Fixed Point Theorems for Infinite Dimensional Holomorphic Functions

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Abstract

This talk discusses conditions on the numerical range of a holomorphic function defined on a bounded convex domain in a complex Banach space that imply that the function has a unique fixed point. In particular, extensions of the Earle-Hamilton Theorem are given for such domains. The theorems are applied to obtain a quantitative version of the inverse function theorem for holomorphic functions and a distortion form of Cartan's uniqueness theorem.

1. Introduction

We begin with a short introduction to the theory of holomorphic functions with domain and range contained in a complex Banach space. We then review some well-known fixed point theorems for holomorphic functions. Perhaps the most basic is the Earle-Hamilton fixed point theorem, which may be viewed as a holomorphic formulation of Banach's contraction mapping theorem.

Next we recall the definition of the numerical range of linear transformations and its extension to holomorphic functions. We prove an extension of the Earle-Hamilton theorem for bounded convex domains which was obtained in joint work with Simeon Reich and David Shoikhet [18]. According

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to this theorem, a holomorphic function on such a domain into the surrounding Banach space has a fixed point if its numerical range lies strictly to the left of the vertical line $x = 1$.

Finally, we apply the above fixed point theorem to obtain quantitative versions of the inverse function theorem and Cartan's uniqueness theorem where norm estimates in the hypotheses are replaced by estimates on the numerical range.

Much of this material has been presented previously at the International Conference on Fixed Point Theory and Its Applications, Technion, Haifa, Israel, June 2001. (See [17].)

2. Holomorphic functions

The infinite dimensional theory of holomorphic functions originated in a series of papers by M. Fréchet and R. Gâteaux that appeared from 1909 to 1929 and was subsequently developed by many others. (See [20, 5, 6, 7].) We discuss two definitions of holomorphy. A strong definition due to Fréchet and a weak definition associated with Gâteaux. Let X and Y be complex Banach spaces and let $\mathcal{D}$ be an open subset of X. In many cases, $\mathcal{D}$ will be the open ball of radius r and center x, i.e.,

$$B_r(x) = \{z \in X : \ \|z - x\| < r\}.$$

STRONG DEFINITION. A function $h : \mathcal{D} \to Y$ is *holomorphic* if for each $x \in \mathcal{D}$ there exists a continuous complex-linear mapping $Dh(x) : X \to Y$ such that

$$\lim_{y \to 0} \frac{\|h(x+y) - h(x) - Dh(x)y\|}{\|y\|} = 0.$$

Clearly every function that is holomorphic in the above sense is continuous and hence locally bounded. There is an another definition which reduces matters to the case of a complex-valued function of a complex variable. (We use λ to denote a complex variable.)

WEAK DEFINITION. A function $h : \mathcal{D} \to Y$ is *holomorphic* if it is locally bounded and if for each $x \in \mathcal{D}$, $y \in X$ and linear functional $\ell \in Y^*$, the function

$$f(\lambda) = \ell(h(x + \lambda y))$$

is holomorphic at $\lambda = 0$.

The nth derivative $D^n h(x)$ of a holomorphic function h at x can be represented as a symmetric multilinear map of degree n. We denote the associated homogeneous polynomial of degree n by $\hat{D}^n h(x)$. (See Example 1 below.)

An advantage of the weak definition is that one can apply the results of classical function theory to f and then use the Hahn-Banach theorem to obtain a similar result for the general case. For example, the Cauchy estimates can be obtained in this way. Specifically, if $h : B_r(x) \to Y$ is holomorphic and bounded, then

$$\|\hat{D}^n h(x)\| \leq \frac{n!}{r^n} \sup\{\|h(y)\| : \ y \in B_r(x)\}.$$

Clearly the strong definition implies the weak definition. In fact, both definitions are equivalent. This was shown for the case where X is the complex plane by N. Dunford in 1938 using the uniform boundedness principle. The general case follows from Theorems 3.10.1 and 3.17.1 of [20].

EXAMPLE 1. Define a mapping $P : X \to Y$ to be a homogeneous polynomial of degree n if

$$P(x) = F(\underbrace{x, \ldots, x}_{n}),$$

where $F : X \times \cdots \times X \to Y$ is a continuous (complex) multilinear map of degree n. Then P is holomorphic on X and

$$DP(x)y = \sum_{k=0}^{n-1} F(\underbrace{x, \ldots, x}_{k}, y, \underbrace{x, \ldots, x}_{n-k-1}), \quad x, y \in X.$$

Let $\{P_n\}_0^\infty$ be a sequence of homogeneous polynomials where P_n is of degree n and put

$$\alpha = \limsup_{n\to\infty} \|P_n\|^{1/n}.$$

Suppose α is finite. The *radius of convergence* of the series $\sum_{n=0}^{\infty} P_n(x)$ is $R = 1/\alpha$ when $\alpha \neq 0$ and $R = \infty$ otherwise. By the Weierstrass M-test this series converges uniformly on $B_r(0)$ whenever $0 < r < R$. It follows from the weak definition that

$$h(x) = \sum_{n=0}^{\infty} P_n(x)$$

converges to a holomorphic function on $B_R(0)$. Conversely, if the series converges uniformly on $B_s(0)$ for some $s > 0$ then $s \leq R$. Thus R may be thought of as the radius of uniform convergence.

EXAMPLE 2. A. E. Taylor was first to point out in 1938 that the power series for an entire function on a Banach space can have a finite radius of convergence. For example, let X be the space c_0 of all sequences of complex numbers converging to zero with the usual sup norm. We exhibit a power series that is holomorphic everywhere in X but does not converge uniformly on $B_1(0)$ although it does converge uniformly on $B_r(0)$ for every r with $0 < r < 1$. Define

$$h(x) = \sum_{k=0}^{\infty} x_k^k \quad \text{for } x = (x_k)_0^\infty$$

and let $S_n(x)$ be the nth partial sum of the series beginning with $n = 0$. Then h is defined and holomorphic everywhere in X by the weak definition. Also, the power series for h converges uniformly on $B_r(0)$ for $0 < r < 1$ by the Weierstrass M-test since $|x_k^k| \leq \|x\|^k$ for any $x \in X$. Now for each positive integer n define an $x \in B_1(0)$ by

$$\begin{cases} x_k &= 1/\sqrt[n]{2} \qquad n < k \leq 2n \\ x_k &= 0 \qquad\qquad \text{otherwise} \end{cases}.$$

Then $S_n(x) = 0$ and

$$S_{2n}(x) \geq \sum_{k=n+1}^{2n} x_k^{2n} = \sum_{k=n+1}^{2n} \frac{1}{4} = \frac{n}{4}.$$

Hence the partial sums $\{S_n(x)\}$ cannot converge uniformly on $B_1(0)$.

The following theorem shows that every holomorphic function can be written as the sum of an infinite series of homogeneous polynomials in some neighborhood of each of the points of its domain. The proof is a consequence of the Cauchy estimates and the weak definition.

Theorem 1 (Taylor's Theorem [20, Th. 3.17.1]) *If h is holomorphic and bounded in $B_r(x)$, then*

$$h(x + y) = \sum_{n=0}^{\infty} \frac{1}{n!} \hat{D}^n h(x)(y)$$

for all y in $B_r(0)$. Moreover, the radius of convergence of the series is at least as large as r.

It follows from Taylor's Theorem and the uniqueness of power series expansions that if h is as in Example 1, then

$$P_n = \frac{1}{n!}\,\hat{D}^n h(0), \quad n = 0, 1, \cdots.$$

3. Holomorphic fixed point theorems

A set S is said to lie *strictly inside* a subset $\mathcal{D}$ of a Banach space if there is some $\epsilon > 0$ such that $B_\epsilon(x) \subseteq \mathcal{D}$ whenever $x \in S$. The following theorem may be viewed as a holomorphic version of the Banach's contraction mapping theorem.

Theorem 2 (Earle-Hamilton [9]) *Let $\mathcal{D}$ be a nonempty domain in a complex Banach space X and let $h : \mathcal{D} \to \mathcal{D}$ be a bounded holomorphic function. If $h(\mathcal{D})$ lies strictly inside $\mathcal{D}$, then h has a unique fixed point in $\mathcal{D}$.*

Proof. We construct a metric ρ, called the CRF-pseudometric, in which h is a contraction. Let Δ be the open unit disc of the complex plane. Define

$$\alpha(x, v) = \sup\{|Dg(x)v| : \ g : \mathcal{D} \to \Delta \text{ holomorphic}\}$$

for $x \in \mathcal{D}$ and $v \in X$, and set

$$L(\gamma) = \int_0^1 \alpha(\gamma(t), \gamma'(t))\, dt$$

for γ in the set Γ of all curves in $\mathcal{D}$ with piecewise continuous derivative. Clearly α specifies a seminorm at each point of $\mathcal{D}$. We view $L(\gamma)$ as the length of the curve γ measured with respect to α. Define

$$\rho(x, y) = \inf\{L(\gamma) : \ \gamma \in \Gamma, \ \gamma(0) = x, \gamma(1) = y\}$$

for $x, y \in \mathcal{D}$. It is easy to verify that ρ is a pseudometric on $\mathcal{D}$.

Let $x \in \mathcal{D}$ and $v \in X$. By the chain rule,

$$D(g \circ h)(x)v = Dg(h(x))\, Dh(x)v$$

for any holomorphic function $g : \mathcal{D} \to \Delta$. Hence,

$$\alpha(h(x), Dh(x)v) \leq \alpha(x, v). \tag{1}$$

By integrating this and applying the chain rule, we obtain $L(h \circ \gamma) \leq L(\gamma)$ for all $\gamma \in \Gamma$ and thus the Schwarz-Pick inequality

$$\rho(h(x), h(y)) \leq \rho(x, y) \tag{2}$$

holds for all $x, y \in \mathcal{D}$.

Now by hypothesis there exists an $\epsilon > 0$ such that $B_\epsilon(h(x)) \subseteq \mathcal{D}$ whenever $x \in \mathcal{D}$. We may assume that $\mathcal{D}$ is bounded by replacing $\mathcal{D}$ by the subset

$$\cup\{B_\epsilon(h(x)) : \ x \in \mathcal{D}\}.$$

Fix t with $0 < t < \epsilon/\delta$, where δ denotes the diameter of $h(\mathcal{D})$. Given $x \in \mathcal{D}$, define

$$\hat{h}(y) = h(y) + t[h(y) - h(x)]$$

and note that $\hat{h} : \mathcal{D} \to \mathcal{D}$ is holomorphic. Given $x \in \mathcal{D}$ and $v \in X$, it follows from

$$D\hat{h}(x)v = (1 + t)Dh(x)v$$

and (1) with h replaced by $\hat{h}$ that

$$\alpha(h(x), Dh(x)v) \leq \frac{1}{1+t}\, \alpha(x, v).$$

Integrating this as before, we obtain

$$\rho(h(x), h(y)) \leq \frac{1}{1+t}\, \rho(x, y)$$

for all $x, y \in \mathcal{D}$.

Now pick a point $x_0 \in \mathcal{D}$ and let $\{x_n\}$ be the sequence of iterates given by $x_n = h^n(x_0)$. Then $\{x_n\}$ is a ρ-Cauchy sequence by the proof of the contraction mapping theorem. Since X is complete in the norm metric, it suffices to show that there exists a constant $m > 0$ such that

$$\rho(x, y) \geq m\, \|x - y\| \tag{3}$$

for all $x, y \in \mathcal{D}$. Since $\mathcal{D}$ is bounded, we may take $m = 1/d$, where d is the diameter of $\mathcal{D}$. Given $x \in \mathcal{D}$ and $v \in X$, define

$$g(y) = m\, \ell(y - x),$$

where $\ell \in X^*$ with $\|\ell\| = 1$. Then $g : \mathcal{D} \to \Delta$ is holomorphic and $Dg(x)v = m\,\ell(v)$. Hence $\alpha(x, v) \geq m\,\|v\|$ by the Hahn-Banach theorem. Integrating as before, we obtain (3).

The Earle-Hamilton theorem still applies in cases where the holomorphic function does not necessarily map its domain strictly inside itself. In fact, the following interesting fixed point theorem is a consequence of two applications of the Earle-Hamilton theorem.

Theorem 3 (Khatskevich-Reich-Shoikhet [22, 28]) *Let $\mathcal{D}$ be a nonempty bounded convex domain in a complex Banach space and let $h : \mathcal{D} \to \mathcal{D}$ be a holomorphic function having a uniformly continuous extension to $\overline{\mathcal{D}}$. If there exists an $\epsilon > 0$ such that $\|h(x) - x\| \geq \epsilon$ whenever $x \in \partial\mathcal{D}$, then h has a unique fixed point in $\mathcal{D}$.*

For example, the hypothesis that $\|h(x)-x\| \geq \epsilon$ for all $x \in \partial\mathcal{D}$ is satisfied when $\mathcal{D}$ contains the origin and

$$\sup_{x \in \partial\mathcal{D}} \frac{\|h(x)\|}{\|x\|} < 1.$$

Proof. Given $0 < t < 1$ and $x \in \mathcal{D}$, define a holomorphic map $f_t : \mathcal{D} \to \mathcal{D}$ by

$$f_t(y) = (1 - t)\,x + t\,h(y)$$

and let $\delta > 0$ be such that $B_\delta(x) \subseteq \mathcal{D}$. To show that $f_t(\mathcal{D})$ lies strictly inside $\mathcal{D}$, take $\epsilon = (1 - t)\delta$. Let $y \in \mathcal{D}$ and let $w \in B_\epsilon(f_t(y))$. Then

$$z = \frac{w - t\,h(y)}{1 - t}$$

is in $\mathcal{D}$ since $z \in B_\delta(x)$, so

$$w = (1 - t)\,z + t\,h(y) \in \mathcal{D}.$$

Hence $B_\epsilon(f_t(y)) \subseteq \mathcal{D}$ for all $y \in \mathcal{D}$.

By the Earle-Hamilton theorem, f_t has a unique fixed point $g_t(x)$ in $\mathcal{D}$. Since the CRF-metric is continuous, the proof of the contraction mapping theorem shows that the iterates of f_t at a chosen point $y_0 \in \mathcal{D}$ are holomorphic and locally uniformly Cauchy in x. Hence the limit function $g_t : \mathcal{D} \to \mathcal{D}$

is holomorphic by [20, Th. 3.18.1]. Now an $x \in \mathcal{D}$ is a fixed point for g_t if and only if x is a fixed point for h. Thus, by the Earle-Hamilton theorem, it suffices to show that $g_t(\mathcal{D})$ lies strictly inside $\mathcal{D}$ for some $t > 0$.

Since h has a uniformly continuous extension to $\overline{\mathcal{D}}$, by hypothesis there exist $\epsilon > 0$ and $\delta > 0$ such that $\|h(x) - x\| \geq \epsilon$ whenever $x \in \mathcal{D}$ and

$$d(x, \partial\mathcal{D}) = \inf\{\|x - y\| : \ y \in \partial\mathcal{D}\} < \delta.$$

Since $\mathcal{D}$ is bounded, there is an M with $\|x\| \leq M$ for all $x \in \mathcal{D}$. If $x \in \mathcal{D}$,

$$h(g_t(x)) - g_t(x) = (1 - t)[h(g_t(x)) - x],$$

so

$$\|h(g_t(x)) - g_t(x)\| \leq 2(1 - t)M.$$

Choose t close enough to 1 so that $2(1 - t)M < \epsilon$. If $d(g_t(x), \partial\mathcal{D}) < \delta$ for some $x \in \mathcal{D}$, then

$$\epsilon \leq \|h(g_t(x)) - g_t(x)\|,$$

a contradiction. Thus, $B_\delta(g_t(x)) \subseteq \mathcal{D}$ for all $x \in \mathcal{D}$, as required.

Example 3. The hypotheses on the behavior of h on $\partial\mathcal{D}$ cannot be omitted in Theorem 3. This follows by considering a translate of the shift operator as in [19]. Specifically, let $X = c_0$ and define

$$h(x) = (\frac{1}{2}, x_1, x_2, \ldots)$$

for $x \in X$. Clearly h is an affine isometry on X and h maps the ball $B_r(0)$ into itself for each $r > \frac{1}{2}$. However, if $h(x) = x$, then

$$\frac{1}{2} = x_1 = x_2 = \cdots,$$

contradicting that x is in c_0. Thus h has no fixed point in X.

It is an open problem whether if B is the open unit ball of a separable reflexive complex Banach space and if $h : B \to B$ is a holomorphic function with a continuous extension to $\overline{B}$ then h has a fixed point in $\overline{B}$. However, Hayden and Suffridge [19] have proved that $e^{i\theta}h$ has a fixed point in $\overline{B}$ for almost every θ. Also, Goebel, Sekowski and Stachura [12] have solved the problem in the affirmative for the case where X is a Hilbert space and this has been extended by Kuczumow [24] to the case where X is a finite product of Hilbert spaces (with the max norm). An example of Kakutani [21] shows that holomorphy is essential in the hypotheses since he exhibited a fixed point free homeomorphism of the closed unit ball of any infinite dimensional Hilbert space.

A related problem is to weaken the hypotheses of the Earle-Hamilton theorem by showing that if $h : B \to B$ is a holomorphic function such that the sequence of iterates $\{h^n(x)\}$ lies strictly inside B for some $x \in B$, then h has a fixed point in B. This has been established when X is a Hilbert space in [12] and when X is a finite product of Hilbert spaces in [23]. These results have been extended to bounded convex domains in a more general class of reflexive Banach spaces by Budzyńska [4]. Example 3 gives a counterexample for the general case.

See [25] for an extensive survey of fixed point theorems for holomorphic mappings. See [16, 6, 10] for more on function-theoretic metrics on infinite dimensional domains.

4. The linear numerical range

The notion of the numerical range was successfully extended from operators on a Hilbert space to operators on an arbitrary complex Banach space X by G. Lumer [26]. To give an equivalent form of his definitions, define

$$J(x) = \{\ell \in X^* : \ \|\ell\| = \ell(x) = 1\}$$

for $x \in X$ with $\|x\| = 1$ and note that $J(x)$ is nonempty by the Hahn-Banach theorem. Let $Q(x)$ be a nonempty subset of $J(x)$ for each $x \in X$ with $\|x\| = 1$.

Now let $A \in \mathcal{L}(X)$. Define numerical ranges of A by

$$V(A) = \{\ell(Ax) : \ \ell \in J(x), \ \|x\| = 1\}, \tag{4}$$

$$W(A) = \{\ell(Ax) : \ \ell \in Q(x), \ \|x\| = 1\}. \tag{5}$$

Clearly, $W(A) \subseteq V(A)$. If $Q(x)$ is taken to be $J(x)$ for all $x \in X$ with $\|x\| = 1$, then $W(A) = V(A)$.

In the case where X is a Hilbert space, the sets $Q(x)$ and $J(x)$ coincide and consist of the single functional $\ell(y) = (y, x)$ by the Riesz representation theorem. Thus

$$W(A) = V(A) = \{(Ax, x) : \ \|x\| = 1, \ x \in H\}$$

in this case.

Since there is in general no adjoint operation on $\mathcal{L}(X)$, an operator $A \in \mathcal{L}(X)$ is defined to be *hermitian* if $V(A)$ is real. The following theorem is well-known. (See Bonsall and Duncan [2, 3] for this and many other properties of the numerical range.)

Theorem 4 *Let $A \in \mathcal{L}(X)$. Then*

1) $V(A)$ is connected,

2) A is hermitian if and only if $\|e^{itA}\| = 1$ for all real t,

3) $\overline{\mathrm{co}}\ \sigma(A) \subseteq \overline{V(A)}$

4) $|V(A)| \leq \|A\| \leq e\,|V(A)|$,

5) $\sup \operatorname{Re} W(A) = \lim_{t\to 0^+} \dfrac{\|I + t\,A\| - 1}{t}$

In the above, the symbol $\overline{\mathrm{co}}$ denotes the closed convex hull and the number e is the best constant in (4). Also,

$$|S| = \sup\{|\lambda| :\ \lambda \in S\}, \qquad \sup \operatorname{Re} S = \sup\{\operatorname{Re}\lambda :\ \lambda \in S\},$$

if S is a set of complex numbers.

Following Lumer, one can deduce from part (5) of the above theorem that the closed convex hulls of $W(A)$ and $V(A)$ are equal no matter what choice of Q is taken. Thus, in particular, $|W(A)| = |V(A)|$.

5. The holomorphic numerical range

We first consider the case where the domain is the open unit ball B of X and $h : B \to X$ is a holomorphic function that is uniformly continuous in B. Then h has a uniformly continuous extension to $\overline{B}$. Moreover, h is bounded in B so

$$\|h\| = \sup\{\|h(x)\| :\ x \in B\}$$

is finite. In analogy with (4) and (5), define numerical ranges of h by

$$\begin{aligned} V(h) &= \{\ell(h(x)) :\ \ell \in J(x),\ \|x\| = 1\}, \\ W(h) &= \{\ell(h(x)) :\ \ell \in Q(x),\ \|x\| = 1\}. \end{aligned}$$

Theorem 5 ([14])

1) $\sup \operatorname{Re} W(h) = \lim_{t\to 0^+} \dfrac{\|I + t\,h\| - 1}{t}$,

2) If P_n is a homogeneous polynomial of degree $n > 1$, then

$$\|P_n\| \le n^{\frac{n}{n-1}}\ |W(P_n)|.$$

As was shown in [14], a consequence of part (1) of Theorem 5 is that the closed convex hulls of $W(h)$ and $V(h)$ are the same no matter what choice of Q is taken. It was also shown in [14] that the constant in part (2) of the theorem is best possible.

Following [18], we now extend the numerical range to functions defined on more general domains. Let $\mathcal{D}$ be a convex domain in X and suppose $\mathcal{D}$ contains the origin. For each $x \in \partial\mathcal{D}$, let

$$J(x) = \{\ell \in X^* :\ \ell(x) = 1,\ \operatorname{Re} \ell(y) \le 1 \text{ for all } y \in \mathcal{D}\}.$$

It follows from [8, Cor. 6, p. 449] that $J(x)$ is nonempty. Let $Q(x)$ be a nonempty subset of $J(x)$ for each $x \in X$ with $x \in \partial\mathcal{D}$. If $h : \mathcal{D} \to X$ has a continuous extension to $\overline{\mathcal{D}}$, then we define

$$\begin{aligned} V(h) &= \{\ell(h(x)) :\ \ell \in J(x),\ x \in \partial\mathcal{D}\}, \\ W(h) &= \{\ell(h(x)) :\ \ell \in Q(x),\ x \in \partial\mathcal{D}\}. \end{aligned}$$

Otherwise, consider

$$h_s(x) = h(s\,x), \quad 0 < s < 1.$$

The function h_s always has a continuous extension to $\overline{\mathcal{D}}$ and hence we may define

$$L(h) = \varliminf_{s\to 1^-} \ \sup \operatorname{Re} W(h_s). \tag{6}$$

If h is uniformly continuous on $\mathcal{D}$, then h has a uniformly continuous extension to $\overline{\mathcal{D}}$. Hence, in this case, $W(h)$ is defined and $L(h) = \sup \operatorname{Re} W(h)$.

Lemma 6 (cf [14, Lemma 2]) *If $h : \mathcal{D} \to X$ is holomorphic and bounded on each domain strictly inside $\mathcal{D}$, then h_s is uniformly continuous on $\overline{\mathcal{D}}$ for each s with $0 < s < 1$.*

The following lemma is the key to our extension of the Earle-Hamilton theorem. Throughout the remainder of this section we assume that the domain $\mathcal{D}$ is bounded as well as convex.

Lemma 7 ([18]) *Let $g : \mathcal{D} \to X$ be holomorphic and bounded on each domain strictly inside $\mathcal{D}$. If $L(g) < 0$, then the equation $g(x) = 0$ has a unique solution $x \in \mathcal{D}$.*

Proof. Our proof is a modification of the proof of part (1) of Theorem 5 given in [14]. It is easy to show as in the proof of Theorem 3 that $s\mathcal{D}$ lies strictly inside $\mathcal{D}$ for each s with $0 < s < 1$. Hence by hypothesis and Lemma 6, we may assume (by considering g_s) that g is bounded and uniformly continuous on $R\mathcal{D}$ for some $R > 1$ and that $\sup \mathrm{Re}\, W(g) < 0$.

Let p be the Minkowski functional for $\mathcal{D}$, i.e.,

$$p(x) = \inf\{r > 0 : \ x \in r\,\mathcal{D}\}.$$

Then $p(x+y) \leq p(x) + p(y)$ and $p(tx) = tp(x)$ for all $x, y \in X$ and $t \geq 0$. Also,

$$\mathcal{D} = \{x \in X : \ p(x) < 1\},$$

and there is a number $M > 0$ such that $p(x) \leq M\|x\|$ for all $x \in X$ since $\mathcal{D}$ contains a ball about the origin.

Define

$$w(r, R) = \sup\left\{\mathrm{Re}\,\frac{\ell(g(x))}{p(x)} : \ \ell \in Q(\frac{x}{p(x)}), \ r \leq p(x) < R\right\}$$

for $0 < r < 1$. By the uniform continuity of g on $R\mathcal{D}$, we may choose an $R > 1$ (at least as small as the previous value) and an r with $0 < r < 1$ so that $w(r, R) < 0$.

If $r \leq p(x) < R$ and $\ell \in Q(x/p(x))$, then

$$p\left((I - tg)(x)\right) \geq \mathrm{Re}\,\ell\left((I - tg)(x)\right) = p(x)\left[1 - t\mathrm{Re}\,\frac{\ell(g(x))}{p(x)}\right] \qquad (7)$$

$$\geq p(x)\left[1 - tw(r, R)\right]$$

for all $t \geq 0$.

It is not difficult to show that there exists an r' with $0 < r' < 1$ and a $\delta > 0$ such that

$$r \le p\left((I + tg)(x)\right) < R$$

whenever $r' \le p(x) < 1$ and $0 < t < \delta$. Hence when these inequalities hold, it follows from (7) that

$$p\left((I + tg)(x)\right)[1 - tw(r, R)] \le p\left((I - tg)(I + tg)(x)\right).$$

Now,

$$(I - tg)(I + tg)(x) = x + t[g(x) - g(x + tg(x))].$$

By the uniform continuity of g on $R\mathcal{D}$, for each given $\epsilon > 0$ there is a $\delta > 0$ (choose it at least as small as the previous one) such that

$$p\left(g(x) - g(x + tg(x))\right) < \epsilon$$

whenever $p(x) < 1$ and $0 < t < \delta$. Hence

$$p\left((I + tg)(x)\right) \le \frac{1 + t\epsilon}{1 - tw(r, R)} \tag{8}$$

whenever $r' \le p(x) < 1$ and $0 < t < \delta$. By the maximum principle (extend [20, p. 115]), inequality (8) holds whenever $p(x) < 1$ and $0 < t < \delta$.

Choose $\epsilon > 0$ with $\epsilon < -w(r, R)$ and fix t with $0 < t < \delta$. Then the right-hand side of (8) is a constant less than one. It follows that $I + tg$ is a holomorphic mapping of $\mathcal{D}$ into $k\mathcal{D}$ for some constant k with $0 < k < 1$. Hence by the Earle-Hamilton Theorem, $I + tg$ has a unique fixed point in $\mathcal{D}$ so $g(x) = 0$ has a unique solution in $\mathcal{D}$.

Theorem 8 ([18]) *Let $h : \mathcal{D} \to X$ be holomorphic and bounded on each domain strictly inside $\mathcal{D}$. If $L(h) < 1$, then h has a unique fixed point in $\mathcal{D}$.*

Proof. This theorem follows from the previous lemma with $g = h - I$ since $L(g) = L(h) - 1 < 0$.

The above theorem is an extension of the Earle-Hamilton fixed point theorem (for the domains we consider) since it is not difficult to show that $L(h) < 1$ when h maps $\mathcal{D}$ strictly inside $\mathcal{D}$.

Increasingly general definitions of $L(h)$ are given in [18, 17] and (6). We show that under mild restrictions all three definitions agree and are independent of the choice of Q.

Theorem 9 *Let $h : \mathcal{D} \to X$ be holomorphic and bounded on each domain strictly inside $\mathcal{D}$ and suppose $L(h) < \infty$ for some choice of Q. Then* $\lim_{s\to 1^-} \sup \mathrm{Re}\, W(h_s)$ *exists and is the same for all choices of Q.*

Proof. Suppose $L(h) < M$. Let $g = h - MI$ and note that $L(g) < 0$ since

$$\sup \mathrm{Re}\, W(g_s) = \sup \mathrm{Re}\, W(h_s) - sM.$$

It follows from the proof of Lemma 7 that $(I + tg)(\mathcal{D}) \subseteq k\mathcal{D}$ where $t > 0$ and $0 < k < 1$. Hence if $x \in \partial\mathcal{D}$ and $\ell \in J(x)$,

$$s + t\mathrm{Re}\, \ell(g_s(x)) = \mathrm{Re}\, \ell((I + tg)(sx) \leq k$$

whenever $0 < s < 1$. Then

$$\overline{\lim_{s\to 1^-}} \sup \mathrm{Re}\, V(g_s) < 0,$$

so

$$\overline{\lim_{s\to 1^-}} \sup \mathrm{Re}\, V(h_s) < M.$$

Hence,

$$\overline{\lim_{s\to 1^-}} \sup \mathrm{Re}\, V(h_s) \leq \underline{\lim_{s\to 1^-}} \sup \mathrm{Re}\, W(h_s).$$

Since $\sup \mathrm{Re}\, W(h_s) \leq \sup \mathrm{Re}\, V(h_s)$ when $0 < s < 1$, it follows that the upper and lower limits of each of these terms are the same. Thus the required limit exists and is the same for all Q.

It follows from Corollary 14 (below) that if $\mathcal{D}$ is the open unit ball of X and if $L(h)$ is finite for the choice $Q = J$, then h is bounded on each domain strictly inside $\mathcal{D}$. Thus, in the case mentioned, this hypothesis may be omitted from Lemma 7, Theorem 8 and Theorem 9.

It is an open problem whether if B is the open unit ball of a separable reflexive Banach space X and if $h : B \to X$ is a holomorphic function with a uniformly continuous extension to $\overline{B}$ satisfying $\sup \mathrm{Re}\, V(h) \leq 1$ then h has a fixed point in $\overline{B}$. When X is a Hilbert space, this follows from a result of Aizenberg, Reich and Shoikhet. (Take $f = I - h$ in Theorem 2 of [1].) This result contains the theorem of Goebel, Sekowski and Stachura mentioned after Example 3.

In the case where the underlying space is finite dimensional, there is a version of Lemma 7 where the domain $\mathcal{D}$ does not need to be convex and the numerical range is computed as in a Hilbert space. (For purposes of comparison, we reformulate the theorem.)

Theorem 10 (M. H. Shih [29]) *Suppose X is finite dimensional and let $(\cdot,\cdot)$ be an inner product on X. Let $\mathcal{D}$ be a bounded domain in X containing the origin and let $g : \mathcal{D} \to X$ be a holomorphic function with a continuous extension to $\overline{\mathcal{D}}$. If $\operatorname{Re}(g(x), x) < 0$ for all $x \in \partial\mathcal{D}$, then the equation $g(x) = 0$ has a unique solution x in $\mathcal{D}$.*

6. Application to Bloch radii

Let B be the open unit ball of a complex Banach space X and let $h : B \to X$ be a holomorphic function satisfying $h(0) = 0$ and $Dh(0) = I$. We say that positive numbers r and P with $r < 1$ are *Bloch radii for h* if h maps a subdomain of $B_r(0)$ biholomorphically onto $B_P(0)$.

It follows from the Cauchy estimate for $Dh^{-1}(0)$ that $P \leq r$ since $Dh^{-1}(0) = I$. Without further restrictions on h, there is no value of P that is independent of h, even in the case $X = \mathbb{C}$. To see this, note that

$$h(z) = \frac{1}{n}[(1+z)^n - 1]$$

is holomorphic in the open unit disc Δ of the complex plane with $h(0) = 0$ and $h'(0) = 1$ but $-1/n$ is not in $h(\Delta)$.

However, when h or its derivative has a given bound in B, it is known that a P can be found that depends only on this bound.

Theorem 11 ([15]) *Let $h : B \to X$ be a holomorphic function with $h(0) = 0$ and $Dh(0) = I$.*

1) *Suppose $\|Dh(x)\| \leq M$ for all $x \in B$. Then*

$$r = \frac{1}{M}, \qquad P = \frac{1}{2M}$$

are Bloch radii for h.

2) *Suppose $\|h(x)\| \leq M$ for all $x \in B$. Then*

$$r = \frac{1}{\sqrt{4M^2+1}}, \qquad P = \frac{1}{2M + \sqrt{4M^2+1}}$$

are Bloch radii for h.

Theorem 12 ([18]) *Let $h : B \to X$ be a holomorphic function with $h(0) = 0$ and $Dh(0) = I$, and suppose $L(h) \le M$ for the choice $Q = J$. Then*

$$r = 1 - \sqrt{1 - \frac{1}{2M-1}}, \qquad P = \left(\sqrt{2M-1} - \sqrt{2(M-1)}\right)^2$$

are Bloch radii for h.

It can be verified that the value of P in Theorem 12 is greater than the value of P in part (2) of Theorem 11 for $M < 1 + 1/\sqrt{3}$ but not otherwise. Thus since $L(h) \le \|h\|$, Theorem 12 improves part (2) of Theorem 11 when $M < 1 + 1/\sqrt{3}$.

Proof. Without loss of generality we may assume that $L(h) < M$. Then for each $\delta > 0$ there is an s with $1 - \delta < s < 1$ and

$$\sup \operatorname{Re} V(h_s) < s\, M.$$

Let $x \in X$ with $\|x\| = 1$ and $\ell \in J(x)$. The function

$$f(\lambda) = \frac{1}{s\lambda} \ell(h(s\lambda x)) - 1$$

is holomorphic in the disc $|\lambda| < 1/s$ and satisfies $f(0) = 0$. Also,

$$\operatorname{Re} f(\lambda) \le \frac{1}{s} \sup \operatorname{Re} V(h_s) - 1 < M - 1$$

when $|\lambda| = 1$ (since then $\ell/\lambda \in J(\lambda x)$). In particular, $M > 1$. Hence by the Borel-Caratheodory lemma [30, p. 175],

$$|f(\lambda)| \le \frac{2(M-1)|\lambda|}{1 - |\lambda|}$$

for $|\lambda| < 1$. Observing that $\operatorname{Re} - f(\lambda) \le |f(\lambda)|$, taking $\lambda = t$ and letting $s \to 1^-$, we obtain

$$\operatorname{Re}\, [t - \ell(h_t(x))] \le \frac{2(M-1)t^2}{1-t}$$

for $0 < t < 1$. Define

$$\Phi(t) = t - \frac{2(M-1)t^2}{1-t}$$

and note that by calculus the maximum of the function Φ in the interval $(0,1)$ is assumed at $t=r$ and $\Phi(r)=P$. If $y \in X$, then

$$\operatorname{Re} \ell(y - h_r(x)) \le \|y\| - \Phi(r)$$

so

$$\sup \operatorname{Re} V(y - h_r) < 0$$

if $\|y\| < P$. Thus by Lemma 7, the equation $y - h_r(x) = 0$ has a unique solution $x \in B$. Moreover, the proof of that lemma shows that x depends holomorphically on y since the fixed point in the Earle-Hamilton theorem depends holomorphically on y. Hence h maps a subdomain of $B_r(0)$ biholomorphically onto $B_P(0)$.

7. Cartan's uniqueness theorem

In this section we use the numerical range to obtain a distortion form of Cartan's uniqueness theorem for the open unit ball B of a complex Banach space. A version of our theorem for holomorphic functions mapping B into the closed unit ball of Y appears in [13].

Theorem 13 ([18]) *Let $h : B \to X$ be a holomorphic function with $h(0) = 0$ and $Dh(0) = I$, and let $L(h)$ be defined with $Q = J$. Then*

$$\|h(x) - x\| \le \frac{8\|x\|^2}{(1 - \|x\|)^2} (L(h) - 1)$$

for all $x \in B$. In particular, $h = I$ when $L(h) \le 1$.

Corollary 14 *If $h : B \to X$ is holomorphic and if $L(h) < \infty$ when $Q = J$, then h is bounded on each $B_r(0)$ with $0 < r < 1$.*

Proof. By Taylor's theorem there exists an $r > 0$ such that

$$h(x) = \sum_{n=1}^{\infty} P_n(x)$$

whenever $x \in B_r(0)$. Let $M > L(h)$. Proceeding as in the proof of Theorem 12, we obtain a function $f(\lambda)$ that is holomorphic on the open unit disc

Δ of the complex plane and satisfies $f(0) = 0$ and $\mathrm{Re}\, f(\lambda) \le M - 1$ for all $\lambda \in \Delta$. Define

$$g(\lambda) = 1 - \frac{f(\lambda)}{M-1}$$

and note that $\mathrm{Re}\, g(\lambda) \ge 0$ for all $\lambda \in \Delta$. Moreover,

$$g(\lambda) = 1 + \sum_{n=1}^{\infty} a_n \lambda^n,$$

where

$$a_n = \frac{\ell(P_{n+1}(x))s^n}{1-M}, \quad n = 1, 2, \ldots .$$

Since $|a_n| \le 2$ for all positive integers n by [27, p. 170], it follows that

$$|V(P_n)| \le 2(M-1), \quad n = 2, 3, \ldots .$$

Applying part (2) of Theorem 5 and the inequality $n^{n/(n-1)} \le 2n$ for $n \ge 2$, we obtain

$$\|P_n\| \le 4n(M-1), \quad n = 2, 3, \ldots .$$

Thus the power series for h has unit radius of convergence and is equal to h on B by the identity theorem [20, Th. 3.16.4]. Therefore,

$$\begin{aligned} \|h(x) - x\| &\le \sum_{n=2}^{\infty} \|P_n\| \, \|x\|^n \\ &\le 4(M-1) \sum_{n=2}^{\infty} n \|x\|^n \\ &\le 4(M-1) \frac{2\|x\|^2}{(1-\|x\|)^2} \end{aligned}$$

for $x \in B$. The required inequality follows since $M > L(h)$ was arbitrary.

One can deduce the Corollary 14 by applying Theorem 13 to $h - h(0) + I - Dh(0)$.

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