \cdot Z_M([x_{k-1}, x_k]; \chi) \cdot \mathrm{wt}(c), \end{aligned} \tag{7.2}$$

where the sum is over all chains $c = \{\widehat{-1} = x_0 < x_1 < \cdots < x_k = \widehat{1}\}$ of length $k \geq 2$ in $\widehat{P}$.

Recall that the operator $\eta$ satisfies the relation (6.4). Similarly, define $\eta_M$ such that

$$\eta_M\big(\Psi(P)\big) = Z_M(P; \chi) \cdot (\mathbf{a} - \mathbf{b})^{\rho(P)-1},$$

where we suppress the dependence of $\eta_M(\Psi(P))$ on the Euler characteristic of the elements in $P$. By expanding equation (5.1) we can rewrite equation (7.2) as

$$\begin{aligned}\Psi(T)^* &= \chi(M)\cdot(\mathbf{a}-\mathbf{b})^{n+1} \\ &\quad + \sum_{c} \kappa\big(\Psi([x_0,x_1])\big)\cdot\mathbf{b}\cdot\eta\big(\Psi([x_1,x_2])\big)\cdot\mathbf{b}\cdots\mathbf{b}\cdot\eta\big(\Psi([x_{k-2},x_{k-1}])\big)\cdot\mathbf{b}\cdot\eta_M\big(\Psi([x_{k-1},x_k])\big)\end{aligned} \tag{7.3}$$

We split the sum by first summing over the element $y = x_{k-1}$ and then over all chains $c'$ in the interval $[\widehat{-1}, y]$. By the definition of the operator $\varphi$ and the fact that the **ab**-index is a coalgebra homeomorphism, we then have

$$\Psi(T)^* = \chi(M)\cdot(\mathbf{a}-\mathbf{b})^{n+1} + \sum_{\widehat{-1}<y<\widehat{1}} \varphi\big(\Psi([\widehat{-1},y])\big)\cdot\mathbf{b}\cdot\eta_M\big(\Psi([y,\widehat{1}])\big). \tag{7.4}$$

Now apply $\eta_M$ to the interval $[y, \widehat{1}]$. We have

$$\begin{aligned}\eta_M\big(\Psi([y,\widehat{1}])\big) &= Z_M([y,\widehat{1}];\chi)\cdot(\mathbf{a}-\mathbf{b})^{\rho(y,\widehat{1})-1} \\ &= \sum_{y\leq x\leq\widehat{1}} (-1)^{\rho(y,x)}\cdot\mu(y,x)\cdot(\mathbf{a}-\mathbf{b})^{\rho(y,x)-1}\cdot\chi(x)\cdot(\mathbf{a}-\mathbf{b})^{\rho(x,\widehat{1})} \\ &= \chi(y)\cdot(\mathbf{a}-\mathbf{b})^{\rho(y,\widehat{1})-1} + \sum_{y<x<\widehat{1}} \overline{\lambda}\big(\Psi([y,x])\big)\cdot\chi(x)\cdot(\mathbf{a}-\mathbf{b})^{\rho(x,\widehat{1})},\end{aligned} \tag{7.5}$$

where the term $x = \widehat{1}$ vanished since the maximal element $\widehat{1}$ is the empty set with $\chi(\widehat{1}) = 0$. Substituting equation (7.5) into equation (7.4) we have

$$\begin{aligned}\Psi(T)^* &= \chi(M)\cdot(\mathbf{a}-\mathbf{b})^{n+1} + \sum_{\widehat{-1}<y<\widehat{1}} \varphi\big(\Psi([\widehat{-1},y])\big)\cdot\mathbf{b}\cdot\chi(y)\cdot(\mathbf{a}-\mathbf{b})^{\rho(y,\widehat{1})-1} \\ &\quad + \sum_{\widehat{-1}<y<\widehat{1}}\sum_{y<x<\widehat{1}} \varphi\big(\Psi([\widehat{-1},y])\big)\cdot\mathbf{b}\cdot\overline{\lambda}\big(\Psi([y,x])\big)\cdot\chi(x)\cdot(\mathbf{a}-\mathbf{b})^{\rho(x,\widehat{1})} \\ &= \chi(M)\cdot(\mathbf{a}-\mathbf{b})^{n+1} + \sum_{\widehat{-1}<x<\widehat{1}} \varphi\big(\Psi([\widehat{-1},x])\big)\cdot\mathbf{b}\cdot\chi(x)\cdot(\mathbf{a}-\mathbf{b})^{\rho(x,\widehat{1})-1} \\ &\quad + \sum_{\widehat{-1}<x<\widehat{1}}\sum_{\widehat{-1}<y<x} \varphi\big(\Psi([\widehat{-1},y])\big)\cdot\mathbf{b}\cdot\overline{\lambda}\big(\Psi([y,x])\big)\cdot\chi(x)\cdot(\mathbf{a}-\mathbf{b})^{\rho(x,\widehat{1})} \\ &= \chi(M)\cdot(\mathbf{a}-\mathbf{b})^{n+1} \\ &\quad + \sum_{\widehat{-1}<x<\widehat{1}} \Big(\varphi\big(\Psi([\widehat{-1},x])\big)\cdot\mathbf{b} + \sum_{\widehat{-1}<y<x} \varphi\big(\Psi([\widehat{-1},y])\big)\cdot\mathbf{b}\cdot\overline{\lambda}\big(\Psi([y,x])\big)\cdot(\mathbf{a}-\mathbf{b})\Big) \\ &\qquad\qquad \cdot\chi(x)\cdot(\mathbf{a}-\mathbf{b})^{\rho(x,\widehat{1})-1} \\ &= \chi(M)\cdot(\mathbf{a}-\mathbf{b})^{n+1} + \sum_{\widehat{-1}<x<\widehat{1}} \mathcal{G}\big(\Psi([\widehat{-1},x])\big)\cdot\chi(x)\cdot(\mathbf{a}-\mathbf{b})^{\rho(x,\widehat{1})-1},\end{aligned}$$

where in the second step we changed the variable in the second term and switched the order of summation in the third term. The last step was to apply the fact that the **ab**-index is a coalgebra map and the definition of the operator $\mathcal{G}$ to $\Psi([\widehat{-1}, x])$; see equation (6.5).

From the last sum we break out the case when $x = \widehat{0}$, that is, the boundary of $M$. Recall that $\mathcal{G}(1) = \mathbf{b}$. For $x > \widehat{0}$ we use that $\Psi([\widehat{-1}, x]) = \mathbf{a} \cdot \Psi([\widehat{0}, x])$ and we can apply Proposition 6.6.

$$\Psi(T)^* = \chi(M) \cdot (\mathbf{a} - \mathbf{b})^{n+1} + \chi(\partial M) \cdot \mathbf{b} \cdot (\mathbf{a} - \mathbf{b})^n \\ + \sum_{\widehat{0} < x < \widehat{1}} 1/2 \cdot \omega\big(\mathbf{a} \cdot \Psi([\widehat{0}, x]) \cdot \mathbf{b}\big) \cdot \chi(x) \cdot (\mathbf{a} - \mathbf{b})^{\rho(x, \widehat{1}) - 1},$$

The result follows now by observing $\chi(\partial M) = \big(1 - (-1)^{\dim(M)}\big) \cdot \chi(M)$, the Euler characteristic of an odd dimensional manifold $x$ is 0 (since $x$ has no boundary), and that $(\mathbf{a} - \mathbf{b})^2 = \mathbf{c}^2 - 2\mathbf{d}$. □

Similar to the **c**-2**d**-index in [4] we have the next result.

**Corollary 7.8.** *Let $M$ be a manifold and $\{N_i\}_{i=1}^m$ be a manifold arrangement in the boundary of $M$, that is, $\partial M$. Let $T$ be the induced Whitney stratification of $M$. Let $w$ be a* **cd***-monomial containing $k$* **d***'s. Then the coefficient of $w$ in the* **cd***-index $\Psi(T)$ is divisible by $2^{k-1}$.*

# 8 Spherical and toric arrangements

We now turn our attention to consequences of the main result. As mentioned in the introduction, we consider two important cases that have been studied earlier, namely spherical and toric arrangements. In both cases, the results have been expanded.

## 8.1 Spherical arrangements

We now extend the original results for spherical arrangements where each sphere has codimension 1 to arrangements without this restriction. By a $k$-dimensional sphere we mean a manifold homeomorphic to the $k$-dimensional unit sphere $\{(x_1, \ldots, x_{k+1}) \in \mathbb{R}^{k+1} : x_1^2 + \cdots + x_{k+1}^2 = 1\}$. Define a *spherical arrangement* to be a collection of spheres $\{N_i\}_{i=1}^m$ on the boundary of a ball such that the intersection $\bigcap_{i \in I} N_i$ is a disjoint union of spheres for all index sets $I \subseteq \{1, \ldots, m\}$. An example of a spherical arrangement is to intersect an arrangement of affine subspaces in Euclidean space with a sphere of large enough radius. Example 3.4, where two 2-dimensional spheres intersect in two points and a circle, is also a spherical arrangement.

Observe that we require every non-empty element of an intersection poset to be a sphere. The next result not only extends the original result in [4, Theorem 3.1], but also [15, Theorem 4.10].

**Theorem 8.1.** *Let $\{N_i\}_{i=1}^m$ be a spherical arrangement in the boundary of an $n$-dimensional ball, with $(P, \rho, \zeta)$ as a quasi-graded intersection poset of the arrangement. Let $T$ be the induced Whitney stratification. Then the* **cd***-index of $T$ is given by*

$$\Psi(T) = \omega\big(\mathbf{a} \cdot \Psi(P, \rho, \zeta)\big)^*.$$

*Proof.* Observe that the Zaslavsky invariant and the manifold Zaslavsky invariant agree for a spherical arrangements. That is, for an interval $[y, \widehat{1}]$ in $P$ we have

$$\begin{aligned} Z_M([y, \widehat{1}]) &= \sum_{y \leq x \leq \widehat{1}} (-1)^{\rho(y,x)} \cdot \mu(y, x) \cdot \Big(1 + (-1)^{\dim(x)}\Big) \\ &= \sum_{y \leq x \leq \widehat{1}} (-1)^{\rho(y,x)} \cdot \mu(y, x) \\ &= Z([y, \widehat{1}]), \end{aligned}$$

where the equality in the second step follows from the fact the product $(-1)^{\rho(y,x)} \cdot (-1)^{\dim(x)}$ does not depend on the element $x$ and thus the term $\sum_{y \leq x \leq \widehat{1}} \mu(y, x)$ vanishes. Hence the two operators $\eta_M$ and $\eta$ agree, and equation (7.3) reduces to $\Psi(T)^* = \varphi\big(\Psi(\widehat{P})\big) = \omega\big(\mathbf{a} \cdot \Psi(P)\big)$. □

**Continuation of Example 7.2.** The spherical intersection poset $P$ has the flag $\overline{f}$-vector $\overline{f}(\emptyset) = 1$ and $\overline{f}(\{1\}) = \overline{f}(\{2\}) = \overline{f}(\{1, 2\}) = 2$. Hence the flag $\overline{h}$-vector is given by $\overline{h}(\emptyset) = \overline{h}(\{1\}) = \overline{h}(\{2\}) = 1$ and $\overline{h}(\{1, 2\}) = -1$. Thus $\Psi(P) = \mathbf{aa} + \mathbf{ba} + \mathbf{ab} - \mathbf{bb}$ and $\Psi(T) = \omega(\mathbf{a} \cdot (\mathbf{aa} + \mathbf{ba} + \mathbf{ab} - \mathbf{bb}))^* = \mathbf{c}^3 + 2 \cdot \mathbf{dc}$.

**Example 8.2.** Let $k$ be a non-negative integer. Let $c$ and $p$ be positive real numbers such that $c < p$ and $c + p < \pi/2$. On the 2-dimensional unit sphere $S^2$ consider the following two closed curves, which intersect each other in $2 \cdot k$ points:

$$\phi = c + p \cdot \sin(k \cdot \theta) \quad \text{and} \quad \phi = -c - p \cdot \sin(k \cdot \theta).$$

Observe that the arrangement of these two curves on the sphere $S^2$ is spherical. Furthermore, there are $(2k-1)!! = (2k-1) \cdot (2k-3) \cdots 1$ ways to divide the points into $k$ zero-dimensional spheres. However, all intersection posets are isomorphic. The spherical intersection poset has rank 3 and consists of 2 atoms and $k$ coatoms, where each coatom covers each atom. The **ab**-index is given by $\Psi(P) = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + (k-1) \cdot \mathbf{b})$. The **cd**-index of the induced subdivision of the sphere is given by

$$\begin{aligned} \Psi(T) &= \omega(\mathbf{a} \cdot (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + (k-1) \cdot \mathbf{b}))^* \\ &= \omega(\mathbf{aaa})^* + (k-1) \cdot \omega(\mathbf{aab})^* + \omega(\mathbf{ab} \cdot (\mathbf{a} + (k-1) \cdot \mathbf{b}))^* \\ &= \mathbf{c}^3 + 2(k-1) \cdot \mathbf{dc} + 2k \cdot \mathbf{cd}. \end{aligned}$$

This can be observed directly in the case $k \geq 1$ by noting that the induced subdivision consists of $2k$ vertices, $2k$ digons and two $2k$-gons. Note that the calculation also holds for the case $k = 0$.

**Example 8.3.** Given a complete flag of subspaces $V_1 \subseteq V_2 \subseteq \cdots \subseteq V_{n-1}$ in $\mathbb{R}^n$ such that $\dim(V_i) = i$, consider the spherical arrangement in the $(n-1)$-dimensional sphere $S^{n-1}$ given by $\{V_i \cap S^{n-1}\}_{1 \leq i \leq n-1}$. The intersection poset is a chain of length $n$, and its **ab**-index is $\mathbf{a}^{n-1}$. The induced stratification consists of two cells for each dimension and cells of different dimensions are comparable. Hence the face poset is the butterfly poset of rank $n+1$, which has the **cd**-index $\mathbf{c}^n$. This checks with $\omega(\mathbf{a} \cdot \mathbf{a}^{n-1}) = \mathbf{c}^n$.

Lastly, we observe that the proof of Theorem 8.1 only used that the Euler characteristic of each manifold was that of a sphere. Hence we have the following direct extension.

**Theorem 8.4.** *Let $M$ be a manifold with Euler characteristic 1. Let $\{N_i\}_{i=1}^m$ be a manifold arrangement with an intersection poset $P$ such that each non-empty element $x$ of $P$ satisfies $\chi(x) = 1 + (-1)^{\dim(x)}$. Then the* **cd**-*index of the induced Whitney stratification $T$ is given by*

$$\Psi(T) = \omega\big(\mathbf{a} \cdot \Psi(P, \rho, \zeta)\big)^*.$$

## 8.2 Toric arrangements

We call an affine subspace $V$ in $\mathbb{R}^n$ *rational* if in the linear system that determines $V = \{\vec{x} : A \cdot \vec{x} = \vec{b}\}$ the matrix $A$ has rational entries. Observe that under the quotient map $\mathbb{R}^n \longrightarrow \mathbb{R}^n/\mathbb{Z}^n = T^n$ a rational affine subspace is sent to a subtorus of $T^n$. Call an affine subspace arrangement $\{V_i\}_{i=1}^m$ in $\mathbb{R}^n$ *rational* if each subspace is rational. A *toric arrangement* is the image of a rational affine subspace arrangement under the quotient map.

Define the map $H' : \mathbb{Z}\langle \mathbf{a}, \mathbf{b}\rangle \longrightarrow \mathbb{Z}\langle \mathbf{a}, \mathbf{b}\rangle$ by $H'(1) = 0$ and $H'(u \cdot \mathbf{a}) = H'(u \cdot \mathbf{b}) = u$. Similar to [4, Proof of Proposition 8.2] (see also [15, Proposition 3.14]) we have

$$H'\big(\Psi(P, \rho, \zeta)\big) = \sum_{x \text{ coatom of } P} \Psi([\widehat{0}, x], \rho, \zeta). \tag{8.1}$$

Theorem 7.3 implies the following.

**Theorem 8.5.** *The* **cd**-*index induced by a toric arrangement on the $n$-dimensional torus $T^n$, for $n \geq 2$, is given by*

$$\Psi(T) = \frac{1}{2} \cdot \omega\big(\mathbf{a} \cdot H'\big(\Psi(P, \rho, \zeta)\big) \cdot \mathbf{b}\big)^*,$$

*where $(P, \rho, \zeta)$ is a quasi-graded intersection poset of the toric arrangement.*

*Proof.* Let $B^2$ denote the two-dimensional ball, that is, a disc. Observe that the $n$-dimensional torus $T^n = T^1 \times T^{n-1}$ is the boundary of $M = B^2 \times T^{n-1}$. Also note that the Euler characteristic is given by $\chi(M) = 0$ for $n \geq 2$. Hence the first term of Theorem 7.3 vanishes. Next, observe that the only elements in the intersection poset with a non-zero Euler characteristic are the points. Hence the sum in Theorem 7.3 reduces to

$$\Psi(T) = \frac{1}{2} \cdot \sum_{x \text{ coatom of } P} \omega\big(\mathbf{a} \cdot \Psi([\widehat{0}, x]) \cdot \mathbf{b}\big)^*,$$

that is, a sum over all coatoms in the intersection poset $P$. Since $w \longmapsto \omega(\mathbf{a} \cdot w \cdot \mathbf{b})$ is linear, by equation (8.1) the sum reduces further to the statement of the theorem. □

**Remark 8.6.** Theorem 8.5 strengthens [15, Theorem 3.12] in a number of directions. First, the previous result was only proved for an arrangement of codimension one tori, that is, the image of a rational (affine) hyperplane arrangement. This is no longer the case. Secondly, [15] required that the toric arrangement induced a regular subdivision of the torus. Again, we no longer require the regularity condition. Lastly, the previous result computes the **ab**-index of the face poset. This is not an Eulerian poset. However, it is almost an Eulerian quasi-graded poset with the weighted zeta function given by the classical zeta function $\zeta$. We only have to change the value of $\overline{\zeta}(\widehat{0}, \widehat{1})$ to $\chi(M) = 0$ to make it Eulerian. This explains why the earlier result in [15] had an extra $(\mathbf{a} - \mathbf{b})^{n+1}$ term.

Similar to Example 8.3, we have a toric analogue.

**Example 8.7.** Let $V_0 \subseteq V_1 \subseteq \cdots \subseteq V_{n-1}$ be a complete flag of rational affine subspaces in $\mathbb{R}^n$ such that $\dim(V_i) = i$. Consider the toric arrangement obtained by the quotient map. The intersection poset is a chain of length $n+1$ whose **ab**-index is $\mathbf{a}^n$. Hence the **cd**-index of the face poset is given by $1/2 \cdot \omega(\mathbf{a} \cdot \mathbf{a}^{n-1} \cdot \mathbf{b})^* = \mathbf{dc}^{n-1}$.

# 9 Concluding remarks

The next natural step is to ask for inequalities for the **cd**-coefficients of arrangements. For the classical case of central hyperplane arrangements, equivalently for zonotopes, the **cd**-index is minimized on the $n$-dimensional cube; see [4, Corollaries 7.5 and 7.6]. This was continued by Nyman and Swartz [29] in special cases. Ehrenborg's lifting technique for polytopes [10] was extended to zonotopes; see [11]. However, the question of determining inequalities is wide open for general manifold arrangements. As a first step one should consider arrangements with codimension one submanifolds.

For results concerning inequalities and unimodality of the $f$-vector of zonotopes, see [17, 18, 19]. The minimal and maximal number of connected components of arrangements has been studied by Shnurnikov [30] in the cases of Euclidean, projective and Lobachevskiĭ spaces. Moci [27] introduced a generalized Tutte polynomial for spherical and toric arrangements. It would be interesting to develop a similar polynomial for manifold arrangements. Is there a natural subclass of manifold arrangements where this would be successful?

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