Ma 113 Recitation Activity for the first period. **Fall 1996**

Recitation dates: Aug. 29, Sep. 3,5,10,12,17,19,24.

The First Skills Test is on September 24 and the relevant sections of the book are 1.4-1.7, 2.1-2.2, 2.5-2.7, 3.1. Here is the outline of the recitation activity for this period.

The problems which are marked as homework have to be submitted individually. The rest of the problems must be submitted as group work. All work assigned in one week is due in the next week during Tuesday's recitation.

All individual work must clearly show the individual's name, the list of problems attempted (including section numbers) and the date of submission.

All group work must clearly show the names of all group members who worked on the project. The name(s) of persons actually writing/typing the work must be clearly indicated. The work must state the problem(s) clearly and show all relevant steps involved.The date of submission is also necessary.

The problems which are useful for the Skills Test are marked as such.

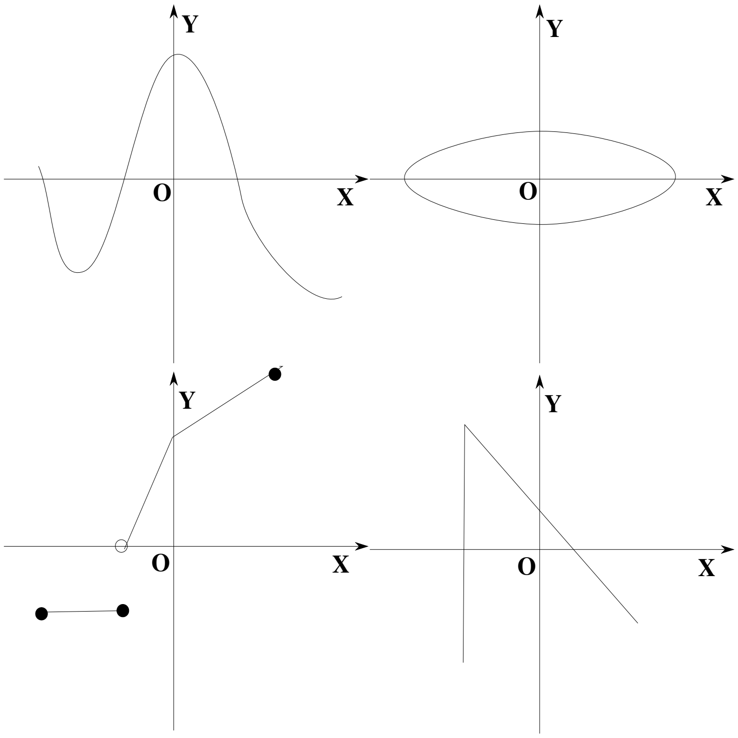
First Period: August 29 Sections up to 1.4 **Due Sep. 5** On section 1.1 to 1.3

Only the following problems on the first three sections need to be submitted. There are additional problems on these sections listed at the end. These should be worked on your own, until you feel confident. Note that there are many which are useful for the Skills Tests.

1. Find out about the various "flat tax" systems and make formulas for them.
2. Q.23 (1.1). Change the manufacturing price (originally 60 cents each) to other numbers and redo the problem. Finally, discuss the best price as a **function of** the manufacturing price. Discuss the domain and the formula for the function.

On section 1.4

1. Consider the function $f(x) = \sqrt{(\frac{1}{x}) - 1}$.
   For each of the following, justify your answer:
   (a) Can x be negative?
   (b) Can $x = 0$?
   (c) Can x be greater than 1?
   (d) What is the domain of f?
   (e) What is the range of f?
2. (a) The vertical line test states, "A curve is the graph of a function only if each vertical line $x = k$ intersects the graph at most once." Explain why this test works.
   (b) Given the graph of a function, is it ever possible that a horizontal line $y = k$ intersects it more than once? If not, explain why not. If so, give an example.
3. State whether the curve is the graph of a function of x. In case it is a function and you are asked for the domain and the range of the function, what answer can you provide? Discuss the validity of the question.



4. (a) **SKILLS** Sketch the graph of the piecewise-defined function

$$g(x) = \begin{cases} 2x+3 & x < 0 \\ x^2 & 0 \le x < 2 \\ 1 & x \ge 3 \end{cases}$$

(b) Use your graph to determine the domain and range of g.

5. (a) **SKILLS** Write a formula for A, the area of a circle, as a function of its radius, r.

(b) What is the domain of A? What is the range of A?

(c) Write a formula for d, the diameter of the circle, as a function of A.

6. **SKILLS** Express the area, A, of an equilateral triangle as a function of the length of a side, s.

7. **SKILLS** If $f(x) = x + 5$ and $g(x) = x^2 - 3$, find

$$f(x^3)\ ,\ g(3x+2)\ ,\ g(f(0))\ ,\ f(g(0))\ ,\ f(f(x))\ ,\ g(g(2))$$

8. Let $f(x) = \dfrac{x-1}{x}$. Show that $f(x) \cdot f(1-x) = 1$.

9. Determine whether the following functions are even, odd, or neither. Justify each answer graphically or algebraically.

$$\cos x\ ,\ \sin x\ ,\ \sin^2 x\ ,\ \tan x\ ,\ x\cos x\ ,\ y = |x|$$

10. **SKILLS** Given the graph of **any** $f(x)$, which of the following choices would produce a graph that is identical in shape to $f(x)$ but that is shifted 2 units up and 2 units to the left?

$$f(x)+2\ ,\ f(x)-2\ ,\ f(x+2)+2\ ,\ f(x-2)+2$$

or *none of these*?

11. Even functions are characterized by the fact that their graphs are symmetric about the y-axis. Are there functions whose graph is symmetric about the x-axis? Justify your answer.

12. **SKILLS** Suppose that f is an even function and that the point $(-3, 5)$ is on the graph of f. Find another point on the graph of f.

13. **SKILLS**

    (a) Suppose that an odd function is known to be increasing on the interval $[-1, 0]$. What can be said about its behavior on $[0, 1]$?

    (b) Suppose that an even function is known to be decreasing on $[-1, 0]$. What can be said about its behavior on $[0, 1]$?

14. **SKILLS** Complete the following table:

$f(x)$	$g(x)$	$(g \circ f)(x)$
$x - 7$	$\sqrt{x}$	
$x + 2$	$3x$	
	$\sqrt{x - 5}$	$\sqrt{x^2 - 5}$
$\dfrac{x}{x - 1}$	$\dfrac{x}{x - 1}$	
	$1 + \dfrac{1}{x}$	x
	$\dfrac{1}{x}$	x

15. **SKILLS** Suggested problems from the book are: Q. 1-7,12-15,16-27,30-32,35. Do the following as homework: **HOMEWORK** Sec. 1.4: Q.4,6,15,17,19,26,31.

Sep. 3,5 section 1.5 to 1.7 **Due Sep. 10** On section 1.5

1. Each of the following statements shows that the sine function is _not_ a polynomial. Explain why.

   (a) The equation $\sin x = 0$ has infinitely many solutions ($x = n\pi$ is a solution for every integer n).

   (b) The sine function is bounded below by -1 and above by 1 (i.e., for every x, $-1 \leq \sin x \leq 1$).

2. What point is common to every logarithmic function? What point is common to every exponential function?

3. **SKILLS** **True** or **false**?

   (a) The functions $y = \log_2 8$ and $8^y = 2$ are inverses.

   (b) Any rational function has a domain of all real numbers.

   (c) If $f(x)$ is a polynomial such that $f(a) = 0$, then $x - a$ is a factor of $f(x)$.

(d) The number 1 can be used as a base for a logarithmic function.

(e) The exponential function $y = 3^x$ grows more slowly than the polynomial $y = x^3$.

(f) The line $y = 0$ is a horizontal asymptote of every function of the form $y = b^x$ (with $b > 0$ and $b \neq 1$).

(g) The function $y = 1^x$ has domain $\{1\}$ and range $(-\infty, \infty)$.

4. $\boxed{\textbf{SKILLS}}$ Determine the domain and range of each of the given functions.

   (a) $y = x^2 + 4$

   (b) $y = \sqrt{x^2 - 4}$

   (c) $y = \dfrac{\cos x}{x}$

   (d) $y = \dfrac{2x}{(x-2)(x+1)}$

   (e) $f(x) = e^{x^2 - 8x + 15}$

   (f) $\ln((x-4)^2 - 1)$

   (g) $y = \dfrac{1}{\sqrt{9 - x^2}}$

5. $\boxed{\textbf{SKILLS}}$ Complete the following "rules" of logarithms and exponents:

   (a) $\log_a(xy) =$

   (b) $\log_a(x^y) =$

   (c) If $b > 0$, then $b^x b^y =$

   (d) If $b > 0$, then $(b^x)^y =$

6. Evaluate each expression:

   (a) $\log_6 \dfrac{1}{36}$

   (b) $e^{3 \ln 2}$

   (c) $4^e 4^{32+x}$

7. $\boxed{\textbf{SKILLS}}$ Without using a calculator or computer, evaluate the following using the properties of logarithms and the information supplied.

   (a) $\boxed{\textbf{SKILLS}}$ Given that $\ln 2 = a$ and $\ln 5 = b$, find $\ln 100$.

   (b) Let $\log_e 3 = 1.099$ and $\log_e 4 = 1.386$. Find $\log_e(\frac{1}{12})$.

8. $\boxed{\textbf{SKILLS}}$

   (a) Solve for x: $5^{2x-3} = \dfrac{1}{125}$ (*Hint*: rewrite using logarithms)

   (b) Let $f(x) = 20e^{.01x}$. Find a value of x for which $f(x) = 40$.

9. Use the laws of exponents $a^x a^y = a^{x+y}$ and $(a^x)^y = a^{xy}$ to explain each of the following:

(a) Explain why $a^0 = 1$ for any positive a. (*Hint*: what is the fundamental property of 1?)

(b) Explain why it should be true that $a^{\frac{1}{2}}$ is the (positive) square root of a. (*Hint*: if a is a number, what does one mean by $\sqrt{a}$?)

(c) Why is it true that $a^{-x} = \frac{1}{a^x}$? (*Hint*: if x is a number other than zero, what is the defining property of $\frac{1}{x}$?)

10. For the next few problems use the facts that $\log(ab) = \log(a) + \log(b)$ and $\log(a^x) = x\log(a)$.

    (a) Why is $\log(1) = 0$ regardless of the base used in the log?

    (b) Why is $\log(\frac{1}{a}) = -1$ for log base a?

    (c) If $\log(a) = 2.1$ and $\log(b^5) = 3.2$, what is $\frac{a}{b}$?

11. $\boxed{\textbf{SKILLS}}$ Find the exact trigonometric ratios for the angle $\theta = \dfrac{3\pi}{4}$.

12. (a) Use properties of the sine and cosine functions to show that $\tan x$ is an odd function.

    (b) Use trigonometric identities to prove that $(\sin x + \cos x)^2 = 1 + \sin(2x)$.

13. Indicate whether the function is even, odd, or neither. If a function is even or odd, show this using the facts that sine is an odd function and cosine is an even function.

$$\sin^2 x\ ,\ \cos^3 x\ ,\ \sin x + \cos x\ ,\ \csc x\ ,\ x\sin x\ ,\ x^2\cos x$$

14. $\boxed{\textbf{SKILLS}}$ Suppose that $y = 20x + 1$ is an equation of the line that intersects the curve $y = b^x$ at $x = 0$ and $x = 4$.

    (a) Find the number b.

    (b) What is the slope of the line that intersects the curve $y = \log_b x$ at $x = 1$ and $x = 81$?

15. $\boxed{\textbf{SKILLS}}$ Suggested problems from the book are: Q. 1-43. $\boxed{\textbf{HOMEWORK}}$ For homework, you should submit: Q. 1,4,7,16,18,21,27,25,30,32,35,42.

On section 1.6

1. $\boxed{\textbf{SKILLS}}$ Let $f(x) = \dfrac{x+1}{x-1}$ and $g(x) = \dfrac{1}{x}$. Find:

$$f+g\ ,\ f-g\ ,\ f\cdot g\ ,\ \frac{f}{g}\ ,\ \frac{g}{f}\ ,\ f\circ g\ ,\ g\circ f$$

2. $\boxed{\textbf{SKILLS}}$ Find the domain and range of each of the functions in problem 1.

3. True or false?

    (a) All functions pass the horizontal line test.

    (b) The inverse of a function $f(x)$ is another function, denoted $f^{-1}(x)$, which is the same as $\dfrac{1}{f(x)}$.

    (c) A constant function is <u>*not*</u> one-to-one.

(d) The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = -x$.

4. $\boxed{\textbf{SKILLS}}$ If $a > 0$ and $a \neq 1$, what is the domain and range of $f(x) = a^x$? What is $f^{-1}(x)$? What is the domain and range of $f^{-1}(x)$?

5. $\boxed{\textbf{SKILLS}}$

   (a) Find the inverse function of $f(x) = \dfrac{3-x}{1-x}$.

   (b) Check your work in part (a) by verifying that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

   (c) What does part (a) tell you about the graph of f?

6. This problem is about inverting the trigonometric function $g(x) = \cos x$.

   (a) Explain why g is <u>not</u> invertible on its full domain, $(-\infty, \infty)$.

   (b) Explain why g <u>does</u> have an inverse function g^{-1} when g is restricted to the domain $[0, \pi]$. What is the range of g on this restricted domain? What is the domain and range of g^{-1}?

   (c) On one set of axes, carefully draw graphs of g (restricted to the domain $[0, \pi]$) and g^{-1}. *Hint: Exploit symmetry to draw g^{-1} using g.

   (d) The function g^{-1} you drew in part (c) is known as the **arccosine function**. It's usually written either $\arccos(x)$ or $\cos^{-1}(x)$. For each value of x in the interval $[-1, 1]$, $y = \cos^{-1}(x)$ is the number in the interval $[0, \pi]$ whose cosine is x. Use your graphing calculator to plot the cosine and arccosine functions together.

7. Since $h(x) = e^x$ and $j(x) = \ln x$ are inverse functions, $e^0 =$ _____ implies that $\ln($_____$) =$ _____.

8. Prove that for any positive number a ($a \neq 1$), we have $\log_a(x) = \dfrac{\ln x}{\ln a}$. (*Hint: Rewrite $y = \log_a(x)$ using exponents, then take the natural log of both sides.)

9. Suppose that f is a periodic function. Explain why f is not invertible.

10. Let $f(x) = \dfrac{1}{x}$ and $g(x) = x$. How does $f \circ f$ differ from g?

11. (a) Find $f \circ g \circ h$ if $f(x) = x^4 + 1$, $g(x) = x - 5$, and $h(x) = \sqrt{x}$. What is the domain of $f \circ g \circ h$?

    (b) Express the function $G(x) = \dfrac{x^2}{x^2 + 4}$ in the form $f \circ g$ in two different ways.

    (c) If $f(x) = x + 4$ and $h(x) = 4x - 1$, find a function g such that $g \circ f = h$.

12. $\boxed{\textbf{SKILLS}}$ The following are general operations commonly employed by calculus students. For operations such as these to be valid they must be true for any possible choice of values for the unknown(s). For each of these state whether or not it is valid. If it is not valid give examples (using specific numbers) for which it fails.

    (a) $(x + y)^2 = x^2 + y^2$

    (b) $\sqrt{x^2} = x$

    (c) $x^a + x^b = x^{a+b}$

    (d) $|x + y| = |x| + |y|$

(e) $\sin(x+y) = \sin(x) + \sin(y)$

(f) $\sqrt{x^2+y^2} = |x| + |y|$

(g) $\sin(2x) = 2\sin x$

13. **SKILLS** Suggested problems from the book are: Q. 1,2,5,7,10-13. **HOMEWORK** For homework, you should submit: Q. 1,2,5,7.

On section 1.7

This section is about modeling. The ideas developed here will be used for the course project. Some of the suggested ideas are below. The project for a group will be determined by discussion with your instructor/TA. The actual project may take longer than the due date.

- "Guess my rule"-type problems. Each group submits a plotted graph to challenge another group to find the function. Creation of the graph should be done by the due date. The function may be found later.
- Modeling functions - each group is responsible to find and model at least two complicated real life functions. Ideas: Tax tables, daily temperatures, rainfalls and other natural data. This can be extended thru the course.
- More modeling - take objects of various shapes and try to guess volume/area functions at sections. Ideas: polygons, cubes with skew sections cylinders, cones, spheres, lengths of varying paths of prescribed types between two points.
- Finish some of the problems 1-7 from the end of the section.

Sep. 10,12 section 2.1,2.2, 2.5 **Due Sep. 17** On section 2.1

1. A particle moves in the plane from $A(-2,5)$ to the y-axis in such a way that $\Delta y = 3\Delta x$. Find its new coordinates.
2. The linear function $y = mx + b$ has a constant slope m. Explain what this conveys – graphically and in terms of rates – about its derivative function.
3. Let $f(x) = 5$, $g(x) = x$, and $h(x) = -2x$. Evaluate $f'(11), g'(-53)$, and $h'(37)$.
4. Suppose that $g(x) = f(x) + 5$ and that $f'(x)$ exists for all x.
   (a) Explain how the graphs of f and g are related.
   (b) How is the graph of g' related to the graph of f'? Explain.
   (c) If $f'(2) = 3$, what is $g'(2)$?
   (d) Let $h(x) = f(x) + k$ where k is a constant. How is the graph of h' related to the graph of f'?
5. Suppose that $f(a) = b$ and that $f'(x) = c$ for all x. Find an algebraic expression for $f(x)$.
6. Water is flowing into a large spherical tank at a constant rate. Let $V(t)$ be the volume of water in the tank and $H(t)$ be the height of the water level at time t.

(a) Give a physical interpretation of $\frac{dV}{dt}$ and $\frac{dH}{dt}$.

(b) Is $\frac{dV}{dt}$ positive, negative, or zero when the tank is one quarter full? Justify your answer.

(c) Is $\frac{dH}{dt}$ positive, negative, or zero when the tank is one quarter full? Justify your answer.

(d) Which of $\frac{dV}{dt}$ and $\frac{dH}{dt}$ is constant? Explain your answer.

7. Given that the function $p(x) = x^3 - 3x + 3$ has derivative $p'(x) = 3x^2 - 3$, answer the following:

   (a) Find an equation for the line tangent to $p(x)$ at the point $(0, 3)$.

   (b) At what points, if any, does the graph of $p(x)$ have horizontal tangent lines?

   (c) Using your graphing calculator, graph $p(x)$ and $p'(x)$ on the same axes. Verify your answers to parts (a) and (b).

8. Use the following information about a function $y = f(x)$ to graph the function for $-1 \le x \le 6$.

   (a) The graph of f is made of line segments joined end to end.

   (b) The graph of f starts at the point $(-1, 1)$.

   (c) The derivative of f is the step function defined by

$$f'(x) = \begin{cases} -2 & \text{if } -1 \le x < 0 \\ 0 & \text{if } 0 < x < 1 \\ 1 & \text{if } 1 < x < 4 \\ -1 & \text{if } 4 < x \le 6 \end{cases}$$

9. In the previous problem, at what points in the interval $[1, 6]$ is the derivative function f' undefined? Why?

10. Suppose that the position of a body at time t is given by the function $P(t)$.

    (a) Write an expression that represents the velocity of the body at time $t = 3.4$.

    (b) If velocity is viewed as an amount function, its rate function represents ____________. How can this rate function be derived from $P(t)$?

11. Use the following position and velocity functions for free-falling objects:
    - $s(t) = -16t^2 + v_0 t + h_0$, where v_0 =initial velocity and h_0 = initial height
    - $v(t) = -32t + v_0$

    (a) A pebble is dropped from a height of 600 feet. Find the pebble's velocity when it hits the ground.

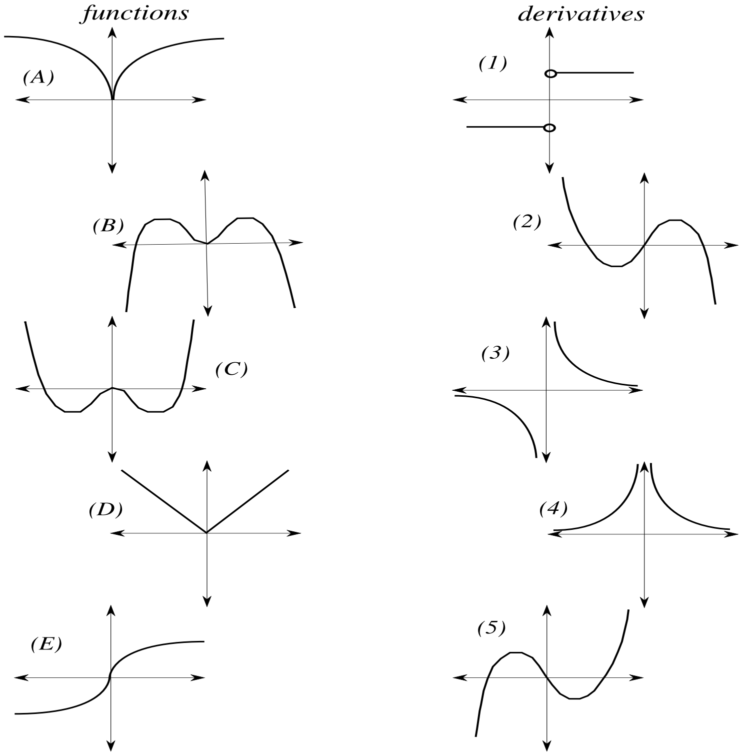
    (b) You and a friend are making a bet. You will drop a ball from a height of 100 feet. One second later your friend will drop another ball from a height of 75 feet. Your friend wants to bet you $100 that the ball he drops will hit the ground first. Should you take the bet?

12. **The racetrack principle:** Let f and g be functions defined for all x in $[a, b]$, and suppose that $f(a) = g(a)$. If $f'(x) \le g'(x)$ for all x in $[a, b]$, then $f(x) \le g(x)$ for all x in $[a, b]$.
    Suppose that $f(1) = 2$ and $f'(x) \le -3$ for all x in $[-5, 5]$.

(a) What does the racetrack principle imply about the value of $f(4)$?

(b) What does the racetrack principle imply about the value of $f(-2)$?

13. Matching graphs of functions with the graphs of their derivatives.



14. The converse of the racetrack principle says that if $f(a) = g(a)$ and $f(x) \leq g(x)$ for all x in $[a, b]$, then $f'(x) \leq g'(x)$ for all x in $[a, b]$. Show that this statement is not true in general by giving graphical examples of functions f and g that satisfy the hypotheses of the statement but not its conclusion.

15. (a) If $f(x) = x - \dfrac{1}{x}$, show that $f(1/x) = -f(x) = f(-x)$.

(b) Is f an even function, an odd function, or neither?

On section 2.2

1. In geometry, we define the tangent to a circle to be a line that meets the circle in exactly one point. Is this an appropriate definition for other curves in the plane? Explain. Illustrate your discussion with graphs.

2. **True** or **false**?

(a) If $f'(x) > 0$, then f is increasing.

(b) If $f''(x) > 0$, then $f' > 0$.

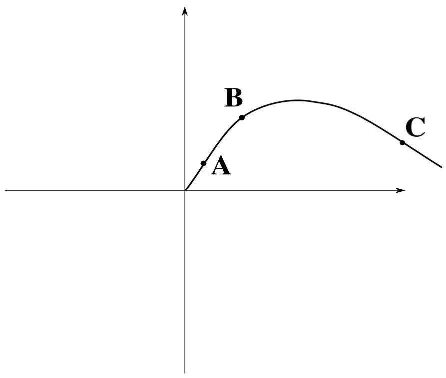
(c) If $f'(x)$ is increasing, then f is increasing.

(d) If $f(x) > 0$, then f' is increasing.

(e) If $f(x) > 0$, then f is increasing.

(f) If $f'(x) > 0$, then f' is increasing.

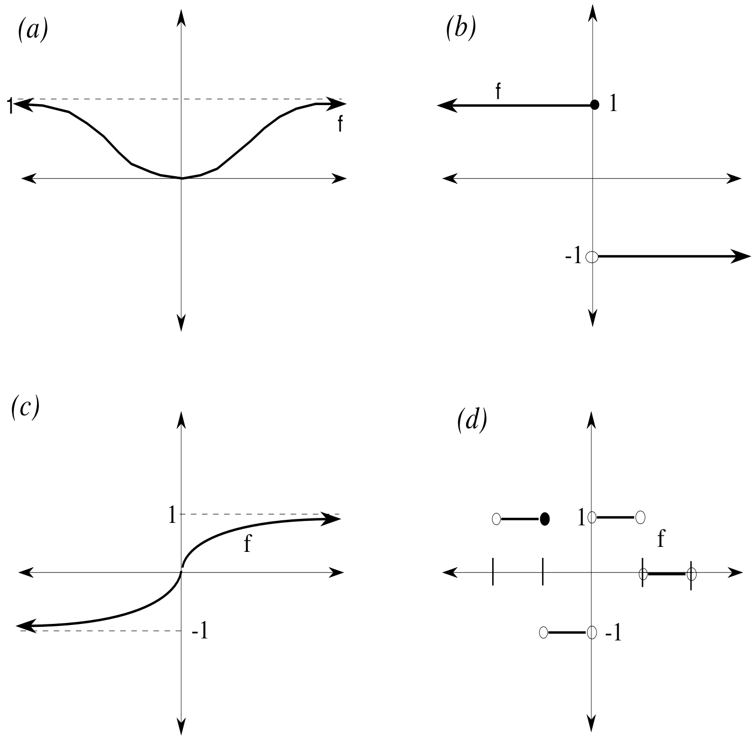
3. For the graph $y = f(x)$ arrange the following numbers in increasing order:
   (a) the slope of the curve at point A
   (b) the slope of the curve at point B
   (c) the slope of the curve at point C
   (d) the slope of the line AB
   (e) the number zero
   (f) the number one



4. Sketch the graph of a function g that is consistent with these data:

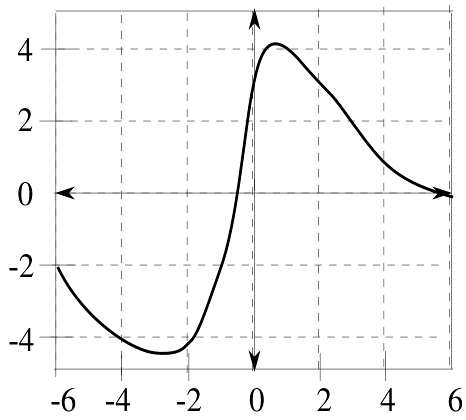
x	$g(x)$	$g'(x)$
-2	1	1
-1	-1	0
1	-1	-1
2	2	-2

5. Sketch the derivative of each of the functions shown below:



6. (a) What does it mean for a function to be **locally linear** at $x = a$?
   (b) Is the function $g(x) = -|x + 2|$ locally linear at $x = -2$? Explain.
   (c) Give an example of a function that is locally linear at $x = 5$.

7. The graph of a function f is shown below:



Graph of f

(a) Estimate $f'(-5)$, $f'(-3)$, $f'(-1)$, $f'(2)$, and $f'(4)$.

(b) Sketch a graph of f'.

8. Let $f(x) = x^3$ and $g(x) = x^{1/3}$.

   (a) Use zooming to estimate $f'(0)$, $f'(1/2)$, $f'(1)$, $f'(3/2)$, and $f'(2)$.

   (b) Use your results from part (a) to sketch a graph of f' over the interval $[-2, 2]$. (*Hint: Why is $f'(-1) = f'(1)$?)

   (c) Use your results from parts (a) and (b) to guess an algebraic formula for $f'(x)$.

   (d) Estimate $g'(-1)$ and $g'(1)$.

   (e) Why is $g'(0)$ undefined? What is the domain of g'? What is the domain of g?

   (f) It can be shown that $f'(4) = 48$. Use this fact to find $g'(64)$. (*Hint: How are the graphs of g and f related? The point $(4, 64)$ is on the graph of f and the point $(64, 4)$ is on the graph of g.)

9. The line tangent to f at $(5, 2)$ passes through the point $(0, 1)$. Find $f(5)$ and $f'(5)$. Justify your answers.

10. Suppose that $f(2) = 4$ and $f'(x) = \sqrt{x^3 + 1}$ for $x \geq -1$.

    (a) Find an equation for the line tangent to f at $x = 2$.

    (b) Let l be the tangent line from part (a). Does $l(0)$ overestimate or underestimate $f(0)$? Justify your answer.

    (c) Does $l(3)$ overestimate or underestimate $f(3)$? Justify your answer.

11. For each of the following pairs of functions, the constant A can be chosen so that $f'(x) = Ag(x)$ for every x in the domain of g. Find this value of A. (*Hint: Examine the graphs of f and g.)

    (a) $f(x) = x^3$, $g(x) = x^2$

    (b) $f(x) = e^x$, $g(x) = e^x$

1. Complete the following definition of the derivative of a function f:
   Let f be a function defined near and at $x = a$. The **derivative of f at $x = a$**, denoted $f'(a)$, is defined by the limit

$$f'(a) =$$

2. (a) The limit $\lim\limits_{h\to 0} \dfrac{(2+h)^3 - 8}{h}$ represents the derivative of some function f at some number a. What are f and a?
   (b) Calculate the limit in part (a).

3. If $g(3) = 5$ and the average rate of change of g from $x = -2$ to $x = 3$ is 6, what is the value of $g(-2)$?

4. Find the derivative of the given function using the definition. State the domain of both the function and its derivative.
   (a) $f(x) = \sqrt{4-x}$
   (b) $g(x) = \dfrac{2x}{x^2 - 9}$

5. Prove, starting from the definition (and drawing a picture to illustrate):
   (a) If $g(x) = f(x) + c$, then $g'(x) = f'(x)$.
   (b) If $g(x) = cf(x)$, then $g'(x) = f'(x)$.

6. **True** or **false**?
   (a) On any graph $y = f(x)$, the difference quotient $\dfrac{f(a+h) - f(a)}{h}$ represents the slope of the tangent line at $x = a$.
   (b) As the limit of slopes of secant lines, the derivative $f'(a)$ represents the slope of the tangent line at $x = a$.
   (c) The difference $\Delta x = (a+h) - a = h$ represents a change in the input variable from $x = a$ to $x = a + h$; the difference $\Delta f = f(a+h) - f(a)$ measures the change in the output $f(x)$.
   (d) Every function is differentiable at each point in its domain.
   (e) If $y = f(x)$ represents the position of a moving object at time t, then the derivative $f'(a)$ represents the average velocity over the interval from $x = a$ to $x = a + h$.
   (f) If $y = f(x)$ is any function, then the tangent line L at the point P may pass through another point P' on the graph of the function.

7. For the function $f(x) = \sin x$, use a difference quotient with $h = 0.001$ to estimate the value of $f'(0)$.

8. Ahmed travelled from Portland to Seattle. Due to road construction, he drove the first ten miles at a constant 30 mph. For the next 100 miles he drove at 75 mph. Then Ahmed took a ten minute break to drink a soda. Finally Ahmed drove the remaining 35 miles at a constant 110 mph.

(a) Plot Ahmed's distance along the road as a function of time.

(b) Plot Ahmed's velocity as a function of time.

(c) What was Ahmed's average speed for the trip (including the stop for the soda)?

(d) If Ahmed received two speeding tickets, each of which took five minutes, what does the average speed for the trip become?

9. The position $p(t)$ of an object at time t (in seconds) along the line with units marked in meters is given by $p(t) = 3t^2 + 1$.

   (a) Find the change in position of the object between time $t = 1$ and $t = 3$.

   (b) Find the average velocity of the object between time $t = 1$ and $t = 3$.

   (c) On one set of axes, sketch a graph of $p(t)$ and the secant line from $t = 1$ to $t = 3$. Find the equation of the secant line.

   (d) Find the average velocity of the object between time $t = 1$ and time $t = 1 + h$. What are the units of your answer?

   (e) Find the instantaneous velocity of the object at time $t = 1$.

   (f) Find an equation of the line tangent to the graph of $p(t)$ at $t = 1$. Add this line to the sketch you made in part (c) above.

10. Let $f(x) = e^x$. Tabulate $f(x)$ at $x = 1.998, 1.999, 2.000, 2.001, 2.002$ and use your results to answer the following questions. (Round numerical values of $f(x)$ to seven decimal places.)

   (a) Explain how these tabulated values show that f is "locally linear" near $x = 2$.

   (b) Estimate $f'(1.999), f'(2)$, and $f'(2.001)$ using the tabulated values of f.

   (c) Does $\dfrac{f(2.001) - f(2)}{0.001}$ overestimate or underestimate the exact value of $f'(2)$? Why? (*Hint: Look at a graph of $f(x)$ near $x = 2$.)

   (d) Estimate $f''(2)$ using the results from part (b).

Sep. 17,19 section 2.6,2.7, 3.1 **Due Sep. 24** On section 2.6

1. Define $g(x) = \begin{cases} 2x^2 + 1 & \text{if } x < 200 \\ 80,000 & \text{if } x = 200 \\ 400x + 1 & \text{if } 200 < x \end{cases}$

   Without graphing $g(x)$, answer the following:

   (a) What does it mean for a function to have a limit at a point a? Give the formal definition.

   (b) Does $g(x)$ have a limit at $x = 200$? Justify your answer.

   (c) Give the formal definition of continuity of a function.

   (d) For what value(s), if any, is the function $g(x)$ discontinuous? Why does the definition fail there?

2. (a) What is wrong with the equation $\dfrac{x^2 + 3x - 10}{x + 5} = x - 2$? (*Hint: What is the difference between the functions $f(x) = \dfrac{x^2 + 3x - 10}{x + 5}$ and $g(x) = x - 2$?)

(b) In view of part (a), explain why the equation $\lim_{x\to -5} \frac{x^2+3x-10}{x+5} = \lim_{x\to -5} x-2$ is correct. (*Hint: You may find it helpful to use your graphing calculator.)

3. Suppose that f is continuous at $x = 3$ and that $\lim_{x\to 3} f(x) = 17$. Indicate which of the following statements about f *must* be true, which *might* be true, and which *cannot* be true. Justify your answers.

   (a) 3 is in the domain of f.

   (b) $f(3) = 17$.

   (c) $\lim_{x\to 3^-} f(x) = 17$.

4. Let $h(x) = \begin{cases} x & \text{if } x < 0 \\ x^2 & \text{if } 0 < x \leq 2 \\ 8 - x & \text{if } x > 2 \end{cases}$

   Find each of the following limits if it exists:

   (a) $\lim_{x\to 0^+} h(x) =$ (b) $\lim_{x\to 0} h(x) =$ (c) $\lim_{x\to 1} h(x) =$

   (d) $\lim_{x\to 2^-} h(x) =$ (e) $\lim_{x\to 2^+} h(x) =$ (f) $\lim_{x\to 2} h(x) =$

5. (a) Evaluate $\lim_{h\to 0} \frac{(7+h)^2 - 49}{h}$.

   (b) What does the expression you evaluated in part (a) represent?

6. (a) The function $f(x)$ is defined by $f(x) = \frac{x^2-1}{x-1}$ when $x \neq 1$ and by $f(1) = 2$. Is f continuous at $x = 1$? Explain.

   (b) Define $g(3)$ in such a way that extends $g(x) = \frac{x^2-9}{x-3}$ to be continuous at $x = 2$.

7. Sketch the functions $g(x)$, $h(x)$, and $f(x)$ such that the following conditions hold:

   (a) $\lim_{x\to 5} g(x) = 10$ however $g(5)$ is not defined.

   (b) $\lim_{x\to 2} h(x) = 3$ however $h(2) \neq 3$.

   (c) $\lim_{x\to 0} f(x)$ does not exist, however $f(0) = 2$.

8. Remember that a function $f(x)$ has a limit at a point a if and only if the right-hand and left-hand limits at a exist and are equal. In symbols,

   $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^+} f(x) = L$ <u>and</u> $\lim_{x\to a^-} f(x) = L$. (L is a <u>finite</u> number!)

   State whether each of the following is true or false and justify your answer:

   (a) $\lim_{x\to 2} f(x) = 3$, where $f(x) = \begin{cases} x & \text{for } x \leq 2 \\ 0 & \text{for } x > 2 \end{cases}$

   (b) $\lim_{x\to 3} f(x) = 1$, where $f(x) = \begin{cases} x - 2 & \text{for } x \leq 3 \\ -x^2 + 8x - 14 & \text{for } x > 3 \end{cases}$

(c) $\lim_{x \to 0} f(x) = 0$, where $f(x) = \begin{cases} 1 & \text{for } x \le 0 \\ 0 & \text{for } x > 0 \end{cases}$

(d) $\lim_{x \to 0} \frac{|x|}{x} = 1$

(e) $\lim_{x \to 0} x^3 = 0$

9. Find the points (if any) at which the following functions are *discontinuous*.

   (a) $y = \frac{x}{x+1}$

   (b) $f(x) = \frac{x+3}{x^2 - 3x - 10}$

   (c) $y = |x-1|$

   (d) $g(x) = \frac{\cos x}{x}$

10. The *signum* (or sign) *function*, denoted by $sgn(x)$, is defined by

$$sgn(x) = \begin{cases} -1 & \text{if x<0} \\ 0 & \text{if x = 0} \\ 1 & \text{if x>0} \end{cases}$$

   (a) Sketch the graph of this function.

   (b) Find each of the following limits or explain why it does not exist:

      i. $\lim_{x \to 0^+} sgn(x) =$

      ii. $\lim_{x \to 0^-} sgn(x) =$

      iii. $\lim_{x \to 0} sgn(x) =$

      iv. $\lim_{x \to 0} |sgn(x)| =$

   (c) Is it true that $\lim_{x \to 0} |sgn(x)| = sgn(0)$? What can you conclude about the continuity of the function $|sgn(x)|$?

11. Given $f(x) = \begin{cases} x^2 & \text{if } x < -5 \\ 20 & \text{if } x = -5 \\ -2x + 15 & \text{if } x > -5 \end{cases}$

    Is $f(x)$ continuous at $x = -5$? *Clearly* explain why or why not.

12. Find all values of a that make the function $f(x)$ continuous:

$$f(x) = \begin{cases} ax^2 & \text{if } x \ge 2 \\ \frac{a^2}{4}x^2 + 3 & \text{if } x < 2 \end{cases}$$

13. (a) Sketch the graph of a function that has a limit at $x = 4$ but is discontinuous there.

    (b) Sketch the graph of a function that has no limit at $x = -1$ and is discontinuous there.

    (c) Is it possible to sketch the graph of a function f that is continuous at a point a but $\lim_{x \to a} f(x)$ does not exist? If yes, do so; else explain why not.

14. For each of the following, use the value of δ given to prove the limit. (Assume $\epsilon < 1$.)

(a) $\lim_{x \to 5} x = 5$; $\delta = \epsilon$

(b) $\lim_{x \to 2} 6 = 6$; $\delta = 0.2$

15. Graph a function with all of the following properties:

(a) $\lim_{x \to 0} f(x) = 5$

(b) $f(0) \neq 5$

(c) $\lim_{x \to 1} f(x)$ is undefined

(d) $\lim_{x \to 1^-} f(x)$ is defined and is greater than 0

(e) $\lim_{x \to 1^+} f(x)$ is defined

(f) $\lim_{x \to 2} f(x) = \lim_{x \to 3} f(x)$ and $f(2) \neq f(3)$

On section 2.7

1. (a) **Definition: (Limit at infinity)** $\lim_{x \to \infty} f(x) = L$ means that $f(x) \to$ __________ as $x \to$ __________.

(b) **Definition: (Infinite limit)** $\lim_{x \to a} f(x) = \infty$ means that $f(x) \to$ __________ as $x \to$ __________.

(c) Sketch graphs of functions that demonstrate each of the above definitions.

2. **True** or **false**?

(a) Rational functions have horizontal asymptotes, if at all, at roots of the denominator.

(b) Polynomials can't have vertical asymptotes.

(c) No polynomial has a horizontal asymptote.

(d) If both f and g are continuous at $x = a$, then each of the functions $f + g$, $f - g$, $f \cdot g$, and $\dfrac{f}{g}$ is also continuous at $x = a$.

(e) $\lim_{x \to \infty} \dfrac{1}{x^n} = 0$ for *any* n.

3. Find all horizontal and vertical asymptotes of each curve. Check your work by graphing the curve (using your calculator) and estimating the asymptotes.

(a) $y = \dfrac{x}{x+4}$

(b) $y = \dfrac{x^3}{x^2 + 3x - 10}$

(c) $f(x) = \dfrac{x^2+4}{x^2-1}$

4. Evaluate the following limits:

(a) $\lim_{x \to 0} \dfrac{x(x^2-1)}{x^2}$

(b) $\lim_{x\to 9} \dfrac{\sqrt{x}-3}{x-9}$

(c) $\lim_{x\to -\infty} \dfrac{3}{x^2}$

(d) $\lim_{x\to(\pi/2)^-} \dfrac{2}{1+e^{\tan x}}$

(e) $\lim_{t\to\infty} \dfrac{3t^2-6t}{4t-8}$

(f) $\lim_{x\to\infty} (\sqrt{x^2+1}-x)$

(g) $\lim_{x\to 4} \dfrac{|x-4|}{x-4}$

(h) $\lim_{x\to -\infty} (\pi/4)^x$

(i) $\lim_{x\to\infty} \dfrac{6x^2+5x}{(1-x)(2x-3)}$

(j) $\lim_{x\to\infty} \dfrac{x}{\sqrt{x-1}}$

5. (a) Sketch the graph of the function g defined by

$$g(x) = \begin{cases} (x-2)^2 & \text{if } x < 3 \\ 2 & \text{if } x = 3 \\ 4-x & \text{if } x > 3 \end{cases}$$

(b) Compute the limits:

$\lim_{x\to 3^+} g(x)$ $\qquad$ $\lim_{x\to 3^-} g(x)$ $\qquad$ $\lim_{x\to 3} g(x)$

(c) State the intervals on which g is continuous and the points where it is discontinuous, if any. Explain why g is discontinuous at any such points.

6. Suppose that $\lim_{x\to 1} f(x) = 5$ and $\lim_{x\to 1} g(x) = -2$. Use this information to evaluate each of the following limits.

(a) $\lim_{x\to 1}(f(x)+g(x))$

(b) $\lim_{x\to 1}(g(x)-f(x))$

(c) $\lim_{x\to 1} \dfrac{f(x)+2}{f(x)\cdot g(x)}$

7. Give examples of polynomials $p(x)$ and $q(x)$ such that $\lim_{x\to\infty} p(x) = \infty$, $\lim_{x\to\infty} q(x) = \infty$, and

(a) $\lim_{x\to\infty} \dfrac{p(x)}{q(x)} = \infty.$

(b) $\lim_{x\to\infty} \dfrac{p(x)}{q(x)} = 2.$

(c) $\lim_{x\to\infty} \dfrac{p(x)}{q(x)} = 0.$

(d) $\lim_{x\to\infty} \frac{p(x)}{q(x)} = 1$ and $\lim_{x\to\infty} (p(x) - q(x)) = 3$.

8. **Theorem (The squeeze principle):** Suppose that $f(x) \leq g(x) \leq h(x)$ for all x near $x = a$. If $f(x) \to L$ and $h(x) \to L$ as $x \to a$, then $g(x) \to L$ as $x \to a$.
   Use the above theorem to evaluate each of the following limits:

   (a) $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$ (*Hint: Use the fact that the sine function is bounded.)

   (b) $\lim_{t\to 0} \frac{2\sin t\cos t}{t}$

   (c) $\lim_{\theta\to\infty} \frac{\cos^2\theta}{\theta^2}$

9. If $3x \leq f(x) \leq x^3 + 2$ for $0 \leq x \leq 2$, evaluate $\lim_{x\to 1} f(x)$. Clearly explain your reasoning.

10. Sketch the graph of a function f that has *all* of the following properties. (Be sure to label axes and indicate units on each axis.)

    (a) f has domain $(-3, \infty)$

    (b) f has range $(-\infty, 2)$

    (c) $f(-1) = 0$; $f(0) = 0$; $f(5) = 0$

    (d) $\lim_{x\to -3^+} f(x) = -\infty$

    (e) $\lim_{x\to 2} f(x) = -3$

    (f) $\lim_{x\to\infty} f(x) = 2$

11. Let P and Q be polynomials. Find $\lim_{x\to\infty} \frac{P(x)}{Q(x)}$ if the degree of P is

    (a) less than the degree of Q

    (b) greater than the degree of Q

    (c) the same as the degree of Q

12. Use the squeeze principle to find $\lim_{x\to c} f(x)$ given that $\left|\frac{f(x) - f(c)}{x - c}\right| \leq M \quad$ for $x \neq c$.

13. Suppose that $|f(x)| \leq g(x)\ \forall\ x$, where $\lim_{x\to a} g(x) = 0$. Find $\lim_{x\to a} f(x)$.

On section 3.1

1. $\boxed{\textbf{SKILLS}}$ Differentiate the following functions:

   (a) $F(y) = \frac{-4y^3}{3}$

   (b) $y = \frac{1}{\sqrt{x}} + \ln 3 + e^2 x$

   (c) $f(x) = (x^3 - 8)(\frac{2}{x} - 1)$

(d) $y = \pi^3$

(e) $h(x) = \dfrac{3}{x^2} + \dfrac{4}{3x}$

(f) $g(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, i.e. a general nth degree polynomial

2. $\boxed{\textbf{SKILLS}}$ If $j(x) = x^2 + 2x - 3$, where is the slope of j positive? negative? Where are the tangent lines to j horizontal?

3. Let $f(x) = -3x^3 + 5x^2 - 6x$.

   (a) Find $\dfrac{dy}{dx}$.

   (b) What is the derivative of f at $x = 2$?

   (c) What is the instantaneous rate of change of f at the point $(2, 16)$?

   (d) What is the slope of the tangent line to f that passes through the point $(2, 16)$?

   (e) Write a sentence describing the difference between parts (a) and (b).

   (f) Write a sentence explaining why parts (c) and (d) are simple after completing part (b).

4. $\boxed{\textbf{SKILLS}}$ **True** or **false**?

   (a) Differentiation and antidifferentiation are inverse processes.

   (b) The domain of f' can be larger than the domain of f.

   (c) The domain of f' can be smaller than the domain of f.

   (d) A function f can have at most one antiderivative.

   (e) If $H'(x) = J'(x)$ for all x, then $H(x) = J(x)$.

   (f) Functions that differ by a constant have the same derivative.

   (g) $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$.

   (h) $\dfrac{d}{dx}(e^x) = xe^{x-1}$.

5. The voltage in a certain electrical circuit is 100 volts. If the current (in Amperes) is I and the resistance (in Ohms) is R, then by Ohm's law, $I = 100/R$. If R is increasing, find the instantaneous rate of change of I with respect to R at:

   (a) any resistance R

   (b) a resistance of 20 Ohms.

6. Find an equation of the tangent line to the graph of $y = x^4$ at the point where the value of the third derivative is 12.

7. Suppose $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$. (That is, P(x) is a polynomial of degree n.)

   (a) What can you say about $P^{(n)}(x)$? (*Hint*: It might be helpful to differentiate some specific polynomials – and don't simplify your coefficients along the way.)

(b) Show that $P^{(k)}(x)$ is the zero polynomial if $k > n$.

8. Suppose f describes the motion of an object. The velocity of the object at time $t = a$ is given by

$$v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

(a) Suppose the displacement of an object moving at time t sec is given, in feet, by the function $f(t) = t^2 - 8t - 4$. What is the (instantaneous) velocity at time $t = 3$?

(b) At what time t will the above object be exactly at rest; that is, have velocity equal to zero?

9. The volume V of a right circular cone of height h and radius r is given by $V = \dfrac{\pi r^2 h}{3}$.

(a) Find the rate of change of the volume with respect to the radius assuming the height is constant.

(b) Find the rate of change of the volume with respect to the height assuming the radius is constant.

(c) Find the rate of change of the height with respect to the volume assuming the radius is constant.

10. $\boxed{\textbf{SKILLS}}$ Find an antiderivative of each of the following functions. Check your answer by differentiating.

(a) $y = 18x^5 - 6x^2 + 4x - 19$

(b) $g(x) = \sqrt[3]{x} + x^{\frac{-1}{3}}$

(c) $f(x) = \frac{2}{x} - \pi^e + 4x^{\frac{2}{5}}$

(d) $H(x) = j'(x)$

11. A gleeful student, elated over acing a Calculus 113 exam, hurls a somewhat large Calculus book directly upward from the ground. It moves according to the law $s(t) = 96t - 16t^2$ where t is the time in seconds after it is thrown and $s(t)$ is the height in feet above the ground at time t. Find:

(a) the velocity of the book after 1.5 seconds

(b) the maximum height the book reaches

(c) the average speed of the book during its upward rise

(d) how long it would take for the 6 foot tall student to be hit on the head by the book.

12. Find points on the graph of $q(x) = \dfrac{1}{2}x^2 - 5x + 1$ at which:

(a) $q''(x) = q(x)$

(b) $q''(x) = q'(x)$

13. According to Boyle's Law, if the temperature of a confined gas is held fixed, then the product of the pressure P and the volume V is a constant. Suppose that, for a certain gas, $PV = 8000$, where P is measured in pounds per square inch and V is measured in cubic inches.

(a) Find the average rate of change of P as V increases from 200 in^3 to 250 in^3.

(b) Express V as a function of P and show that the instantaneous rate of change of V with respect to P is proportional to the inverse square of P.

14. Let $m(s) = 2s^6 - 15s^4 + 24s^2 + 3$. Find all points at which $m'(s) = 0$. (*Hint*: you should be able to factor $m'(s)$ just by looking at it for a while.)

15. Find $f(x)$ if $f''(x) = \sqrt{x} + \frac{1}{x^3}$, $f(1) = 0$, and $f(0) = -1$.

Additional problems from the first three sections.

These three sections are not expected to be covered in regular lectures and these topics were covered in pre-calculus. They will, however, be included in skills tests and may form parts of various projects. You are not required to submit any of these, but you should make sure that you learn the material.

Problems from 1.1

1. Q. 13. What do you need to do if the triangle is replaced by a circle of radius 5, centered at the origin? Find the formula, or try to work it out yourself!

2. $\boxed{\textbf{SKILLS}}$ Q. 2-5,10,11,21

Problems from 1.2

1. $\boxed{\textbf{SKILLS}}$ Q. 1-11,16-24,38-41

2. Work out the problem 35 by common sense and verify by finding the appropriate formula.

3. Q. 48,49

4. Q. 78,79

Problems from 1.3

1. Q. 1. The sin function has well known approximations of higher order. Find them and redo the exercise.

2. $\boxed{\textbf{SKILLS}}$ Q. 2,3.

3. Q. 8-11.

4. Q. 19-24. This can be done with other functions. Try some and try to analyze when you get an interesting question.