Advances in Mathematics **134**, 32–42 (1998)
Article No. AI971693

On Valuations, the Characteristic Polynomial, and Complex Subspace Arrangements

Richard Ehrenborg and Margaret A. Readdy

Department of Mathematics, White Hall, Cornell University, Ithaca, New York 14853-7901
E-mail: jrge@math.cornell.edu and readdy@math.cornell.edu

Received September 1, 1997

We present a new combinatorial method to determine the characteristic polynomial of any subspace arrangement that is defined over an infinite field, generalizing the work of Blass and Sagan. Our methods stem from the theory of valuations and Groemer's integral theorem. As a corollary of our main theorem, we obtain a result of Zaslavsky about the number of chambers of a real hyperplane arrangement. The examples we consider include a family of complex subspace arrangements, which we call the divisor Dowling arrangement, whose intersection lattice generalizes that of the Dowling lattice. We also determine the characteristic polynomial of interpolations between subarrangements of the divisor Dowling arrangement, generalizing the work of Józefiak and Sagan.

1. INTRODUCTION

This paper is motivated by a result of Blass and Sagan on how to evaluate the characteristic polynomial of subarrangements from the braid arrangement B_n. Before reviewing their result, recall that for a subspace arrangement $\mathscr{A}$, the *intersection lattice* $L(\mathscr{A})$ is the lattice formed by all intersections of subspaces in $\mathscr{A}$ ordered by reverse inclusion. Thus the minimal element $\hat{0}$ of the intersection lattice is the entire space and the maximal element $\hat{1}$ is the intersection of all the subspaces in $\mathscr{A}$. For $\mathscr{A}$ a subspace arrangement with intersection lattice $L(\mathscr{A})$, the *characteristic polynomial* of $\mathscr{A}$ is given by

$$\chi(\mathscr{A}) = \sum_{\substack{x \in L(\mathscr{A}) \\ x \neq \varnothing}} \mu(\hat{0}, x) \cdot t^{\dim(x)},$$

where μ denotes the Möbius function on the intersection lattice $L(\mathscr{A})$.

The Blass–Sagan method consists of counting lattice points in a dilated hypercube of side length t which do not lie on any of the subspaces in a subarrangement of B_n. This enumeration is shown to give a combinatorial

0001-8708/98 $25.00

interpretation for evaluating the characteristic polynomial at an odd positive integer t. They use their method to show the characteristic polynomial of certain subarrangements from B_n factors into linear terms over the integers. Zhang [16] continued their investigations to interpolation arrangements between braid arrangements, while Athanasiadis [1] used modulo q arguments to study arrangements of the reals having integer coefficients, including the Shi arrangement. It is known that the characteristic polynomial factors completely into linear terms for supersolvable lattices and free arrangements; see [2, 6, 11–13].

We generalize the Blass–Sagan result to all possible subspace arrangements over any infinite field. To do this, we prove the existence of a valuation on the Boolean algebra generated by affine subspaces. The existence of such a valuation follows from Groemer's integral theorem [8, 10].

Our main theorem is that the characteristic polynomial of a subspace arrangement over any infinite field is simply the valuation of the complement of the arrangement. In order to facilitate the calculation of valuations, we prove a Fubini-type theorem. This enables us to determine the valuation of a set one coordinate at a time, making our valuation computations very much in the spirit of those of Blass–Sagan.

A real hyperplane arrangement cuts the space $\mathbb{R}^n$ into chambers. Zaslavsky showed that the number of chambers equals, up to a sign, the characteristic polynomial evaluated at -1. Using the fact that the Euler characteristic is a well-defined valuation, we obtain Zaslavsky's result as a corollary to our main theorem.

We end this paper by considering a subspace arrangement, defined over the complex field, which we call the divisor Dowling arrangement. This family of examples unifies previously-studied arrangements, including the Dowling arrangement, that is, the arrangement whose intersection lattice is the Dowling lattice, and braid arrangement interpolations from B_{n-1} to B_n and from D_n to B_n due to Józefiak–Sagan [9]. We also extend the Blass–Sagan result to a complex analogue by considering sets of complex numbers which are invariant under rotation by $2\pi/m$ radians.

2. THE VALUATION ν

Let V be a set. Let G be a collection of subsets of V such that G is closed under finite intersections and the empty set $\varnothing$ belongs to G. Let $B(G)$ be the Boolean algebra generated by G, that is, $B(G)$ is the smallest collection of subsets of V such that $B(G)$ contains G and is closed under finite intersections, finite unions and complements. Observe that $B(G)$ forms a lattice where the meet and join are intersection and union.

Let L be a collection of sets closed under intersection and union. A *valuation* v on L is a function $v: L \to R$, where R is a commutative ring, satisfying

$$v(A) + v(B) = v(A \cap B) + v(A \cup B),$$
$$v(\varnothing) = 0.$$

For a more detailed account of valuations, we refer the reader to [10, Chapter 2].

In this paper we will study the case when V is a finite-dimensional vector space over an infinite field $\mathbf{k}$. Take G to be the collection of affine subspaces of V. Moreover, let G also contain the empty set.

THEOREM 2.1. *There exists a unique valuation $v: B(G) \to \mathbb{Z}[t]$ such that for A a nonempty affine subspace we have*

$$v(A) = t^{\dim(A)}.$$

Proof. Define a function $\lambda: G \to \mathbb{Z}[t]$ by $\lambda(A) = t^{\dim(A)}$ for a nonempty affine subspace A and $\lambda(\varnothing) = 0$. By Groemer's integral theorem [10, Theorem 2.2.1] and one of its corollaries [10, Corollary 2.2.2], the function λ extends uniquely to a valuation on $B(G)$ if and only if λ satisfies the inclusion-exclusion formula

$$\lambda(A_1 \cup \cdots \cup A_n) = \sum_i \lambda(A_i) - \sum_{i<j} \lambda(A_i \cap A_j) + \cdots \tag{2.1}$$

whenever $A_i \in G$ and $A_1 \cup \cdots \cup A_n \in G$. Since the field $\mathbf{k}$ is infinite, when $A_1, ..., A_n$ and $A_1 \cup \cdots \cup A_n$ are affine subspaces then for some index, say n, we have $A_n = A_1 \cup \cdots \cup A_n$. That is, for all indices $j = 1, ..., n$ the affine subspace A_j is contained in A_n. Now it is straightforward to verify the inclusion-exclusion formula (2.1), since every term on the right-hand side, except $\lambda(A_n)$, will cancel pairwise with a term having the opposite sign. ∎

As an example, if A is a finite subset of V then the valuation of A is the cardinality of A, that is, $v(A) = |A|$. Similarly, if A is a cofinite subset of V then $v(A) = t^{\dim(V)} - |V - A|$.

One consequence of Groemer's integral theorem is that one can define integrals on simple functions. Let I_A denote the *indicator function* on the set A, that is, $I_A(x) = 1$ if $x \in A$ and $I_A(x) = 0$ if $x \in V - A$. A function $f: V \to \mathbb{Z}[t]$ is said to be *simple* if f can be written as a linear combination of indicator functions of sets from the Boolean algebra $B(G)$. That is, f can be written as

$$f = \sum_{i=1}^{k} \alpha_i \cdot I_{A_i}$$

where $\alpha_i \in \mathbb{Z}[t]$ and $A_i \in B(G)$. The *integral* of a simple function is defined to be the sum

$$\int f \, dv = \sum_{i=1}^{k} \alpha_i \cdot v(A_i).$$

Observe that the integral is a linear functional and we have $\int I_A \, dv = v(A)$.

Since any set in $B(G)$ can be written in terms of sets in G with a finite number of unions, intersections and complements, we obtain that any simple function f can be written in the form

$$f = \sum_{i=1}^{n} \beta_i \cdot I_{A_i}$$

where A_i belongs to G, that is, A_i is an affine subspace. This implies that the indicator functions of affine subspaces form a linear basis for all simple functions.

We will now introduce a "Fubini-type" theorem that will help us to compute the valuation v, or equivalently, to compute the integral. Let V_1 and V_2 be two vector spaces over the field $\mathbf{k}$ and let V be the vector space $V_1 \times V_2$. A function f defined on V is viewed as a function of two variables $f(x, y)$, where $x \in V_1$ and $y \in V_2$. For a fixed element x in V_1 and f a function on V, let $f_x(y)$ denote the function f viewed as a function of the variable y. Moreover, let $\int f \, dv_i$ denote the integral on the space V_i.

PROPOSITION 2.2. *If $f(x, y)$ is a simple function on $V = V_1 \times V_2$ then $f_x(y)$ is a simple function on V_2 and $\int f_x(y) \, dv_2$ is a simple function on V_1. Moreover,*

$$\int f \, dv = \iint f_x(y) \, dv_2 \, dv_1.$$

Proof. It is enough to consider the proposition in the case when f is the indicator function $f = I_A$ for A an affine subspace of V. For $x \in V_1$ the set $A_x = \{ y \in V_2 \colon (x, y) \in A \}$ is the intersection of two affine subspaces, and hence is an affine subspace of V_2. Thus the function f_x is a simple function on V_2.

Let $\pi\colon V \to V_1$ denote the projection onto V_1. Then

$$\int f_x(y)\, dv_2 = t^{\dim(A_z)} \cdot I_{\pi(A)}(x),$$

for some $z \in V_1$ such that A_z is non-empty. Hence we obtain $\int f_x(y)\, dv_2$ is a simple function on V_1. Integrating with respect to x gives

$$\begin{aligned}\iint f_x(y)\, dv_2\, dv_1 &= t^{\dim(A_z)} \cdot t^{\dim(\pi(A))}\\ &= t^{\dim(A)}\\ &= \int f\, dv. \quad \blacksquare\end{aligned}$$

Now we present our main result.

THEOREM 2.3. *Let $\mathscr{A} = \{A_1, ..., A_k\}$ be a subspace arrangement in V. Then the characteristic function of $\mathscr{A}$ is given by*

$$\chi(\mathscr{A}) = v\left(V - \bigcup_{i=1}^{k} A_i\right).$$

Proof. Our proof mirrors that of Blass–Sagan [3], replacing cardinality with the valuation v. In what follows all the unions and sums are taken over elements Y from the intersection lattice $L(\mathscr{A})$. For $X \in L(\mathscr{A})$, let

$$r(X) = X - \bigcup_{Y \subset X} Y$$

Here $Y \subset X$ means Y is strictly contained in X. Recall that $Y \subseteq X$ means $Y \geqslant X$ in the lattice $L(\mathscr{A})$. Now define $g(X) = v(r(X))$. We would like to compute $g(V)$. Observe that

$$X = \bigcup_{Y \subseteq X} r(Y).$$

On applying the valuation v to this equation, we obtain that

$$t^{\dim(X)} = \sum_{Y \subseteq X} g(Y).$$

By the Möbius inversion theorem, see for instance [14], and the definition of the characteristic polynomial, we obtain the result. $\blacksquare$

Our first example shows Theorem 2.3 in action. We will carefully apply Proposition 2.2 to facilitate the computation.

EXAMPLE 2.4 [11, Figure 11]. Consider the hyperplane arrangement $\mathscr{H}$ in three-dimensional space V over a field $\mathbf{k}$ of characteristic different from 2, defined by the seven hyperplanes

$$x=0, \qquad y=0, \qquad z=0, \qquad x \pm y \pm z = 0.$$

We would like to compute the valuation v of the complementary set $C = V - \bigcup_{i=1}^{7} H_i$. Consider the partition of the set C into two disjoint sets

$$E = \{(x, y, z) \in C : x \neq \pm y\} \qquad \text{and} \qquad F = \{(x, y, z) \in C : x = \pm y\}.$$

We have by the Fubini result, Proposition 2.2, that

$$v(E) = \iiint (I_E)_{x,\,y}\,(z)\, dv_3\, dv_2\, dv_1.$$

Let $E' = \{(x, y) : x \neq 0,\ y \neq 0,\ \pm x\}$ and $E'' = \{x : x \neq 0\}$. Then we have that

$$\begin{aligned} \iiint (I_E)_{x,\,y}\,(z)\, dv_3\, dv_2\, dv_1 &= (t-5) \cdot \iint (I_{E'})_x\,(y)\, dv_2\, dv_1 \\ &= (t-3) \cdot (t-5) \cdot \int I_{E''}(x)\, dv_1 \\ &= (t-1) \cdot (t-3) \cdot (t-5). \end{aligned}$$

We interpret this calculation as there are $t-1$ ways to choose the x coordinate so that $x \neq 0$. Then there are $t-3$ ways to choose the y coordinate so that $y \neq 0, -x, x$. Finally there are $t-5$ ways to choose z such that $z \neq 0, x+y, x-y, -x+y, -x-y$. To compute $v(F)$ there are $t-1$ ways to choose x, two to choose y and $t-3$ to choose z. Hence the characteristic polynomial for this arrangement is given by

$$\begin{aligned} \chi(\mathscr{H}) &= (t-1) \cdot (t-3) \cdot (t-5) + (t-1) \cdot 2 \cdot (t-3) \\ &= (t-1) \cdot (t-3) \cdot (t-3). \end{aligned}$$

If the field $\mathbf{k}$ had characteristic 2, then the characteristic polynomial would be

$$\begin{aligned} \chi(\mathscr{H}) &= (t-1) \cdot (t-2) \cdot (t-2) + (t-1) \cdot 1 \cdot (t-1) \\ &= (t-1) \cdot (t^2 - 3 \cdot t + 3). \end{aligned}$$

In the rest of the paper we will not refer explicitly to Proposition 2.2 when computing the characteristic polynomial.

We end this section with a result due to Zaslavsky [15, Theorem A]. A hyperplane arrangement $\mathscr{H} = \{H_1, ..., H_k\}$ in $\mathbb{R}^n$ cuts the space $\mathbb{R}^n$ into *chambers*, also know as maximal regions. Zaslavsky showed how to compute the number of chambers in terms of the characteristic polynomial of the hyperplane arrangement. We prove his result using valuations.

The *Euler characteristic* is a valuation on certain subsets of $\mathbb{R}^n$. We denote this valuation by ε so as not to confuse it with the characteristic polynomial χ. For more on the Euler characteristic as a valuation, see [10]. The important property of the Euler characteristic is that if the set A is homeomorphic to a k-dimensional open ball then $\varepsilon(A) = (-1)^k$. Thus if A is a k-dimensional affine subspace then $\varepsilon(A) = v(A)|_{t=-1}$. Hence for any set A in the Boolean algebra $B(G)$ the Euler characteristic of A is equal to the valuation $v(A)$ evaluated at $t = -1$. With this observation we obtain Zaslavsky's result:

Corollary 2.5 (Zaslavsky). *Let $\mathscr{H} = \{H_1, ..., H_k\}$ be a hyperplane arrangement in $\mathbb{R}^n$. Then the number of chambers of the hyperplane arrangement $\mathscr{H}$ is equal to $(-1)^n \cdot \chi(\mathscr{H})|_{t=-1}$.*

Proof. By Theorem 2.3 we know that $\chi(\mathscr{H})|_{t=-1} = \varepsilon(\mathbb{R}^n - \bigcup_{i=1}^k H_i)$. The set $\mathbb{R}^n - \bigcup_{i=1}^k H_i$ is a disjoint union of chambers. Each chamber is homeomorphic to an n-dimensional open ball, and hence the Euler characteristic of a chamber is $(-1)^n$. Using the fact that the Euler characteristic is a valuation, we obtain that $\varepsilon(\mathbb{R}^n - \bigcup_{i=1}^k H_i) = (-1)^n \cdot (\#\text{chambers})$, which proves the result. ∎

3. THE DIVISOR DOWLING ARRANGEMENT

In this section we will apply our main theorem to a family of complex hyperplane arrangements. Recall that the *Dowling arrangement* $B_n(m)$ is the complex hyperplane arrangement

$$\zeta^h \cdot x_i = x_j \qquad \text{for} \quad 1 \leqslant i < j \leqslant n \qquad \text{and} \qquad 1 \leqslant h \leqslant m,$$
$$x_i = 0 \qquad \text{for} \quad 1 \leqslant i \leqslant n,$$

where m is a positive integer and ζ is a primitive mth root of unity. We give this arrangement the name Dowling since the associated intersection lattice is the Dowling lattice; see [4, 5].

We now consider a generalization of the Dowling arrangement. Let $\mathbf{m} = (m_1, ..., m_{n-1})$ be a list of positive integers such that m_{i+1} divides m_i.

Let ζ_i be a primitive m_ith root of unity. The *divisor Dowling arrangement* $\mathscr{B}(\mathbf{m})$ is the complex arrangement defined as

$$(\zeta_i)^h \cdot x_i = x_j \quad \text{for} \quad 1 \leqslant i < j \leqslant n \quad \text{and} \quad 1 \leqslant h \leqslant m_i,$$
$$x_i = 0 \quad \text{for} \quad 1 \leqslant i \leqslant n.$$

The arrangement $\mathscr{B}(\mathbf{m})$ reduces to the Dowling arrangement when all the m_i's are equal. Furthermore, when $m_1 = \cdots = m_{n-1} = 1$, respectively $m_1 = \cdots = m_{n-1} = 2$, the arrangement is the braid arrangement A_n, respectively B_n.

Proposition 3.1. *The characteristic polynomial of the divisor Dowling arrangement is given by*

$$\chi(\mathscr{B}(\mathbf{m})) = \prod_{i=1}^{n} \left(t - 1 - \sum_{j=1}^{i-1} m_j\right).$$

Proof. By Theorem 2.3 we need to compute the valuation of the complement of the arrangement. The number of ways to choose x_1 is $(t-1)$, the number of ways to choose x_2 is $(t-1-m_1)$, and in general, the number of ways to choose x_i is $(t-1-\sum_{j=1}^{i-1} m_j)$. ∎

Our next example is an interpolation between $\mathscr{B}(m_1, ..., m_{n-2})$ and $\mathscr{B}(m_1, ..., m_{n-1})$. Let $\mathscr{B}_k(\mathbf{m})$ denote the subspace arrangement

$$(\zeta_i)^h \cdot x_i = x_j \quad \text{for} \quad 1 \leqslant i < j \leqslant n-1 \quad \text{and} \quad 1 \leqslant h \leqslant m_i,$$
$$x_i = 0 \quad \text{for} \quad 1 \leqslant i \leqslant n-1,$$

and any k of the hyperplanes

$$(\zeta_i)^h \cdot x_i = x_n \quad \text{for} \quad 1 \leqslant i \leqslant n-1 \quad \text{and} \quad 1 \leqslant h \leqslant m_i,$$
$$x_n = 0.$$

Proposition 3.2. *The characteristic polynomial of the arrangement $\mathscr{B}_k(\mathbf{m})$ interpolating between the divisor Dowling arrangements $\mathscr{B}(m_1, ..., m_{n-2})$ and $\mathscr{B}(m_1, ..., m_{n-1})$ is*

$$\chi(\mathscr{B}_k(\mathbf{m})) = (t-k) \cdot \prod_{i=1}^{n-1} \left(t - 1 - \sum_{j=1}^{i-1} m_j\right).$$

Proof. The argument is the same as the previous proposition, except for when we choose the last coordinate, which can be done in $(t-k)$ ways. ∎

Proposition 3.2 generalizes Józefiak and Sagan's interpolations between the Dowling arrangements $B_{n-1}(m)$ and $B_n(m)$ [9, Theorem 5.2].

We next turn to a divisor analogue of interpolations between between the Coxeter arrangements D_n and B_n. Consider now a divisor list $\mathbf{m} = (m_1, ..., m_{n-1})$ such that $m_{k+1} = m_{k+2} = \cdots = m_{n-1}$, that is, the last $n-k-1$ entries of $\mathbf{m}$ are equal. Define $\mathscr{D}_k(\mathbf{m})$ to be the arrangement

$$(\zeta_i)^h \cdot x_i = x_j \qquad \text{for} \quad 1 \leqslant i < j \leqslant n \qquad \text{and} \qquad 1 \leqslant h \leqslant m_i,$$

$$x_i = 0 \qquad \text{for} \quad 1 \leqslant i \leqslant k.$$

When $m_1 = \cdots = m_{n-1}$ and $k = 0$, the arrangement $\mathscr{D}_k(\mathbf{m})$ is a complex D_n-analogue. For this special case, interpolations were considered by Józefiak and Sagan [9, Theorem 2.4].

PROPOSITION 3.3. *Let* $\mathbf{m} = (m_1, ..., m_{n-1})$ *be a divisor list such that* $m_{k+1} = m_{k+2} = \cdots = m_{n-1}$. *Then the characteristic polynomial of the arrangement* $\mathscr{D}_k(\mathbf{m})$ *is*

$$\chi(\mathscr{D}_k(\mathbf{m})) = \left(t + n - k - 1 - \sum_{j=1}^{n-1} m_j\right) \cdot \prod_{i=1}^{n-1} \left(t - 1 - \sum_{j=1}^{i-1} m_j\right).$$

Proof. At most one of the coordinates $x_{k+1}, ..., x_n$ can be equal to zero. If there is no coordinate equal to zero, this is the arrangement $\mathscr{B}(\mathbf{m})$ and is counted by the polynomial $\prod_{i=1}^{n} (t - 1 - \sum_{j=1}^{i-1} m_j)$. To have one coordinate equal to zero there are $n-k$ choices. The remaining variables behave like the arrangement $\mathscr{B}(m_1, ..., m_{n-2})$, which has characteristic polynomial $\prod_{i=1}^{n-1} (t - 1 - \sum_{j=1}^{i-1} m_j)$. Hence the characteristic polynomial of $\mathscr{D}_k(\mathbf{m})$ is given by the sum

$$\begin{aligned}\chi(\mathscr{D}_k(\mathbf{m})) &= \prod_{i=1}^{n} \left(t - 1 - \sum_{j=1}^{i-1} m_j\right) + (n-k) \cdot \prod_{i=1}^{n-1} \left(t - 1 - \sum_{j=1}^{i-1} m_j\right) \\ &= \left(t + n - k - 1 - \sum_{j=1}^{n-1} m_j\right) \cdot \prod_{i=1}^{n-1} \left(t - 1 - \sum_{j=1}^{i-1} m_j\right). \quad \blacksquare\end{aligned}$$

The results in this section may also be obtained by extending the theorem of Blass and Sagan [3, Theorem 2.1] to a complex analogue. Recall that a subspace arrangement $\mathscr{A}$ is *embedded* in an arrangement $\mathscr{B}$ if every element of the intersection lattice $L(\mathscr{A})$ is a member of $L(\mathscr{B})$. The three arrangements discussed in this section are all embedded in the Dowling arrangement $B_n(m)$.

We call a finite subset S of the complex numbers *m-invariant* if $0 \in S$ and $x \in S$ implies $\zeta \cdot x \in S$ for ζ a primitive mth root of unity. Observe that S is invariant under a rotation of the complex plane by an angle of $2\pi/m$. The cardinality of an m-invariant set is congruent to 1 modulo m.

We now obtain the following result, whose proof we leave to the reader.

PROPOSITION 3.4. *Let* $\mathscr{A} = \{A_1, ..., A_k\}$ *be an arrangement embedded in the Dowling arrangement* $B_n(m)$ *and let* S *be an* m*-invariant subset of the complex numbers* $\mathbb{C}$. *Then the characteristic polynomial of* $\mathscr{A}$ *evaluated at the cardinality of* S *is given by*:

$$\chi(\mathscr{A})|_{t=|S|} = \left| S^n - \bigcup_{i=1}^{k} A_i \right|.$$

The original result of Blass and Sagan is obtained by setting $m = 2$ and $S = \{-s, -s+1, ..., s\}$.

ACKNOWLEDGMENTS

The authors thank Dan Klain and Bruce Sagan for their comments on an earlier version of this paper.

REFERENCES

1. C. A. Athanasiadis, Characteristic polynomials of subspace arrangements and finite fields, *Adv. in Math.* **122** (1996), 193–233.
2. C. A. Athanasiadis, On free deformations of the braid arrangement, *Europ. J. Combin.*, to appear.
3. A. Blass and B. Sagan, Characteristic and Ehrhart polynomials, *J. Combin. Theory Ser. A* **72** (1995), 209–231.
4. T. A. Dowling, A q-analog of the partition lattice, *in* “A Survey of Combinatorial Theory” (J. N. Srivastava *et al.*, Eds.), pp. 101–115, North-Holland, Amsterdam, 1973.
5. T. A. Dowling, A class of geometric lattices based on finite groups, *J. Combin. Theory Ser. B* **14** (1973), 61–86.
6. P. H. Edelman and V. Reiner, Free hyperplane arrangements between A_{n-1} and B_n, *Math. Z.* **215** (1994), 347–365.
7. R. Ehrenborg and M. Readdy, On flag vectors, the Dowling arrangement and braid arrangements, *Discrete Comput. Geom.*, to appear.
8. H. Groemer, On the extension of additive functionals on classes of convex sets, *Pacific J. Math.* **75** (1978), 397–410.
9. T. Józefiak and B. Sagan, Basic derivations for subarrangements of Coxeter arrangements, *J. Algebraic Combin.* **2** (1993), 291–320.
10. D. A. Klain and G.-C. Rota, “Introduction to Geometric Probability,” Cambridge Univ. Press, Cambridge, 1997.
11. P. Orlik, “Introduction to Arrangements,” CBMS Series, Vol. 72, American Math. Society, Providence, RI, 1989.

12. P. Orlik and H. Terao, "Arrangements of Hyperplanes," Grundlehren der matematischen Wissenschaften, Vol. 300, Springer-Verlag, New York, 1992.
13. R. P. Stanley, Supersolvable lattices, *Alg. Univ.* **2** (1972), 197–217.
14. R. P. Stanley, "Enumerative Combinatorics," Vol. I, Wadsworth and Brooks/Cole, Pacific Grove, 1986.
15. T. Zaslavsky, Facing up to arrangements: Face count formulas for partitions of space by hyperplanes, *Memoirs Amer. Math. Soc.* **154** (1975).
16. P. Zhang, The characteristic polynomials of interpolations between Coxeter arrangements, preprint, 1997.