Inequalities for Zonotopes*

RICHARD EHRENBORG

Dedicated to Louis Billera on his 60th birthday

Abstract

We present two classes of linear inequalities that the flag f-vectors of zonotopes satisfy. These inequalities strengthen inequalities for polytopes obtained by the lifting technique of Ehrenborg [13].

1 Introduction

The systematic study of flag f-vectors of polytopes was initiated by Bayer and Billera [2]. Billera then suggested the study of flag f-vectors of zonotopes; see the dissertation of his student Liu [22]. The essential computational results of the field appeared in the two papers by Billera, Ehrenborg and Readdy [7, 8]. In this paper, we present two classes of linear inequalities for the flag f-vectors of zonotopes. These classes are motivated by Ehrenborg's recent results for polytopes [13].

The flag f-vector of a convex polytope contains all the enumerative incidence information between the faces of the polytope. For an n-dimensional polytope the flag f-vector consists of 2^n entries, in other words, the flag f-vector lies in the vector space $\mathbb{R}^{2^n}$. Bayer and Billera [2] showed that the flag vectors of n-dimensional polytopes span a subspace of $\mathbb{R}^{2^n}$, called the generalized Dehn-Sommerville subspace and denoted by GDSS_n. Bayer and Klapper [5] proved that GDSS_n is naturally isomorphic to the nth homogeneous component of the non-commutative ring $\mathbb{R}\langle \mathbf{c}, \mathbf{d} \rangle$, where the grading is given by $\deg(\mathbf{c}) = 1$ and $\deg(\mathbf{d}) = 2$. Hence, the flag f-vector of a polytope P can be encoded by a non-commutative polynomial $\Psi(P)$ in the variables $\mathbf{c}$ and $\mathbf{d}$, called the **cd**-index.

The next essential step is to consider linear inequalities that the flag f-vector of polytopes satisfy. The known linear inequalities are: the non-negativity of the toric g-vector [19, 21, 26], inequalities obtained by the Kalai convolution [20], and that the **cd**-index is minimized coefficientwise on the n-dimensional simplex Σ_n [6]. Recently, Ehrenborg [13] introduced a lifting technique that allows one to use lower dimensional inequalities to obtain higher dimensional inequalities. A special case of this lifting technique is the following inequality:

*Appeared in Mathematical Sciences Research Institute Publication on Combinatorial and Computational Geometry, (J. E. Goodman, J. Pach and E. Welzl, eds.), Cambridge University Press, Cambridge, England, 2005, pages 277–286.

Theorem 1.1 *Let u, q and v be three* **cd***-monomials such that the sum of the degrees of u, q and v is n and the degree of q is k. Let Δ_q denote the coefficient of the* **cd***-monomial q in the* **cd***-index of a k-dimensional simplex Σ_k. Then for all n-dimensional polytopes P we have*

$$\left\langle u \cdot (q - \Delta_q \cdot \mathbf{c}^k) \cdot v \mid \Psi(P) \right\rangle \geq 0$$

where the bracket $\langle \cdot | \cdot \rangle$ is the standard inner product on $\mathbb{R}\langle \mathbf{c}, \mathbf{d} \rangle$.

The purpose of this paper is to improve Theorem 1.1 for zonotopes.

Recall that a zonotope is a polytope obtained as the Minkowski sum of line segments. In particular, the flag f-vectors of n-dimensional zonotopes lie in the subspace GDSS_n. Billera, Ehrenborg and Readdy [8] proved that flag f-vectors of zonotopes do not lie in any proper subspace of GDSS_n. They later showed that among all n-dimensional zonotopes (and more generally, the dual of the lattice of regions of oriented matroids), the n-dimensional cube minimizes the **cd**-index coefficientwise [7]. This is the zonotopal analogue of Stanley's Gorenstein* lattice conjecture [28, Conjecture 2.7].

We continue this vein of research by introducing further classes of linear inequalities for flag f-vectors of zonotopes. We develop two sharper versions of the inequality appearing in Theorem 1.1. For an n-dimensional zonotope we show that the expression in Theorem 1.1 is at least the value obtained by the n-dimensional cube C_n; see Theorem 3.1. The second improvement is the case when $u = 1$. We can replace the factor Δ_q by the larger factor $\Box_q$, where $\Box_q$ denotes the coefficient of q in the **cd**-index of the k-dimensional cube C_k; see Theorem 3.6.

2 Preliminaries

For standard terminology for posets we refer the reader to [25]. A partially ordered set (poset) P is ranked if there is a rank function $\rho : P \longrightarrow \mathbb{Z}$ such that when x is covered by y then $\rho(y) = \rho(x) + 1$. Furthermore, the poset P is graded of rank n if it is ranked and has a minimal element $\hat{0}$ and a maximal element $\hat{1}$ such that $\rho(\hat{0}) = 0$ and $\rho(\hat{1}) = n$. Define the interval $[x, y]$ to be the subposet $\{z \in P : x \leq z \leq y\}$. Observe that the interval $[x, y]$ is also a graded poset of rank $\rho(y) - \rho(x)$.

Let P be a graded poset of rank $n+1$. For $S = \{s_1 < s_2 < \cdots < s_k\}$ a subset of $\{1, \ldots, n\}$, define f_S to be the number of chains $\hat{0} = x_0 < x_1 < \cdots < x_{k+1} = \hat{1}$, where the rank of the element x_i is s_i for $1 \leq i \leq k$. These 2^n values constitute the *flag f-vector* of the poset P. Define the *flag h-vector* of P by the two equivalent relations $h_S = \sum_{T \subseteq S} (-1)^{|S-T|} f_T$ and $f_S = \sum_{T \subseteq S} h_T$. There has been a lot of recent work in understanding the flag f-vectors of graded posets and Eulerian posets. For example, see [1, 4, 9].

For S a subset of $\{1, \ldots, n\}$ define the monomial $u_S = u_1 u_2 \cdots u_n$, where $u_i = \mathbf{a}$ if $i \notin S$ and $u_i = \mathbf{b}$ if $i \in S$. Define the **ab**-*index* of a graded poset P of rank $n+1$ to be the sum

$$\Psi(P) = \sum_S h_S \cdot u_S.$$

A poset P is Eulerian if every interval $[x, y]$, where $x \neq y$, has the same number of elements of odd rank as the number of elements of even rank. This condition states that every interval $[x, y]$ satisfies the Euler-Poincaré relation. The condition of being Eulerian is equivalent to the condition that the Möbius function $\mu(x, y)$ is $(-1)^{\rho(x,y)}$. The two main examples of Eulerian posets are the strong Bruhat order and face lattices of convex polytopes.

The following result was conjectured by Fine and proved by Bayer and Klapper [5]. It states that the generalized Dehn-Sommerville subspace GDSS_n is naturally isomorphic to the space of **cd**-polynomials of degree n.

Theorem 2.1 *The* **ab***-index of an Eulerian poset P, $\Psi(P)$, can be written in terms of* $\mathbf{c} = \mathbf{a} + \mathbf{b}$ *and* $\mathbf{d} = \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a}$.

When $\Psi(P)$ is expressed in terms of **c** and **d** it is called the **cd**-*index* of the poset P. There exist several proofs of this result in the literature; see [5, 10, 12, 17, 27]. The **cd**-index has been extraordinarily useful for flag vector computations; see [3, 7, 16]. Moreover, this basis is now emerging as a key tool for obtaining linear inequalities for the entries of the flag f-vector; see [6, 13, 14, 27].

Define an inner product $\langle \cdot | \cdot \rangle$ on $\mathbb{R}\langle \mathbf{c}, \mathbf{d} \rangle$ by $\langle u | v \rangle = \delta_{u,v}$ for all **cd**-monomials u and v, and extend this relation by linearity. Using this notation any linear inequality on the flag f-vector of an n-dimensional polytope can be expressed as $\langle H | \Psi(P) \rangle \geq 0$, where H is homogeneous **cd**-polynomial of degree n.

In the remainder of this section we will focus upon the **cd**-index of zonotopes. However, all the results carry over to oriented matroids. In order to keep the statements of the results explicit, we will use the geometric language of zonotopes and their hyperplane arrangements.

A zonotope Z is a polytope obtained by the Minkowski sum of line segments, that is, $Z = [\mathbf{0}, \mathbf{v}_1] + \cdots + [\mathbf{0}, \mathbf{v}_m]$. For each line segment $[\mathbf{0}, \mathbf{v}_i]$ let H_i be the hyperplane through the origin that is orthogonal to $\mathbf{v}_i$. The collection of these hyperplanes $\mathcal{H} = \{H_1, \ldots, H_m\}$ is the central hyperplane arrangement associated to the zonotope Z. The intersection lattice L of the arrangement $\mathcal{H}$ is the collection of all the intersections of the hyperplanes $H_1, \ldots, H_m$ ordered by reverse inclusion.

Let ω be the linear map from $\mathbb{R}\langle \mathbf{a}, \mathbf{b} \rangle$ to $\mathbb{R}\langle \mathbf{c}, \mathbf{d} \rangle$ defined on an **ab**-monomial by replacing each occurrence of **ab** with 2**d** and then replacing the remaining variables by **c**. The fundamental theorem of computing the **cd**-index of a zonotope is the following [7]:

Theorem 2.2 *Let Z be a zonotope (and more generally, let Z be the dual of the lattice of regions of an oriented matroid). Let L be the intersection lattice of the associated central hyperplane arrangement $\mathcal{H}$ and $\Psi(L)$ the* **ab***-index of the lattice L. Then the* **cd***-index of the zonotope and the sum of the* **cd***-indices of all the vertex figures of the zonotope are given by*

$$\Psi(Z) = \omega(\mathbf{a} \cdot \Psi(L)), \tag{2.1}$$

$$\sum_{v} \Psi(Z/v) = 2 \cdot \omega(\Psi(L)), \tag{2.2}$$

where v ranges over all vertices of the zonotope Z.

The identity (2.1) is [7, Theorem 3.1]. The identity (2.2) follows from (2.1) and using the linear map h defined in Section 8 in [7].

It remains to compute the **ab**-index of the intersection lattice L. We do this using R-labelings. For more details, see [7, Section 7] and [11, 24, 25]. Linearly order the hyperplanes in the arrangement $\mathcal{H}$ as $\mathcal{H} = \{H_1, \ldots, H_m\}$. Mark each edge $x \prec y$ in the Hasse diagram of the lattice L with the smallest (in the given linear order) hyperplane H such that intersecting x with H gives y. That is,

$$\lambda(x, y) = \min\{i \,:\, x \cap H_i = y\}.$$

For a maximal chain $c = \{\hat{0} = x_0 \prec x_1 \prec \cdots \prec x_n = \hat{1}\}$ in the intersection lattice L define its *descent set* $D(c)$ by

$$D(c) = \{i \,:\, \lambda(x_{i-1}, x_i) > \lambda(x_i, x_{i+1})\}.$$

We then have the following result; see Section 7 in [7].

Theorem 2.3 *The* **ab***-index of intersection lattice L is given by*

$$\Psi(L) = \sum_{c} u_{D(c)},$$

where the sum ranges over all maximal chains c in the lattice L.

3 Inequalities for zonotopes

In this section we will improve Theorem 1.1 for zonotopes. Let C_n denote the n-dimensional cube.

Theorem 3.1 *Let Z be an n-dimensional zonotope (and more generally, let Z be the dual of the lattice of regions of an oriented matroid). Let q be a* **cd***-monomial of degree k that contains at least one* **d***. Then the* **cd***-index $\Psi(Z)$ satisfies the inequality*

$$\left\langle u \cdot \left(q - \Delta_q \cdot \mathbf{c}^k\right) \cdot v \mid \Psi(Z) - \Psi(C_n)\right\rangle \geq 0.$$

for any two **cd***-monomials u and v such that $\deg(u) + \deg(v) = n - k$.*

Definition 3.2 *Let q be a* **cd***-monomial of degree k that contains at least one* **d***. For two* **cd***-polynomials z and w define the order relation $z \preceq_q w$ if the inequality $\langle u \cdot (q - \Delta_q \cdot \mathbf{c}^k) \cdot v | w - z\rangle \geq 0$ holds for all* **cd***-monomials u and v.*

In this notation Theorem 3.1 becomes $\Psi(Z) \succeq_q \Psi(C_n)$ and that of Theorem 1.1 becomes $\Psi(P) \succeq_q 0$. Note that this order relation differs slightly from the order relation used in [13].

Lemma 3.3 *Let z and w be non-negative* **cd***-polynomials such that $z \succeq_q 0$ and $w \succeq_q 0$. Then we have $z \cdot \mathbf{d} \cdot w \succeq_q 0$.*

Proof: Without loss of generality, we may assume that z and w are homogeneous polynomials. We would like to prove

$$\left\langle u \cdot (q - \Delta_q \cdot \mathbf{c}^k) \cdot v \mid z \cdot \mathbf{d} \cdot w \right\rangle \geq 0,$$

for all **cd**-monomials u and v such that $\deg(u) + \deg(v) = \deg(z\mathbf{d}w) - k$, where k is the degree of q. We do this in three cases. The first case is $\deg(u\mathbf{c}^k) \leq \deg(z)$. Try to factor $v = v_1 \cdot v_2$ such that $\deg(u\mathbf{c}^k v_1) = \deg(z)$. If such factoring is not possible, both sides of the inequality are equal to zero. If factoring is possible then $\left\langle u(q - \Delta_q \mathbf{c}^k)v | z\mathbf{d}w \right\rangle = \left\langle u(q - \Delta_q \mathbf{c}^k)v_1 | z \right\rangle \cdot \langle v_2 | \mathbf{d}w \rangle \geq 0$. The second case is $\deg(u) \geq \deg(z\mathbf{d})$, which is symmetric to the first case.

The third is $\deg(u\mathbf{c}^k) > \deg(z)$ and $\deg(u) < \deg(z\mathbf{d})$. Since z and w have non-negative coefficients we have $\langle uqv | z\mathbf{d}w \rangle \geq 0$. Moreover, $\left\langle u\mathbf{c}^k v | z\mathbf{d}w \right\rangle = 0$. This completes the third case. □

Proposition 3.4 *Let Z be an n-dimensional zonotope and let Z' be the zonotope obtained by taking the Minkowski sum of Z with a line segment in the affine span of Z. Then we have $\Psi(Z') \succeq_q \Psi(Z)$.*

Proof: Let $\mathcal{H}$ and $\mathcal{H}'$ be the associated hyperplane arrangements and let H be the new hyperplane. Let $\mathcal{H}'$ inherit the linear order of $\mathcal{H}$ with the new hyperplane H inserted at the end of the linear order. Similarly, let L and L' be the corresponding intersection lattices. Observe that every maximal chain in L is also a maximal chain in L'. Also observe that there is no maximal chain in L' whose last label is H. Hence the difference in the **ab**-indices between the two intersection lattices is

$$\Psi(L') - \Psi(L) = \sum_{c} u_{D(c)} \tag{3.1}$$

$$= \sum_{\hat{0} < x \prec y} \Psi([\hat{0}, x]) \cdot \mathbf{ab} \cdot \Psi([y, \hat{1}]) + \sum_{\hat{0} = x \prec y} \mathbf{b} \cdot \Psi([y, \hat{1}]), \tag{3.2}$$

where the first sum (3.1) is over all maximal chains c containing the label H and the two sums in (3.2) are over edges $x \prec y$ in the Hasse diagram of L' having the label H. Applying the map $w \longmapsto \omega(\mathbf{a} \cdot w)$ we obtain

$$\Psi(Z') - \Psi(Z) = \sum_{\hat{0} < x \prec y} \omega(\mathbf{a} \cdot \Psi([\hat{0}, x])) \cdot 2\mathbf{d} \cdot \omega(\Psi([y, \hat{1}])) + \sum_{\hat{0} \prec y} 2\mathbf{d} \cdot \omega(\Psi([y, \hat{1}])). \tag{3.3}$$

The term $\omega(\mathbf{a} \cdot \Psi([\hat{0}, x]))$ is the **cd**-index of a zonotope and hence is non-negative in the order $\succeq_q$ by Theorem 1.1. Similarly, the term $\omega(\Psi([y, \hat{1}]))$ is one half of the sum of **cd**-indices of the vertex figures of a zonotope and hence is also $\succeq_q$-non-negative. The result now follows by Lemma 3.3 and the property that the order $\succeq_q$ is preserved under addition. □

Proof of Theorem 3.1: Observe that any n-dimensional zonotope is obtained from the n-dimensional cube C_n by Minkowski adding line segments. Thus the result follows from Proposition 3.4. □

The second improvement of the zonotopal inequalities is when comparing the coefficients of $\mathbf{c}^k v$ and qv, that is, when u is equal to 1. Let $\square_q$ denote the coefficient of the monomial q in the **cd**-index of the k-dimensional cube C_k, that is, $\Box_q = \langle q|\Psi(C_k)\rangle$. For ease in notation, we introduce a second order relation.

Definition 3.5 *Let q be a* **cd***-monomial of degree k that contains at least one* **d** *and let z and w be two* **cd***-polynomials. Define the order relation $z \preceq'_q w$ on the* **cd***-polynomials z and w by $\left\langle (q - \Box_q \cdot \mathbf{c}^k) \cdot v | w - z \right\rangle \geq 0$ for all* **cd***-monomials v.*

Theorem 3.6 *Let Z be an n-dimensional zonotope (and more generally, let Z be the dual of the lattice of regions of an oriented matroid). Let q be a* **cd***-monomial of degree k that contains at least one* **d***. Then the* **cd***-index $\Psi(Z)$ satisfies the inequality $\Psi(Z) \succeq'_q \Psi(C_n)$. That is, for all* **cd***-monomials v of degree $n-k$ we have*

$$\left\langle (q - \Box_q \cdot \mathbf{c}^k) \cdot v \mid \Psi(Z) - \Psi(C_n) \right\rangle \geq 0.$$

The proof of Theorem 3.6 consists of the following lemma and two propositions.

Lemma 3.7 *Let z and w be two non-negative* **cd***-polynomials such that $z \succeq'_q 0$. Then we have $z \cdot \mathbf{d} \cdot w \succeq'_q 0$. Furthermore if $\deg(q) \leq \deg(z)$ we have that $z \cdot w \succeq'_q 0$.*

Proof: We would like to show for all **cd**-monomials v that $\left\langle (q - \Box_q \mathbf{c}^k)v | z\mathbf{d}w \right\rangle \geq 0$, where $k = \deg(q)$. Consider first the case when $k \leq \deg(z)$. Try to write $v = v_1 \cdot v_2$ such that $k + \deg(v_1) = \deg(z)$. If this is not possible both sides are equal to zero. If this is possible we have $\left\langle (q - \Box_q \mathbf{c}^k)v | z\mathbf{d}w \right\rangle = \left\langle (q - \Box_q \mathbf{c}^k)v_1 | z \right\rangle \cdot \langle v_2 | \mathbf{d}w \rangle \geq 0$. The second case is $k > \deg(z)$. Then directly we have $\left\langle \mathbf{c}^k v | z\mathbf{d}w \right\rangle = 0$. Also $\langle qv | z\mathbf{d}w \rangle \geq 0$, since both z and w have non-negative coefficients. The second statement of the lemma is proved by similar reasoning, where there is only the case: $\left\langle (q - \Box_q \mathbf{c}^k)v | zw \right\rangle = \left\langle (q - \Box_q \mathbf{c}^k)v_1 | z \right\rangle \cdot \langle v_2 | w \rangle \geq 0$. $\Box$

Proposition 3.8 *The* **cd***-index of the n-dimensional cube C_n satisfies $\Psi(C_n) \succeq'_q 0$.*

Proof: The proof is by induction on n. Observe that when $n < \deg(q)$ there is nothing to prove. When $n = \deg(q)$ the result is directly true. The induction step is based on the Purtill recursion for the **cd**-index of the n-dimensional cube; see [15, 23] or [16, Proposition 4.2]:

$$\Psi(C_{n+1}) = \Psi(C_n) \cdot \mathbf{c} + \sum_{i=0}^{n-1} 2^{n-i} \cdot \binom{n}{i} \cdot \Psi(C_i) \cdot \mathbf{d} \cdot \Psi(\Sigma_{n-i-1}).$$

By Lemma 3.7 we observe that all the terms in this expression are greater than 0 in the order $\succeq'_q$. $\Box$

Proposition 3.9 *Let Z be an n-dimensional zonotope and let Z' be the zonotope obtained by taking the Minkowski sum of Z with a line segment in the affine span of Z. Assume that all zonotopes W of dimension $n-1$ and less satisfy the relation $0 \preceq'_q \Psi(W)$. Then the order relation $\Psi(Z) \preceq'_q \Psi(Z')$ holds.*

Proof: The proof follows the same outline as the proof of Proposition 3.4. By Lemma 3.7 each term in equation (3.3) is non-negative in the order $\preceq'_q$. Since the property of being non-negative is preserved under addition, the result follows. □

We now prove Theorem 3.6.

Proof of Theorem 3.6: The proof is by induction. The base of the induction is $n = 0$ which is straightforward. For the induction step assume that every zonotope W of dimension k less than n satisfies the inequality $\Psi(C_k) \preceq'_q \Psi(W)$. Especially, we know that the **cd**-index of a lower dimensional zonotope is non-negative in the order $\preceq'_q$. Thus by Proposition 3.9 we know that $\Psi(Z) \preceq'_q \Psi(Z')$ holds for n-dimensional zonotopes. Now the theorem follows from Propositions 3.8. □

4 Concluding remarks

In the view of the lifting technique in [13], it is natural to consider the following conjecture.

Conjecture 4.1 *Let H be a* **cd***-polynomial homogeneous of degree k such that for all k-dimensional polytopes P the inequality $\langle H \mid \Psi(P)\rangle \geq 0$ holds. Then for all n-dimensional zonotopes (and more generally, the dual of the lattice of regions of an oriented matroid) the inequality*

$$\langle u \cdot H \cdot v \mid \Psi(Z) - \Psi(C_n)\rangle \geq 0$$

holds for all **cd***-monomials u and v such that the sum of their degrees is $n-k$, u does not end with* **c** *and v does not begin with* **c**.

Conjecture 4.1 is the zonotopal analogue of Conjecture 6.1 in [13]. Theorem 3.1 is the verification of Conjecture 4.1 in the case when $H = q - \Delta_q \cdot \mathbf{c}^k$. Moreover, in the light of Theorem 3.6 we also suggest the next conjecture.

Conjecture 4.2 *Let H be a* **cd***-polynomial homogeneous of degree k such that for all k-dimensional zonotopes Z (and more generally, the dual of the lattice of regions of an oriented matroid) the inequality $\langle H \mid \Psi(Z) - \Psi(C_k)\rangle \geq 0$ holds. Then for all n-dimensional zonotopes (oriented matroids) the inequality*

$$\langle H \cdot v \mid \Psi(Z) - \Psi(C_n)\rangle \geq 0$$

holds for all **cd***-monomials v of degree $n-k$.*

There are other natural questions that arise. For instance, is there a way to interpolate between Theorems 3.1 and 3.6? Such an interpolation would let the factor vary between the constants Δ_q and $\Box_q$, depending on the degree of the monomial u. Another inequality to consider is the following multiplicative version of Theorem 3.1:

Conjecture 4.3 *The* **cd***-index of a zonotope Z (and more generally, the dual of the lattice of regions of an oriented matroid) satisfies the inequality*

$$\frac{\langle uqv|\Psi(Z)\rangle}{\langle u\mathbf{c}^k v|\Psi(Z)\rangle} \geq \frac{\langle uqv|\Psi(C_n)\rangle}{\langle u\mathbf{c}^k v|\Psi(C_n)\rangle}.$$

More linear inequalities for the flag f-vector of zonotopes can be obtained by the Kalai convolution [20]. That is, if the two inequalities $\langle H_1|\Psi(Z)\rangle \geq 0$ and $\langle H_2|\Psi(P)\rangle \geq 0$ hold for all m-dimensional zonotopes, respectively all n-dimensional polytopes, then the inequality $\langle H_1 * H_2|\Psi(Z)\rangle \geq 0$ holds for all $(m+n+1)$-dimensional zonotopes. For an explicit description of the convolution on **cd**-polynomials, see [13, Proposition 2.2].

Finally, another class of linear inequalities for the flag f-vector of zonotopes have been obtained by Varchenko and Liu; see [18, 22, 30]. Recently, this class has been sharpened by Stenson [29].

Acknowledgements

I would like to thank the MIT Mathematics Department for their kind support where this research was initiated while the author was a Visiting Scholar. The author also thanks Margaret Readdy for many helpful discussions. This research was partially supported by National Science Foundation grant 0200624.

References

[1] M. Bayer, Signs in the *cd*-index of Eulerian partially ordered sets, *Proc. Amer. Math. Soc.* **129** (2001), 2219–2225.

[2] M. Bayer and L. Billera, Generalized Dehn-Sommerville relations for polytopes, spheres and Eulerian partially ordered sets, *Invent. Math.* **79** (1985), 143–157.

[3] M. Bayer and R. Ehrenborg, The toric h-vectors of partially ordered sets, *Trans. Amer. Math. Soc.* **352** (2000), 4515–4531.

[4] M. Bayer and G. Hetyei, Flag vectors of Eulerian partially ordered sets, *European J. Combin.* **22** (2001), 5–26.

[5] M. Bayer and A. Klapper, A new index for polytopes, *Discrete Comput. Geom.* **6** (1991), 33–47.

[6] L. J. Billera and R. Ehrenborg, Monotonicity of the **cd**-index for polytopes, *Math. Z.* **233** (2000), 421–441.

[7] L. J. Billera, R. Ehrenborg, and M. Readdy, The **c**-2**d**-index of oriented matroids, *J. Combin. Theory Ser. A* **80** (1997), 79–105.

[8] L. J. Billera, R. Ehrenborg, and M. Readdy, The **cd**-index of zonotopes and arrangements, *Mathematical essays in honor of Gian-Carlo Rota* (B. E. Sagan and R. P. Stanley, eds.), Birkhäuser, Boston, 1998, pp. 23–40.

[9] L. J. Billera and G. Hetyei, Linear inequalities for flags in graded partially ordered sets, *J. Combin. Theory Ser. A* **89** (2000), 77–104.

[10] L. J. Billera and N. Liu, Noncommutative enumeration in graded posets, *J. Algebraic Combin.* **12** (2000), 7–24.

[11] A. Björner, Shellable and Cohen-Macaulay partially ordered sets, *Trans. Amer. Math. Soc.* **260** (1980), 159–183.

[12] R. Ehrenborg, k-Eulerian posets, *Order* **18** (2001), 227–236.

[13] R. Ehrenborg, Lifting inequalities for polytopes, *Adv. Math* **193** (2005), 205–222.

[14] R. Ehrenborg and H. Fox, Inequalities for **cd**-indices of joins and products of polytopes, *Combinatorica* **23** (2003), 427–452.

[15] R. Ehrenborg and M. Readdy, The **r**-cubical lattice and a generalization of the **cd**-index, *European J. Combin.* **17** (1996), 709–725.

[16] R. Ehrenborg and M. Readdy, Coproducts and the **cd**-index, *J. Algebraic Combin.* **8** (1998), 273–299.

[17] R. Ehrenborg and M. Readdy, Homology of Newtonian coalgebras, *European J. Combin.* **23** (2002), 919–927.

[18] K. Fukuda, S. Saito, and A. Tamura, Combinatorial face enumeration in arrangements of hyperplanes and oriented matroids, *Discrete Appl. Math.* **31** (1991), 141–149.

[19] G. Kalai, Rigidity and the lower bound theorem. I, *Invent. Math.* **88** (1987), 125–151.

[20] G. Kalai, A new basis of polytopes, *J. Combin. Theory Ser. A* **49** (1988), 191–209.

[21] K. Karu, Hard Lefschetz theorem for nonrational polytopes, arXiv: math.AG/0112087 v4.

[22] N. Liu, "Algebraic and combinatorial methods for face enumeration in polytopes," Doctoral dissertation, Cornell University, Ithaca, New York, 1995.

[23] M. Purtill, André permutations, lexicographic shellability and the *cd*-index of a convex polytope, *Trans. Amer. Math. Soc.* **338** (1993), 77–104.

[24] R. P. Stanley, Finite lattices and Jordan-Hölder sets, *Algebra Universalis* **4** (1974), 361–371.

[25] R. P. Stanley, "Enumerative Combinatorics, Vol. I," Wadsworth and Brooks/Cole, Pacific Grove, 1986.

[26] R. P. Stanley, Generalized h-vectors, intersection cohomology of toric varieties, and related results, in: "Commutative Algebra and Combinatorics" (M. Nagata and H. Matsumura, eds.), *Advanced Studies in Pure Mathematics* **11**, Kinokuniya, Tokyo and North-Holland, Amsterdam/New York 1987, pp. 187–213.

[27] R. P. Stanley, Flag f-vectors and the *cd*-index, *Math. Z.* **216** (1994), 483–499.

[28] R. P. Stanley, A survey of Eulerian posets, in: "Polytopes: Abstract, Convex, and Computational," (T. Bisztriczky, P. McMullen, R. Schneider, A. I. Weiss, eds.), NATO ASI Series C, vol. 440, Kluwer Academic Publishers, 1994.

[29] Stenson, Families of tight inequalities for polytopes, preprint 2003.

[30] A. Varchenko, On the numbers of faces of a configuration of hyperplanes, *Soviet Math. Dokl.* **38** (1989), 291–295.

R. Ehrenborg, Department of Mathematics, University of Kentucky, Lexington, KY 40506, jrge@ms.uky.edu