

# Alternation points and bivariate Lagrange interpolation

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## Abstract

Given  $m + 1$  strictly decreasing numbers  $h_0, h_1, \dots, h_m$ , we give an algorithm to construct a corresponding finite sequence of orthogonal polynomials  $p_0, p_1, \dots, p_m$  such that  $p_0 = 1$ ,  $p_j$  has degree  $j$  and  $p_{m-j}(h_n) = (-1)^n p_j(h_n)$  for all  $j, n = 0, 1, \dots, m$ . Using these polynomials, we construct bivariate Lagrange polynomials and cubature formulas for nodes that are points in  $\mathbb{R}^2$  where the coordinates are taken from given finite decreasing sequences of the same length and where the indices have the same (or opposite) parity.

*Keywords:* orthogonal polynomials, cubature, Christoffel-Darboux formula, Bézout identity

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## 1. Introduction

Our object is to show that every decreasing finite sequence of real numbers is the set of alternation points of a finite sequence of orthogonal polynomials and to apply this to construct Lagrange polynomials and cubature formulas for the even and odd nodes of the Cartesian product of the points.

A motivating example of alternation points is the Chebyshev points  $h_n = \cos(n\pi/m)$  and the corresponding polynomials are the Chebyshev polynomials  $T_n$ , where  $T_n(\cos \theta) = \cos(n\theta)$ . In previous papers [19, 23, 3], the alternation property (given in the Abstract) was used implicitly to construct two sets of bivariate polynomials having common zeros. The zeros were pairs of Chebyshev points where both indices of all pairs have the same or have opposite parity. We call two such sets of common zeros the even and odd product nodes, respectively.

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They were shown to be nodes of bivariate Lagrange polynomials of degree  $m$  and nodes of minimal or near minimal cubature formulas of degree  $2m - 1$ .

Our discussion allows us to replace the Chebyshev points by any decreasing finite sequence of real numbers. We do this by constructing orthogonal polynomials for which the numbers are alternation points. Our algorithm to compute these polynomials follows an argument of Wendroff [21] and has structure similar to the extended Euclidean algorithm [18]. Lagrange polynomials are constructed from the associated bivariate reproducing Kernel and are used to obtain a cubature formula as in [12, 13, 14].

## 2. Alternation points

Let  $m$  be a positive integer and let  $p_0, p_1, \dots, p_m$  be real polynomials satisfying a truncated three-term recurrence relation

$$p_0(x) = 1, \quad p_1(x) = a_0x + b_0, \quad (1)$$

$$p_{j+1}(x) = (a_jx + b_j)p_j(x) - c_jp_{j-1}(x), \quad j = 1, \dots, m - 1, \quad (2)$$

where  $a_0 > 0$  and  $a_j, c_j > 0$  for  $j = 1, \dots, m - 1$ .

It follows from Favard's theorem [6, 16, 2] that the polynomials  $p_0, p_1, \dots, p_m$  extend to a sequence of polynomials that are orthogonal with respect to a positive definite moment functional, which in turn can be represented by a (positive) finite Borel measure. Note that there are many choices for the moment functional and the measure. (For example, one can define  $a_n = a_{m-1}$ ,  $b_n = b_{m-1}$  and  $c_n = c_{m-1}$  for all  $n \geq m$ .) Conversely, it is well known that every set of polynomials  $p_0, \dots, p_m$  satisfies (1) and (2) when the polynomials are orthogonal with respect to some finite Borel measure [1, p. 244].

**Definition 1.** We say that real numbers  $h_0 > h_1 > \dots > h_m$  are alternation points for polynomials  $p_0, p_1, \dots, p_m$  satisfying (1) and (2) if

$$p_{m-j}(h_n) = (-1)^n p_j(h_n), \quad j, n = 0, 1, \dots, m. \quad (3)$$

Note that the cases  $j = 0$  and  $j = 1$  imply that  $p_m(h_n) = (-1)^n$  and  $\pi_m(h_n) = 0$  for  $n = 0, 1, \dots, m$ , where  $\pi_m = p_1p_m - p_{m-1}$ . We shall see that there are many possible choices for the polynomials  $p_j$ . In particular, we may assume that  $c_j = 1$  for  $j = 1, \dots, m - 1$ , but we may not assume that the polynomials  $p_j$  are monic.

Basic examples are the alternation points for the four kinds of Chebyshev polynomials. These are given for each  $m$  in the last column of Table 1.

**Table 1: The four kinds of Chebyshev polynomials**

Kind	Definition for $x = \cos \theta$	weight	$h_i = \cos \theta_i$
1st	$T_n(x) = \cos n\theta$	$w_1(x) = \frac{2}{\pi\sqrt{1-x^2}}$	$\theta_i = \frac{i\pi}{m}$
2nd	$U_n(x) = \frac{\sin(n+1)\theta}{\sin \theta}$	$w_2(x) = \frac{2}{\pi} \sqrt{1-x^2}$	$\theta_i = \frac{(i+1)\pi}{m+2}$
3rd	$V_n(x) = \frac{\cos(n+1/2)\theta}{\cos(\theta/2)}$	$w_3(x) = \frac{1}{\pi} \sqrt{\frac{1+x}{1-x}}$	$\theta_i = \frac{i\pi}{m+1}$
4th	$W_n(x) = \frac{\sin(n+1/2)\theta}{\sin(\theta/2)}$	$w_4(x) = \frac{1}{\pi} \sqrt{\frac{1-x}{1+x}}$	$\theta_i = \frac{(i+1)\pi}{m+1}$

More generally, recall from [12] that Geronimus polynomials are a sequence  $\{p_n\}_{n=0}^\infty$  of polynomials satisfying recursive equations

$$p_0(x) = 1, \quad p_1(x) = ax + b,$$

$$p_{n+1}(x) = (cx + d)p_n(x) - p_{n-1}(x), \quad n \geq 1,$$

where  $a, b, c$  and  $d$  are constants with  $a, c > 0$ . The case  $c = 2$  and  $d = 0$  includes all four kinds of Chebyshev polynomials. In particular,  $T_n$  is obtained when  $p_1(x) = x$ ,  $U_n$  is obtained when  $p_1(x) = 2x$ ,  $V_n$  is obtained when  $p_1(x) = 2x - 1$  and  $W_n$  is obtained when  $p_1(x) = 2x + 1$ . It was shown in [12] that for any sequence of Geronimus polynomials  $\{p_n\}_{n=0}^\infty$  and for any positive integer  $m$  there exist alternation points for  $p_0, p_1, \dots, p_m$ . The following shows that a converse holds.

**Proposition 1.** *If  $\{p_n\}$  is a sequence of orthogonal polynomials such that alternation points exist for each positive integer  $m$ , then  $\{p_n\}$  is a sequence of Geronimus polynomials.*

This proposition is an easy consequence of a more general fact.

**Lemma 2.** *If  $p_0, p_1, \dots, p_m$  are polynomials satisfying (1) and (2) for which there exist alternation points, then*

$$a_{m-j} = a_j c_{m-j}, \quad b_{m-j} = b_j c_{m-j}, \quad c_{m-j} c_j = 1, \quad (4)$$

for  $j = 1, \dots, m-1$ .

PROOF. Let  $0 \leq n \leq m$ . By (2), we have

$$p_{j+1}(h_n) = (a_j h_n + b_j) p_j(h_n) - c_j p_{j-1}(h_n), \quad (5)$$

$$p_{m-j+1}(h_n) = (a_{m-j} h_n + b_{m-j}) p_{m-j}(h_n) - c_{m-j} p_{m-j-1}(h_n). \quad (6)$$

Applying (3) in (6) and dividing by  $(-1)^n$ , we obtain

$$p_{j-1}(h_n) = (a_{m-j}h_n + b_{m-j})p_j(h_n) - c_{m-j}p_{j+1}(h_n) \quad (7)$$

Let  $q_j(x) = (A_jx + B_j)p_j(x) + C_jp_{j-1}(x)$ , where

$$A_j = a_{m-j} - c_{m-j}a_j, \quad B_j = b_{m-j} - c_{m-j}b_j, \quad C_j = c_{m-j}c_j - 1.$$

Substituting (5) into (7), we see that  $q_j(h_n) = 0$  for all  $n$ . Since  $q_j$  is a polynomial of degree at most  $m$  with  $m+1$  roots, it follows that  $q_j$  is the zero polynomial and hence (4) holds.

Another interesting example of alternation points is the case of successive integers  $0, 1, \dots, m$ . In reverse order, these are alternation points for the polynomials  $p_j(x) = k_j(m-x)$ ,  $j = 0, 1, \dots, m$ , where  $k_j$  is the binary Krawtchouk polynomial of degree  $j$  given by

$$k_j(x) = \sum_{i=0}^j (-1)^i \binom{x}{i} \binom{m-x}{j-i}.$$

The polynomials  $p_j$  satisfy (1) and (2) with

$$a_j = \frac{2}{j+1}, \quad b_j = -\frac{m}{j+1}, \quad c_j = \frac{m-j+1}{j+1},$$

and are orthogonal in the discrete inner product

$$(p, q) = \frac{1}{2^m} \sum_{j=0}^m \binom{m}{j} p(j)q(j). \quad (8)$$

The alternation property for  $h_n = m-n$  follows from the identity  $k_{m-j}(n) = (-1)^n k_j(n)$  for  $j, n = 0, 1, \dots, m$ . The above facts about Krawtchouk polynomials are consequences of equations that are given along with references in [15]. Note that the hypothesis of Proposition 1 does not apply in this case.

Apparently, finding polynomials so that a given set of decreasing numbers are alternation points with respect to the polynomials requires finding a solution of a complicated set of algebraic equations. However, we show that this can be accomplished by a natural extension of the Euclidean algorithm to obtain equivalent Bézout identities.

To begin, let  $A$  and  $B$  be polynomials with positive leading coefficients and simple real zeros. Suppose that  $A$  and  $B$  have degree  $n$  and  $n-1$ , respectively,

and that their zeros interlace, i.e., exactly one zero of  $B$  lies between any two consecutive zeros of  $A$ . In [21], Wendroff showed that then  $A$  and  $B$  are part of a sequence of orthogonal polynomials by deriving a decreasing sequence of polynomials beginning with  $A$  that satisfies a three-term recurrence relation. The following adds some easily verified extra details needed in our proof of Theorem 4.

**Lemma 3.** *Let  $c_1, \dots, c_{n-1}$  be any given positive numbers. Then there exist polynomials  $p_0, \dots, p_n$  satisfying (1) and (2) with  $m = n$  such that  $p_n = A$  and  $p_{n-1} = B$ . Further, there are polynomials  $s_0, \dots, s_n$  and  $t_0, \dots, t_n$  satisfying*

$$p_{n-j} = s_j B - t_j A, \quad j = 0, \dots, n,$$

where  $s_0 = 0$ ,  $t_0 = -1$ ,  $s_1 = 1$ ,  $t_1 = 0$ , and

$$s_{j+1} = \frac{q_{n-j}s_j - s_{j-1}}{c_{n-j}}, \quad t_{j+1} = \frac{q_{n-j}t_j - t_{j-1}}{c_{n-j}}, \quad j = 1, \dots, n-1. \quad (9)$$

Here  $q_j(x) = a_j x + b_j$  for some positive  $a_j$  and real  $b_j$ .

**Theorem 4.** *Any  $m + 1$  numbers  $h_0 > h_1 > \dots > h_m$  are alternation points for some polynomials  $p_0, p_1, \dots, p_m$  satisfying (1) and (2).*

PROOF. Let

$$P = \prod_{n \text{ even}} (x - h_n), \quad Q = \prod_{n \text{ odd}} (x - h_n), \quad S = \sum_{n=0}^m (-1)^n h_n,$$

and take  $k = [(m + 1)/2]$ . Clearly  $Q$  has degree  $k$  and  $P$  has degree  $k + 1$  or  $k$  according as  $m$  is even or odd.

Suppose  $m$  is even. Then  $m = 2k$ . Since the zeros of  $A = P$  and  $B = Q$  interlace, by Lemma 3 with  $n = k + 1$  we obtain polynomials  $p_0, \dots, p_k$ , where we have dropped  $p_{k+1}$ . After shifting the index of  $s$  and  $t$  down by one, we have that  $p_{k-j} = s_j Q - t_j P$  for  $j = 0, \dots, k$ , where now

$$s_0 = 1, \quad t_0 = 0, \quad s_1 = x - S, \quad t_1 = 1.$$

We also shift the index of  $q$  and  $c$  down by one so that (9) holds with  $n = k$ . The remaining polynomials are obtained by defining  $p_{k+j} = s_j Q + t_j P$  for  $j = 1, \dots, k$ . (Note that this redefines  $p_{k+1}$ .)

If  $m$  is odd, then  $m = 2k - 1$ . It can be shown that  $A = Q + P$  and  $B = Q - P$  have interlacing zeros. (Apply the method of [20, Theorem VIII].) By Lemma 3 with  $n = k$ , there exist polynomials  $p_0, \dots, p_k$  with  $p_{k-j} = s_j Q - t_j P$

for  $j = 0, \dots, k$  and  $s_0 = 1, t_0 = -1, s_1 = 1$  and  $t_1 = 1$ . (Here we have used that sums and differences of finite sequences satisfying (9) still satisfy these equations.) The remaining polynomials are obtained by defining  $p_{k+j-1} = s_j Q + t_j P$  for  $j = 2, \dots, k$ .

Note that except when  $j = 0$  the polynomial  $s_j$  has degree  $j$  when  $m$  is even and degree  $j - 1$  when  $m$  is odd, and  $t_j$  always has degree  $j - 1$ . In both cases we have  $p_{m-k+j} = s_j Q + t_j P$  for  $j = 0, \dots, k$ . Replacing  $k - j$  by  $j$  in our previous equations and adding and subtracting them, we obtain

$$p_{m-j} + p_j = 2s_{k-j}Q, \quad p_{m-j} - p_j = 2t_{k-j}P$$

for  $j = 0, \dots, k$ . Thus (3) holds.

To verify that  $p_0, \dots, p_m$  satisfy (1) and (2), first divide each of them by the constant  $p_0$ , which is nonzero since  $P$  and  $Q$  are relatively prime. Note that  $q_j$  and  $c_j$  have been defined for  $j = 1, \dots, k$  except for the case where  $m$  is even and  $j = k$ . In this case, define  $q_k = 2(x - S)$  and  $c_k = 1$ . Then by Lemma 3 and our definitions, we have

$$p_{j+1} = \begin{cases} q_j p_j - c_j p_{j-1} & \text{when } 1 \leq j < k \\ \frac{q_{m-j} p_j - p_{j-1}}{c_{m-j}} & \text{when } k \leq j < m \end{cases} \quad (10)$$

Algorithm 1 shows how to produce a computer program that constructs the orthogonal polynomials of Theorem 4. Note that the polynomials  $q_1, \dots, q_{k-1}$  are computed as quotients. (Arrays have been used to clarify the role of the variables  $s$  and  $t$  although it is necessary to store only their previous two values.)

### Algorithm 1

```
# Produce a set of orthogonal polynomials that has
# given decreasing numbers as its alternation points.
input  $m \in \mathbb{N}, h_0 > \dots > h_m$ 
        $c_1, \dots, c_{k-1} > 0$ , where  $k = \lfloor (m+1)/2 \rfloor$ 
output  $p_0, \dots, p_m$ 
# Initialize
 $P := \prod_{n \text{ even}} (x - h_n)$ ;    $Q := \prod_{n \text{ odd}} (x - h_n)$ ;
 $S := \sum_{n=0}^m (-1)^n h_n$ ;
if  $m$  is even then
     $k := m/2$ ;  $s_0 := 1$ ;  $t_0 := 0$ ;  $s_1 := x - S$ ;  $t_1 := 1$ 
else
     $k := (m+1)/2$ ;  $s_0 := 1$ ;  $t_0 := -1$ ;  $s_1 := 1$ ;  $t_1 := 1$ 
end(if)
```

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 $p_k := s_0Q - t_0P;$ 
 $p_{k-1} := s_1Q - t_1P;$ 
if  $m$  is even then  $p_{k+1} := s_1Q + t_1P$  end(if)
# Main loop
for  $j := 2$  to  $k$  do
   $q := \text{quotient}(p_{k-j+2}, p_{k-j+1});$  #  $q = q_{k-j+1}$ 
   $s_j := (qs_{j-1} - s_{j-2})/c_{k-j+1};$ 
   $t_j := (qt_{j-1} - t_{j-2})/c_{k-j+1};$ 
   $p_{k-j} := s_jQ - t_jP;$ 
   $p_{m-k+j} := s_jQ + t_jP$ 
end(do)
for  $j := 0$  to  $m$  do  $p_j := p_j/p_0$  end(do)

```

It has been the author's experience that if rational numerical values or other values that can be computed exactly are assigned to the  $h_i$ 's then the algorithm runs quickly requiring little memory. However, if the  $h_i$ 's are treated as symbols then the algorithm requires a large amount of time and memory. For example, if the  $h_i$ 's are the equispaced points given by  $h_i = m/2 - i$  or the Chebyshev points  $h_i = \cos(i\pi/m)$ , then the computing time when  $m = 20$  was seen to be 0.032 and 0.203 seconds, respectively, using the Maple 17 programming language and the *time* command. However, Table 2 shows the dramatically larger numbers when the  $h_i$ 's are left unevaluated and treated symbolically.

**Table 2: Algorithm cost for symbolic computation**

$m$	2	3	4	5	6	7
seconds	0	0	.031	.156	1.953	7,550
terms in $p_1$	2	38	313	8,195	130,648	15,618,708
terms in $p_m$	7	36	167	6,269	107,532	4,537,496

### 3. The even and odd product nodes

Let  $h_0 > h_1 > \dots > h_m$  and  $\tilde{h}_0 > \tilde{h}_1 > \dots > \tilde{h}_m$  be any given numbers and let

$$\mathcal{N} = \{(h_n, \tilde{h}_q) : 0 \leq n, q \leq m\}$$

be the Cartesian product of these two sets. We consider two subsets of this product that we use as the nodes of bivariate Lagrange polynomials and of cubature formulas. Define the even nodes to be the set  $\mathcal{N}_0$  of all ordered pairs  $(h_n, \tilde{h}_q)$  in  $\mathcal{N}$  where  $n$  and  $q$  are both even or both odd and define the odd nodes to be the set  $\mathcal{N}_1$  of all ordered pairs  $(h_n, \tilde{h}_q)$  in  $\mathcal{N}$  where  $n$  is even and  $q$  is odd or  $n$  is odd

and  $q$  is even. Clearly  $\mathcal{N}_0$  and  $\mathcal{N}_1$  are disjoint and  $\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_1$ . We define the index set

$$Q_k = \{(n, q) : 0 \leq n, q \leq m, \quad n - q = k \pmod{2}\}$$

so that we can consider both sets of nodes simultaneously as

$$\mathcal{N}_k = \{(h_n, \tilde{h}_q) : (n, q) \in Q_k\}, \quad k = 0, 1.$$

When  $m$  is odd, the number of even nodes is the same as the number of odd nodes but when  $m$  is even the number of even nodes is one more than the number of odd nodes. Specific formulas are given in [13, p. 50].

The even and odd nodes for the case  $h_n = \tilde{h}_n = \cos(n\pi/m)$ ,  $n = 0, 1, \dots, m$ , are referred to as the Chebyshev points in [23], as the Chebyshev nodes in [3] and as the Xu points in [5]. These nodes were first considered in 1978 by Morrow and Patterson [19] in connection with cubature formulas and required  $m$  to be even. One way they obtained the odd nodes is by using alternately an  $m/2$  node Gauss-Chebyshev rule of degree  $m - 1$  followed by a Lobatto-Chebyshev rule of degree  $m - 1$  for the  $y$  coordinates at each successive  $x$  coordinate. Using a lower bound of Möller, they observed that their cubature formula has a minimal number of nodes for polynomials of degree  $2m - 1$ . They obtained the even nodes in the same way by starting with the Lobatto-Chebyshev rule first and observed that their cubature formula in this case has one more node than Möller's lower bound. (The author was led to these nodes in connection with Markov's theorem. See [11, p. 353].)

In 1996, Yuan Xu [23] extended the Morrow-Patterson cubature formulas to the case where  $m$  is odd and defined Lagrange polynomials for the even and odd nodes in terms of the reproducing kernel. His proofs applied a general theory of cubature that he gave in [22].

Using a different approach, Bojanov and Petrova [3] extended the results of Xu to the case where  $h_0, h_1, \dots, h_n$  are nodes of a quadrature formula with respect to any weight that satisfies certain symmetry conditions. They gave an argument involving even and odd sums that motivates the definition of the even and odd nodes. See also [4, p.200].

The author extended the results of Yuan Xu to the case where the  $h_n$ 's are alternation points of the same Geronimus polynomials in [12] and of different Geronimus polynomials in [14]. The author also extended the cubature formula of Bojanov and Petrova in [13] and replaced the symmetry condition by an alternation property. It was shown in [13] and [7] that the author's methods also apply to the Morrow-Patterson nodes, which include the Padua points.

#### 4. Lagrange polynomials and cubature

In this section we obtain Lagrange polynomials and a cubature theorem for any even or odd nodes  $\mathcal{N}_k$ . We first apply Theorem 4 to obtain orthogonal polynomials that we use to define a large family of polynomials that all vanish on  $\mathcal{N}_k$ . We then apply a bivariate Christoffel-Darboux formula involving this family to obtain a formula for Lagrange polynomials on  $\mathcal{N}_k$  in terms of a bivariate reproducing kernel. By Lemma 6 (given below), the form we obtain for the Lagrange polynomials is just what is needed to obtain a cubature formula for the nodes. Proofs are given at the end of the section.

To begin with construction of Lagrange polynomials, we first observe that, by Theorem 4, there exist polynomials  $p_0, p_1, \dots, p_m$  and  $\tilde{p}_0, \tilde{p}_1, \dots, \tilde{p}_m$  satisfying (1) and (2) such that

$$p_{m-j}(h_n) = (-1)^n p_j(h_n), \quad \tilde{p}_{m-j}(\tilde{h}_n) = (-1)^n \tilde{p}_j(\tilde{h}_n), \quad j, n = 0, 1, \dots, m. \quad (11)$$

It follows from (10) that the polynomials can be chosen so that  $c_j = \tilde{c}_j = 1$  for  $j = 1, \dots, m-1$ . The corresponding bivariate reproducing kernels are given by

$$K_n(s, t, u, v) = \sum_{i=0}^n \sum_{j=0}^i \frac{p_{i-j}(s) \tilde{p}_j(t) p_{i-j}(u) \tilde{p}_j(v)}{H_{i-j} \tilde{H}_j}, \quad 0 \leq n \leq m,$$

where

$$H_j = \begin{cases} \frac{a_0 H_0}{a_j} & j = 0, \dots, m-1, \\ \frac{a_0 c_m H_0}{a_m} & j = m, \end{cases} \quad (12)$$

and where  $\tilde{H}_j$  is defined similarly. Here  $H_0$  and  $\tilde{H}_0$  are given positive numbers. (See [8, pp. 97-100] for a discussion of multivariate reproducing kernels.) Motivated by similar definitions given in [23] and [11]–[14], we define  $G_m$  by

$$\begin{aligned} 2G_m(s, t, u, v) &= K_{m-1}(s, t, u, v) + K_m(s, t, u, v) \\ &\quad + A_m p_m(s) p_m(u) + B_m \tilde{p}_m(t) \tilde{p}_m(v), \end{aligned}$$

where

$$A_m = \frac{1}{\tilde{H}_0} \left( \frac{1}{H_0} - \frac{1}{H_m} \right), \quad B_m = \frac{1}{H_0} \left( \frac{1}{\tilde{H}_0} - \frac{1}{\tilde{H}_m} \right).$$

Clearly  $G_m(s, t, u, v)$  is a polynomial of degree at most  $m$  in  $(s, t)$  and in  $(u, v)$ . Also, combining like terms in the reproducing kernels, we obtain

$$G_m(s, t, u, v) = K_{m-1}(s, t, u, v) + \frac{1}{2} S_m(s, t, u, v), \quad (13)$$

where

$$S_m(s, t, u, v) = \frac{p_m(s)p_m(u) + \tilde{p}_m(t)\tilde{p}_m(v)}{H_0\tilde{H}_0} + \sum_{j=1}^{m-1} \frac{p_{m-j}(s)\tilde{p}_j(t)p_{m-j}(u)\tilde{p}_j(v)}{H_{m-j}\tilde{H}_j}.$$

Hence  $G_m(s, t, s, t) > 0$  for all  $(s, t) \in \mathbb{R}^2$ . (Note that (13) shows that  $G_m$  does not depend on  $H_m$  or  $\tilde{H}_m$ .) Finally, given  $(n, q) \in Q_k$ , define

$$P_{n,q}(s, t) = \lambda_{n,q}G_m(s, t, h_n, \tilde{h}_q), \text{ where } \lambda_{n,q} = \frac{1}{G_m(h_n, \tilde{h}_q, h_n, \tilde{h}_q)}. \quad (14)$$

We shall see that a bivariate Christoffel-Darboux formula given later in (21) expresses the polynomials

$$(s - u)G(s, t, u, v) \text{ and } (t - v)G(s, t, u, v)$$

as a sum of terms where each term has a factor that is one of the polynomials vanishing on  $\mathcal{N}_k$ . Thus each of these two expressions vanish when both  $(s, t)$  and  $(u, v)$  are in  $\mathcal{N}_k$  and we shall see that then the polynomials  $P_{n,q}$  with  $(n, q) \in Q_k$  are Lagrange polynomials for  $\mathcal{N}_k$ . Note that the Lagrange polynomials are computed without reference to the underlying measures.

**Theorem 5.** *Let  $k = 0$  or  $k = 1$  and let  $(n, q) \in Q_k$ . Then  $P_{n,q}$  is a polynomial of degree  $m$  satisfying  $P_{n,q}(h_n, \tilde{h}_q) = 1$  and  $P_{n,q}(x) = 0$  for all  $x \in \mathcal{N}_k$  with  $x \neq (h_n, \tilde{h}_q)$ .*

Gasca and Lebrón [10] have obtained Lagrange polynomials for all the nodes of a product of two finite sets of numbers using a recursive formula involving determinants of rectangular sublattices. In the case of  $\mathcal{N}$ , their polynomials can have twice the degree of ours but include more nodes.

Next we describe a connection between Lagrange polynomials, reproducing kernels and cubature formulas. Let  $\mathcal{P}_m(\mathbb{R}^k)$  denote the space of all real-valued polynomials in  $k$  variables with degree at most  $m$  and let  $\mathcal{S}_m$  be the space of all orthogonal polynomials of degree  $m$  with respect to a measure  $\mu$  on  $\mathbb{R}^k$ , i.e.,

$$\mathcal{S}_m = \{p \in \mathcal{P}_m(\mathbb{R}^k) : (p, q) = 0 \text{ for all } q \in \mathcal{P}_{m-1}(\mathbb{R}^k)\},$$

where  $(p, q)$  is an inner product on  $\mathcal{P}_m(\mathbb{R}^k)$  induced by  $\mu$ . In view of the form of our construction of Lagrange polynomials, the following lemma from [13] applies to obtain a cubature formula for the nodes  $\mathcal{N}_k$ .

**Lemma 6.** Let  $\{x_i\}_{i=1}^n$  be  $n$  distinct points of  $\mathbb{R}^k$  having Lagrange polynomials  $\{P_i\}_{i=1}^n$  of degree at most  $m$ .

Conditions (a) and (b) below are equivalent.

a) If  $p \in \mathcal{P}_m(\mathbb{R}^k)$  then there is an  $S \in \mathcal{S}_m$  with

$$p = \sum_{i=1}^n p(x_i)P_i + S.$$

Also, for each  $i$ , there is an  $S_i \in \mathcal{S}_m$  with

$$P_i(x) = \lambda_i K_{m-1}(x, x_i) + S_i(x), \quad x \in \mathbb{R}^k.$$

b)

$$\int_{\mathbb{R}^k} p(x) d\mu(x) = \sum_{i=1}^n \lambda_i p(x_i)$$

for all  $p \in \mathcal{P}_{2m-1}(\mathbb{R}^k)$ .

**Theorem 7.** Let  $h_0 > h_1 > \dots > h_m$  and  $\tilde{h}_0 > \tilde{h}_1 > \dots > \tilde{h}_m$ , and let  $\mu$  and  $\tilde{\mu}$  be two corresponding finite Borel measures on  $\mathbb{R}$  for which the polynomials of Theorem 4 are orthogonal. Then

$$\int \int_{\mathbb{R}^2} p(x, y) d(\mu \times \tilde{\mu})(x, y) = \sum_{(n,q) \in Q_k} \lambda_{n,q} p(h_n, \tilde{h}_q)$$

for all bivariate polynomials  $p$  of degree at most  $2m - 1$  and for  $k = 0, 1$ . The values of  $\lambda_{n,q}$  are given by (14) and  $H_0$  and  $\tilde{H}_0$  are the integrals of the constant polynomial 1 with respect to  $d\mu$  and  $d\tilde{\mu}$ , respectively.

The case where the given decreasing sequences are alternation points of two sequences of Geronimus polynomials is given in [14]. By a lower bound of Möller [17], if the integral in Theorem 7 vanishes on all odd polynomials then the number of nodes in the theorem is at most one more than the minimal number of nodes over all cubature formulas for polynomials of the same degree and it is minimal when  $m$  is even and  $k = 1$ . Thus Theorem 7 expands the known examples of minimal cubature formulas.

Unlike most cubature formulas, we can choose any coordinates for the nodes  $\mathcal{N}_k$  in Theorem 7. There are many measures in which the corresponding polynomials of Theorem 4 are orthogonal. Weight functions for the measures are given

in Table 1 when the alternation points are those for any of the four kinds of Chebyshev polynomials. A discrete measure always exists.

Specifically, let

$$\nu = \sum_{n=0}^m w_n \delta_{h_n},$$

where

$$w_n = \frac{(-1)^n}{\prod_{i \neq n} (h_n - h_i)} > 0$$

and  $\delta_x$  denotes the Dirac measure at  $x$ .

**Theorem 8.** *The polynomials of Theorem 4 are orthogonal with respect to the measure  $\nu$  with inner product*

$$(p, q) = C_m \sum_{n=0}^m w_n p(h_n) q(h_n), \quad (15)$$

where

$$\frac{1}{C_m} = \sum_{n=0}^m w_n.$$

For example, when the  $h_n$ 's are evenly spaced a distance  $\delta$  apart, one can show that  $1/w_n = (m-n)!n!\delta^m$  and hence the weights  $w_n$  are a constant multiple of the Krawtchouk weights  $\binom{m}{n}$ . Also, constant multiples of the weights  $\{w_n\}$  for the alternation points of the four kinds of Chebyshev polynomials are given after theorem 2.2 of [12]. By uniqueness of orthogonal polynomials, the polynomials of Theorem 4 are scalar multiples of any orthogonal polynomials obtained for the inner product (15) as, for example, by Gram-Schmidt orthogonalization.

If  $a$  and  $b$  are defined by  $p_1(x) = ax + b$ , it follows from theorem 3 of [13] that

$$\int_{\mathbb{R}} p(x) d\mu = \sum_{n=0}^m \lambda_n p(h_n), \quad (16)$$

where

$$\lambda_n = \frac{a_0 H_0}{(-1)^n \pi_m'(h_n)}$$

and  $p$  is any polynomial of degree at most  $2m - 1$ . (The alternation property for  $j = 0$  and (12) have been applied here.) Note that since  $\pi_m$  is a polynomial of degree  $m + 1$  with roots  $h_0, h_1, \dots, h_m$ , there exists a constant  $A$  such that  $\lambda_n = Aw_n$  for  $n = 0, 1, \dots, m$ . Hence polynomials orthogonal with respect to  $\mu$

are also orthogonal with respect to  $\nu$  and its inner product (15). In particular, if  $\mu = \nu$  and  $p$  is the  $n$ th Lagrange polynomial for the points  $h_0, h_1, \dots, h_m$ , then by (16) we have  $\lambda_n = w_n$  for all  $n$ . In general,

$$H_0 = \int_{\mathbb{R}} d\mu = \sum_{n=0}^m \lambda_n = \frac{A}{C_m}$$

so  $A$  is positive and

$$\lambda_n = H_0 C_m w_n, \quad n = 0, 1, \dots, m. \quad (17)$$

There is another approach to cubature for the even and odd product nodes besides Lemma 6 and a consequence is given in theorem 7 of [13]. Since the weights of cubature with nodes  $\mathcal{N}_k$  are unique (by Theorem 5), it follows from this theorem and Theorem 7 that the product formula  $\lambda_{n,q} = 2\lambda_n \tilde{\lambda}_q$  holds, where  $\tilde{\lambda}_q$  is defined similarly to  $\lambda_n$ . Hence, by Theorem 7 with  $\mu = \nu$ ,

$$\sum_{n=0}^m \sum_{q=0}^m w_n \tilde{w}_q p(h_n, \tilde{h}_q) = 2 \sum_{(n,q) \in Q_k} w_n \tilde{w}_q p(h_n, \tilde{h}_q) \quad (18)$$

for all polynomials  $p$  of degree at most  $2m-1$  and  $k = 0, 1$ . (This identity no longer holds when  $p(s, t) = s^m t^m$ . See [9] for further details.) Another consequence of the product formula and (17) is

$$\lambda_{n,q} = 2H_0 \tilde{H}_0 C_m \tilde{C}_m w_n \tilde{w}_q, \quad 0 \leq n, q \leq m, \quad (19)$$

where  $\tilde{H}_0$ ,  $\tilde{C}_m$  and  $\tilde{w}_n$  have analogous definitions. A similar identity can be obtained for the Morrow-Patterson nodes from [7].

**PROOF OF THEOREMS 5 AND 7.** Our proof of Theorem 5 follows easily from a Christoffel-Darboux identity for bivariate polynomials, which is an extension of the identity given in proposition 2 of [14]. To state the identity, first define polynomials

$$\begin{aligned} X_j(s, t) &= p_{m-j}(s) \tilde{p}_j(t) - \epsilon p_j(s) \tilde{p}_{m-j}(t), \quad j = 0, 1, \dots, m, \\ Y_0(s, t) &= p_1(s) p_m(s) - p_{m-1}(s) = \pi_m(s), \\ Y_j(s, t) &= p_{m-j+1}(s) \tilde{p}_j(t) - \epsilon p_{j-1}(s) \tilde{p}_{m-j}(t), \quad j = 1, \dots, m, \end{aligned} \quad (20)$$

where  $\epsilon$  is a constant. It follows from (11) that all of these polynomials vanish at the nodes of  $\mathcal{N}_k$  when  $\epsilon = (-1)^k$  and  $k = 0, 1$ . Define

$$F_j(s, t, u, v) = [X_j(s, t) p_{m-j-1}(u) + Y_j(s, t) p_{m-j}(u)] \tilde{p}_j(v), \quad j = 0, \dots, m-1.$$

Recall that the  $p_j$ 's are chosen so that  $c_j = 1$  for  $j = 1, \dots, m-1$ . Also, the recursion coefficients of the  $\tilde{p}_j$ 's satisfy  $\tilde{a}_{m-j} = \tilde{a}_j$  for  $j = 1, \dots, [m/2]$  by Lemma 2. This is enough to apply the arguments in [14] to obtain the identity

$$2a_0H_0(s-u)G_m(s,t,u,v) = \frac{Y_m(s,t)\tilde{p}_m(v) - Y_m(u,v)\tilde{p}_m(t)}{\tilde{H}_0} \quad (21)$$

$$+ \sum_{j=0}^{m-1} \frac{F_j(s,t,u,v) - F_j(u,v,s,t)}{\tilde{H}_j}.$$

Clearly  $P_{n,q}(h_n, \tilde{h}_q) = 1$  by definition. Let  $x = (h_{n'}, \tilde{h}_{q'})$  be a node in  $\mathcal{N}_k$ . It follows from (21) that  $(h_{n'} - h_n)P_{n,q}(x) = 0$ . Now let  $\tilde{G}_m$  be as in the definition of  $G_m$  but with  $p$  and  $\tilde{p}$  interchanged,  $s$  and  $t$  interchanged, and  $u$  and  $v$  interchanged. Then  $\tilde{G}_m(t, s, v, u) = G_m(s, t, u, v)$  by the symmetry of the definitions of  $K_m$  and  $G_m$ . Hence, also  $(\tilde{h}_{q'} - \tilde{h}_q)P_{n,q}(x) = 0$ . Thus  $P_{n,q}(x) = 0$  whenever  $x \neq (h_n, \tilde{h}_q)$ . Hence Theorem 5 holds.

To prove Theorem 7, it suffices to verify condition (a) of Lemma 6. The first statement of (a) is a consequence of proposition A-1 of [12] and the second statement of (a) follows from (13) and (14).

## 5. An example

To illustrate the computations described in the previous section, we consider the case where the given set of decreasing numbers is equispaced a distance  $\delta$  apart. In this case we may assume that  $h_i = m/2 - i$ ,  $i = 0, 1, \dots, m$ , since if these are alternation points for orthogonal polynomials  $p_0, p_1, \dots, p_m$ , then the given equispaced points  $\tilde{h}_0, \tilde{h}_1, \dots, \tilde{h}_m$  are alternation points for the orthogonal polynomials  $\tilde{p}_i(x) = p_i(Ax + B)$ ,  $i = 0, 1, \dots, m$ , where

$$A = \frac{1}{\delta}, \quad B = -\frac{\tilde{h}_0 + \tilde{h}_m}{2\delta}.$$

For simplicity, we take  $m = 5$  and hence

$$h_0 = \frac{5}{2}, \quad h_1 = \frac{3}{2}, \quad h_2 = \frac{1}{2}, \quad h_3 = -\frac{1}{2}, \quad h_4 = -\frac{3}{2}, \quad h_5 = -\frac{5}{2}.$$

Algorithm 1 with  $c_1 = c_2 = 1$  produces the polynomials

$$\begin{aligned} p_0(x) &= 1 & p_3(x) &= \frac{2}{15}x(4x^2 - 13) \\ p_1(x) &= \frac{16x}{15} & p_4(x) &= \frac{16}{45}x^4 - \frac{88}{45}x^2 + 1 \\ p_2(x) &= \frac{4x^2}{5} - 1 & p_5(x) &= \frac{x(16x^4 - 120x^2 + 149)}{60}. \end{aligned}$$

It is easy to verify that each  $p_i(x)$  is a multiple of the translated Krawtchouk polynomial  $k_i(m/2 - x)$  for  $0 \leq i \leq 5$ . Also,

$$\pi_5(x) = p_1(x)p_5(x) - p_4(x) = \frac{64}{225}x^6 - \frac{112}{45}x^4 + \frac{1036}{225}x^2 - 1$$

and  $\pi_5$  has roots  $h_0, h_1, \dots, h_5$ . The coefficients in (1) and (2) are  $b_i = 0$  for  $0 \leq i \leq 4$ ,  $c_i = 1$  for  $1 \leq i \leq 4$ , and

$$a_0 = \frac{16}{15}, \quad a_1 = a_4 = \frac{3}{4}, \quad a_2 = a_3 = \frac{2}{3}.$$

Both the even nodes  $\mathcal{N}_0$  and the odd nodes  $\mathcal{N}_1$  have 18 nodes. After Lagrange polynomials  $P_{n,q}$  for each of these sets of nodes were generated using (13) and (14), it was observed that all 36 polynomials were simple modifications of the following six polynomials

$$\begin{aligned} P_{0,0} &= (t+5+s)(t-1+s)(t+3+s)(t-3+s)(t+1+s)/3840 \\ P_{0,1} &= (t+s)(t-2+s)(t+4+s)(-t+1+s)(t+2+s)/768 \\ P_{0,2} &= (-t+2+s)(-t+s)(t+3+s)(t+1+s)(t-1+s)/384 \\ P_{1,1} &= -(t+3+s)(t+1+s)(t-1+s)(5s^2 - 6st + 5t^2 - 25)/768 \\ P_{1,2} &= -(t+2+s)(t+s)(-t+1+s)(5s^2 - 2st + 5t^2 - 35)/384 \\ P_{2,2} &= (t+1+s)(10s^4 - 8s^3t + 12s^2t^2 - 8st^3 + 10t^4 - 100s^2 + 64st - 100t^2 + 225)/384 \end{aligned}$$

and that  $P_{q,n}(s, t) = P_{n,q}(t, s)$  whenever  $0 \leq n, q \leq m$ . The remaining polynomials are given in terms of these in Table 3, where

$$\alpha(s, t) = (s, -t), \quad \beta(s, t) = (-t, s), \quad \gamma(s, t) = (-t, -s).$$

The Lagrange polynomials are not unique because by (20) there exist many polynomials of degree  $m$  that vanish on  $\mathcal{N}_k$  for  $k = 0$  and  $k = 1$ .

**Table 3: Lagrange polynomials for  $m = 5$**

$n \setminus q$	0	1	2	3	4	5
0	$P_{0,0}$	$P_{0,1}$	$P_{0,2}$	$P_{0,2} \circ \alpha$	$P_{0,1} \circ \alpha$	$P_{0,0} \circ \alpha$
1		$P_{1,1}$	$P_{1,2}$	$P_{1,2} \circ \alpha$	$P_{1,1} \circ \alpha$	$P_{0,1} \circ \beta$
2			$P_{2,2}$	$P_{2,2} \circ \alpha$	$P_{1,2} \circ \beta$	$P_{0,2} \circ \beta$
3				$P_{2,2} \circ \gamma$	$P_{1,2} \circ \gamma$	$P_{0,2} \circ \gamma$
4					$P_{1,1} \circ \gamma$	$P_{0,1} \circ \gamma$
5						$P_{0,0} \circ \gamma$

The polynomials  $p_0, p_1, \dots, p_5$  are orthogonal with respect to the inner product (15), which in this case is the Krawtchouk inner product (8) with  $m = 5$  since  $C_5 = 15/4$ . Note also that  $H_0 = (1, 1) = 1$ . The cubature weights are easiest to compute using (19) and are given by

$$\lambda_{p,q} = \frac{1}{2^9} \binom{5}{p} \binom{5}{q}, \quad p, q = 0, 1, \dots, 5.$$

## 6. Coefficient formulas

Another approach to finding orthogonal polynomials that have a given set of decreasing numbers as alternation points is to solve the algebraic equations of the three-term recurrence relations together with the equations defining the alternation points using the Maple 17 “solve” command. This was done for the coefficients in (1) and (2) with  $c_j = 1$  for  $j = 1, \dots, m - 1$ , to create Table 4, where  $h_0 > h_1 > \dots > h_m$  are given and  $S_n = \sum_{j=0}^n (-1)^j h_j$ . When  $m = 1$  or  $m = 3$ , the coefficients are the same as obtained by Algorithm 1. When  $m$  is even, an extra parameter  $C > 0$  can be introduced into Algorithm 1 at initialization by

$$s_1 = \frac{x - S}{C}, \quad t_1 = \frac{1}{C}.$$

The coefficients given in Table 4 are then the same as those obtained by Algorithm 1 with  $C = D_2 a_0$  when  $m = 2$  and  $C = A^3 / (D_4 a_0)$  when  $m = 4$ . As in Table 2, the complexity of our expressions increases rapidly with  $m$ .

**Table 4: Coefficients for the three-term recurrences**

$m = 1)$	$a_0 = \frac{2}{D_1},$	$b_0 = -\frac{h_0 + h_1}{D_1},$	$D_1 = h_0 - h_1$
$m = 2)$	$a_0 > 0,$	$b_0 = -h_1 a_0$	
	$a_1 = \frac{2}{D_2 a_0},$	$b_1 = -S_2 a_1,$	$D_2 = D_1 (h_1 - h_2)$
$m = 3)$	$a_0 = \frac{S_3^3}{2D_3},$	$b_0 = \frac{(h_1 h_3 - h_0 h_2) a_0}{S_3},$	$D_3 = D_2 (h_0 - h_3) (h_2 - h_3)$
	$a_1 = \frac{2}{S_3},$	$b_1 = \frac{h_3^2 - h_2^2 + h_1^2 - h_0^2}{S_3^2}$	
	$a_2 = a_1,$	$b_2 = b_1$	
$m = 4)$	$a_0 > 0,$	$b_0 = \frac{(h_0 h_2 h_4 - h_1 h_3 S_4) a_0}{A},$	$A = (h_3 - h_4) S_3 + D_2$
	$a_1 = \frac{A^2}{D_4 a_0},$	$b_1 = \frac{h_1 h_3 A^2 + D_4}{D_4 b_0},$	$D_4 = D_3 (h_1 - h_4) (h_3 - h_4)$
	$a_2 = \frac{2}{A a_1},$	$b_2 = -S_4 a_2$	
	$a_3 = a_1,$	$b_3 = b_1$	

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