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Markov's Inequality for Polynomials on Normed Linear Spaces

Lawrence A. Harris

This article is dedicated to the 70th anniversary of Acad. Bl. Sendov

It is a longstanding problem to determine the best bounds for each of the derivatives of a polynomial with unit bound on the closed unit ball of a real normed linear space. In this note we review the progress made on this problem and give an exposition of the main results. Most of these are due to Sarantopoulos and Muñoz. We include their elegant proofs.

1. Introduction

The following celebrated theorem of V. A. Markov appeared in 1892. It extended a result in 1889 of his older brother, A. A. Markov, who considered only the case of the first derivative.

Theorem 1. (V. A. Markov) *If $p(x)$ is a polynomial of degree at most m and if $|p(x)| \leq 1$ whenever $-1 \leq x \leq 1$, then $|p^{(k)}(x)| \leq T_m^{(k)}(1)$ whenever $-1 \leq x \leq 1$ and $1 \leq k \leq m$.*

Here $T_m(x)$ is the m-th Chebyshev polynomial and is given by $T_m(x) = \cos(m \cos^{-1}(x))$. It is easy to verify that $T_m(x)$ is a polynomial of degree m satisfying $|T_m(x)| \leq 1$ whenever $-1 \leq x \leq 1$. Thus the constants in Theorem 1 cannot be improved. Explicitly,

$$T_m^{(k)}(1) = \frac{m^2(m^2-1^2)(m^2-2^2)\cdots(m^2-(k-1)^2)}{1 \cdot 3 \cdot 5 \cdots (2k-1)}$$

and, in particular, $T_m'(1) = m^2$ and $T_m^{(m-1)}(1) = T_m^{(m)}(1) = 2^{m-1}m!$.

Our object is to review recent progress in extending Markov's theorem to polynomials on any real normed linear space X where the closed unit ball B

of X plays the role of the interval $-1 \leq x \leq 1$. Specifically, the problem is to determine the best bound $M_{m,k}$ for the k-th order directional derivative on B of polynomials with unit bound on B and with degree at most m. (For polynomials, the k-th order directional derivative is the same as the homogeneous polynomial associated with the k-th order Fréchet derivative.)

Y. Sarantopoulos [17] has shown that the best bound on the first derivative is the same for real normed linear spaces as it is for the real line, i.e., $M_{m,1} = m^2$. Also, G. Muñoz and Y. Sarantopoulos [14] have shown that the best bounds on each derivative for real Hilbert spaces are the same as they are for the real line, i.e., $M_{m,k} = T_n^{(k)}(1)$ for $1 \leq k \leq m$ when X is restricted to real Hilbert spaces.

For general real normed linear spaces X, the author has established in [5] that

$$T_m^{(k)}(1) \leq M_{m,k} \leq 2^{2k-1} T_m^{(k)}(1)$$

for $1 \leq k \leq m$ but it is not known whether $M_{m,k} = T_m^{(k)}(1)$ when $1 < k < m$. Many years ago, A. D. Michal announced this equality as a theorem in [12] and [13, p. 72-78] (both published posthumously) but unfortunately his arguments contain an elementary error. (See the review of [12] by R. G. Bartle in Math. Reviews 15,630f.)

We first list classical results that are used in the proofs of the extended Markov theorems. Then we review definitions and notation for polynomials on normed linear spaces and their derivatives. Next, following [17, 14], we give elementary proofs of the mentioned Markov theorems.

A useful lemma shows that the constants $M_{m,k}$ are no larger for general normed linear spaces than they are for the case where X is $\ell^1(\mathbb{R}^2)$, i.e., the space $\mathbb{R}^2$ with norm $\|(s,t)\| = |s| + |t|$.

Finally, we review the main facts about Markov's inequality for homogeneous polynomials on real normed linear spaces. In this case, obtaining a Markov inequality reduces to finding the best bounds on the coefficients of a polynomial of a single variable t that is dominated by the function $(1 + |t|)^m$. A table of these best bounds is given.

2. Classical inequalities

In this section we collect classical inequalities that will be used in the proofs below.

A proof of the case $k = 1$ of Theorem 1 due to A. A. Markov can be deduced easily from the following two elementary results. (See, for example, [11] or [15].)

Proposition 2. (Bernstein) *If $T(\theta)$ is a trigonometric polynomial of degree at most m and if $|T(\theta)| \le 1$ for all real θ, then $|T'(\theta)| \le m$ for all real θ.*

Proposition 3. (Schur) *If $p(x)$ is a polynomial of degree at most $m-1$ satisfying*

$$|p(x)| \le \frac{M}{\sqrt{1-x^2}}$$

whenever $-1 < x < 1$, then $|p(x)| \le mM$ whenever $-1 \le x \le 1$.

It is not difficult to prove that the Chebyshev polynomials give the best bounds for polynomial derivatives at points outside the the open unit interval (see [16]).

Proposition 4. (Schur) *If $p(x)$ is a polynomial of degree at most m and if $|p(x)| \le 1$ whenever $-1 \le x \le 1$, then $|p^{(k)}(x)| \le |T_m^{(k)}(x)|$ whenever $|x| \ge 1$ and $1 \le k \le m$.*

Corollary 5. (Chebyshev) *If $p(x) = a_0 + a_1 x + \cdots + a_m x^m$ satisfies $|p(x)| \le 1$ whenever $-1 \le x \le 1$, then $|a_m| \le 2^{m-1}$.*

Note that the above corollary and the case $k = m$ of Theorem 1 follow immediately from Proposition 4.

Duffin and Schaeffer [2, 3] have obtained elementary proofs of Theorem 1 although they are still considerably deeper than the proofs of any of the above results. Their proof in [2] is essentially establishing the following two propositions. Define

$$\mathcal{M}_{m,k}(x) = T_m^{(k)}(x)^2 + S_m^{(k)}(x)^2,$$

where $S_m(x) = \sin(m \cos^{-1}(x))$.

Proposition 6. (Duffin and Schaeffer) *If $p(x)$ is a polynomial of degree at most m and if $|p(x)| \le 1$ whenever $-1 \le x \le 1$, then $|p^{(k)}(x)|^2 \le \mathcal{M}_{m,k}(x)$ whenever $-1 \le x \le 1$ and $1 \le k \le m$.*

Proposition 7. ([14]) *If $p(x)$ is a polynomial of degree at most $m-k$, where $1 \le k \le m$, and if $|p(x)|^2 \le \mathcal{M}_{m,k}(x)$ whenever $-1 \le x \le 1$, then $|p(x)| \le T_m^{(k)}(1)$ whenever $-1 \le x \le 1$.*

Note that Proposition 3 is a special case of Proposition 7 since $\mathcal{M}_{m,1}(x) = m^2/(1-x^2)$.

3. Definitions and notation

In this section we recall the definitions of polynomial and its Fréchet and directional derivatives.

Let X and Y be real normed linear spaces. Given a positive integer m, a mapping $P : X \to Y$ is called a *homogeneous polynomial of degree* m if there exists a continuous symmetric m-linear mapping $F : X \times \cdots \times X \to Y$ such that $P(x) = F(x, \ldots, x)$ for all $x \in X$. In this case we write $P = \hat{F}$ and call P the homogeneous polynomial associated with F. A mapping $P : X \to Y$ is called a *polynomial of degree at most* m if

$$P = P_0 + P_1 + \cdots + P_m, \tag{1}$$

where $P_k : X \to Y$ is a homogeneous polynomial of degree k for $k = 1, \ldots, m$ and a constant function for $k = 0$.

Given $x \in X$, the *Fréchet derivative of* P *at* x, denoted by $DP(x)$, is a continuous linear map $L : X \to Y$ such that

$$\lim_{y \to 0} \frac{\|P(x+y) - P(x) - L(y)\|}{\|y\|} = 0.$$

Clearly, $DP(x)y = \frac{d}{dt} P(x+ty)\Big|_{t=0}$ when $DP(x)$ exists.

If F is a continuous symmetric m-linear mapping, we write $F(x^j y^k)$ for $F(\underbrace{x, \ldots, x}_{j}, \underbrace{y, \ldots, y}_{k})$. Then the binomial theorem for F can be written as

$$\hat{F}(x+y) = \sum_{k=0}^{m} \binom{m}{k} F(x^{m-k} y^k).$$

It follows from the continuity of F that $D\hat{F}(x)y = mF(x^{m-1}y)$. Hence if $P : X \to Y$ is a polynomial of degree at most m then $DP : X \to \mathcal{L}(X, Y)$ is a polynomial of degree at most $m-1$, where $\mathcal{L}(X, Y)$ denotes the space of all continuous linear mappings $L : X \to Y$ with the operator norm, i.e., $\|L\| = \sup\{\|L(x)\| : \|x\| \le 1\}$.

It is an elementary fact that the k-th order Fréchet derivative $D^k P(x)$ may be identified with a continuous symmetric k-linear map. We denote the associated homogeneous polynomial of degree k by $\hat{D}^k P(x)$ and call this the k-th order directional derivative of P at x. Thus,

$$\hat{D}^k P(x)y = \frac{d^k}{dt^k} P(x+ty)\Big|_{t=0} \tag{2}$$

and

$$\frac{1}{k!}\hat{D}^k\hat{F}(x)y = \binom{m}{k} F(x^{m-k}y^k) \tag{3}$$

for all integers k with $0 \le k \le m$.

As usual, if $P : X \to Y$ is a polynomial, we define

$$\|P\| = \sup\{\|P(x)\| : \ \|x\| \le 1\}$$

and

$$\|D^k P(x)\| = \sup\{\|D^k P(x)(x_1, \dots, x_k)\| : \ \|x_1\| \le 1, \dots, \|x_k\| \le 1\}$$

for $x \in X$. By an inequality of R. S. Martin [1, Th. 1.7],

$$\|D^k P(x)\| \le \frac{k^k}{k!}\|\hat{D}^k P(x)\|$$

for $x \in X$. If X is a real Hilbert space, by an equality due to Banach and others (see [4, Th. 4] or [1, Ex. 1.9]),

$$\|D^k P(x)\| = \|\hat{D}^k P(x)\|$$

for $x \in X$.

For further discussion of concepts in this section, see [1] and [9].

4. Extension of Markov's inequalities

Let m and k be positive integers with $1 \le k \le m$. The fundamental problem we consider is to find the smallest number $M_{m,k}$ such that

$$\|\hat{D}^k P\| \le M_{m,k}\|P\|$$

whenever $P : X \to Y$ is a polynomial of degree at most m and X and Y are any real normed linear spaces. We shall always assume that $Y = \mathbb{R}$ since by the Hahn-Banach theorem the numbers $M_{m,k}$ do not change when we restrict to this case.

The proof of the following estimates requires only Proposition 4 (above).

Proposition 8. ([5, p. 149])

$$T_m^{(k)}(1) \le M_{m,k} \le 2^{2k-1}T_m^{(k)}(1).$$

P r o o f. By taking $X = Y = \mathbb{R}$ and $P = T_m$, we obtain the lower estimate. To obtain the upper estimate, let $P : X \to \mathbb{R}$ be a polynomial of degree at most m and suppose $\|P\| \le 1$. Let $x, y \in X$ with $\|x\| \le 1$ and $\|y\| \le 1$ and let $-1 \le s \le 1$. Define $q(t) = P(\phi(t))$, where

$$\phi(t) = \frac{x - sy + t(x + sy)}{2}.$$

Since

$$\|\phi(t)\| \le \frac{1+t}{2} + \frac{1-t}{2} = 1$$

when $-1 \le t \le 1$, it follows that $q(t)$ is a polynomial of degree at most m satisfying $|q(t)| \le 1$ whenever $-1 \le t \le 1$. Then $|q^{(k)}(1)| \le |T_m^{(k)}(1)|$ by Proposition 4 and

$$q^{(k)}(1) = \frac{1}{2^k} \hat{D}^k P(x)(x + sy)$$

by (2). Put $p(s) = \hat{D}^k P(x)(x + sy)$ and $M = 2^k T_m^{(k)}(1)$. Then p is a polynomial of degree at most k with $|p(s)| \le M$ whenever $-1 \le s \le 1$. Since $\hat{D}^k P(x)(y)$ is the coefficient of s^k in $p(s)$, we have

$$|\hat{D}^k P(x)(y)| \le 2^{k-1} M$$

by Corollary 5. Thus $\|\hat{D}^k P\| \le 2^{2k-1} T_m^{(k)}(1)$. ■

The following reduces the computation of the numbers $M_{m,k}$ to the case where X is $\ell^1(\mathbb{R}^2)$, i.e., the space $\mathbb{R}^2$ with norm $\|(s,t)\| = |s| + |t|$.

Lemma 9. *The number $M_{m,k}$ is the supremum of the absolute values of $\frac{d^k}{dt^k} q(1,t)\Big|_{t=0}$ where $q : \mathbb{R}^2 \to \mathbb{R}$ is any polynomial of degree at most m such that $|q(s,t)| \le 1$ whenever $|s| + |t| \le 1$.*

P r o o f. Let $P : X \to \mathbb{R}$ be a polynomial of degree at most m. Without loss of generality we may suppose that $\|P\| \le 1$. Given $x, y \in X$ with $\|x\| \le 1$ and $\|y\| \le 1$, define

$$q(s,t) = P(sx + ty)$$

for all real numbers s and t. Clearly q satisfies the mentioned conditions and

$$q((1,0) + t(0,1)) = P(x + ty)$$

so

$$\hat{D}^k q(1,0)(0,1) = \frac{d^k}{dt^k} q(1,t)\Big|_{t=0} = \hat{D}^k P(x) y$$

by (2). ■

For the case of the first and highest derivative, Y. Sarantopoulos [17] and G. Muñoz and Y. Sarantopoulos [14] have shown that the best constant in Markov's theorem is no larger for real normed linear spaces than it is for the real line.

Theorem 10. ([17, 14])

$$M_{m,1} = m^2, \quad M_{m,m} = 2^{m-1} m!.$$

Proof. We first prove $M_{m,m} = 2^{m-1} m!$. Let $P : X \to \mathbb{R}$ be a polynomial of degree at most m and note that $\hat{D}^m P(x)y = m! P_m(y)$ for all $x, y \in X$ by (1) and (3). If $\|P\| \le 1$ and $\|y\| \le 1$, then $p(t) = P(ty)$ is a polynomial of degree at most m with $|p(t)| \le 1$ whenever $-1 \le t \le 1$ and $p^{(m)}(0) = \hat{D}^m P(0)y$. Hence

$$|\hat{D}^m P(x)y| \le 2^{m-1} m!$$

by Corollary 5. Therefore, $M_{m,m} \le 2^{m-1} m!$ and equality follows from Proposition 8.

We use Lemma 9 to prove that $M_{m,1} = m^2$. Let $q : \mathbb{R}^2 \to \mathbb{R}$ be a polynomial of degree at most m with $|q(s,t)| \le 1$ whenever $|s| + |t| \le 1$. Given $-1 < r < 1$, define

$$\begin{aligned} T(\theta) &= q(r\cos\theta, \sqrt{1-r^2}\sin\theta), \\ p(r) &= \left.\frac{d}{dt} q(r,t)\right|_{t=0}. \end{aligned}$$

It suffices to show that $|p(1)| \le m^2$. It is easy to show that $T(\theta)$ is a trigonometric polynomial of degree at most m. Also, $|T(\theta)| \le 1$ for all real θ since

$$|r\cos\theta| + |\sqrt{1-r^2}\sin\theta| \le 1$$

by the Cauchy-Schwarz inequality. Hence $|T'(\theta)| \le m$ by Proposition 2. Since $T'(0) = p(r)\sqrt{1-r^2}$, it follows that

$$|p(r)| \le \frac{m}{\sqrt{1-r^2}}$$

whenever $-1 < r < 1$. Thus $|p(r)| \le m^2$ whenever $-1 \le r \le 1$ by Proposition 3. ■

G. Muñoz and Y. Sarantopoulos [14] have completely solved the problem of determining the Markov constants for the case of Hilbert spaces. The case of the first derivative was done in finite dimensions by Kellogg [10] and in infinite dimensions by the author [4].

Theorem 11 ([14]) *If the spaces X in the definition of $M_{m,k}$ are restricted to real Hilbert spaces, then $M_{m,k} = T_m^{(k)}(1)$ whenever $1 \leq k \leq m$.*

P r o o f. Let $P : X \to \mathbb{R}$ be a polynomial of degree at most m and suppose $\|P\| \leq 1$. Given $x, y \in X$ with $\|x\| < 1$ and $\|y\| \leq 1$, define

$$p(t) = P(x + \alpha(t - \beta)y),$$

where

$$\alpha = \sqrt{1 - \|x\|^2 + (x, y)^2}, \quad \beta = \frac{(x, y)}{\alpha}.$$

Note that $\|x + \alpha(t - \beta)y\| \leq 1$ whenever $-1 \leq t \leq 1$ since $0 < \alpha \leq 1$ and

$$\|x + \alpha(t - \beta)y\|^2 = \|x - (x, y)y + t\alpha y\|^2 = 1 - \alpha^2 + t^2\alpha^2.$$

Hence $p(t)$ is a polynomial of degree at most m with $|p(t)| \leq 1$ whenever $-1 \leq t \leq 1$ and $p^{(k)}(\beta) = \alpha^k \hat{D}^k P(x)y$ by (2). Then by Proposition 6 and the identity $\alpha^2(1 - \beta^2) = 1 - \|x\|^2$, we have

$$|\hat{D}^k P(x)y|^2 \leq \frac{\mathcal{M}_{m,k}(\beta)}{\alpha^{2k}} = \frac{\mathcal{M}_{m,k}(\beta)(1 - \beta^2)^k}{(1 - \|x\|^2)^k}.$$

It is evident from [14, Lemma 2] that $\mathcal{M}_{m,k}(\beta)(1-\beta^2)^k$ is an increasing function of β on the interval $[0, 1)$ and it is easy to show that $\mathcal{M}_{m,k}(\beta)$ is an even function of β. Since $|\beta| \leq \|x\|$, it follows that

$$|\hat{D}^k P(x)y|^2 \leq \mathcal{M}_{m,k}(\|x\|).$$

(Compare [4, §6].)

Now define $q(t) = \hat{D}^k P(tx/\|x\|)y$. Then $q(t)$ is a polynomial of degree at most $m - k$ satisfying $|q(t)|^2 \leq \mathcal{M}_{m,k}(t)$ whenever $-1 \leq t \leq 1$. Hence by Proposition 7, we have $|q(\|x\|)| \leq T_n^{(k)}(1)$, which is the required inequality. ■

5. Markov's inequalities for homogeneous polynomials

Let m and k be positive integers with $1 \leq k \leq m$. Define $c_{m,k}$ to be the smallest number satisfying

$$\|\hat{D}^k P\| \leq c_{m,k}\|P\| \tag{4}$$

whenever $P : X \to Y$ is a homogeneous polynomial of degree m and X and Y are any real normed linear spaces. As before, we may take $Y = \mathbb{R}$ without loss of generality.

The following theorem and proposition have been established in [6].

Theorem 12.

$$\frac{m^m}{k^k(m-k)^{m-k}} \le \frac{c_{m,k}}{k!} \le \binom{m}{k} \frac{m^{m/2}}{k^{k/2}(m-k)^{(m-k)/2}},$$

$$c_{m,k} \le (Mm \log m)^k, \quad m > 1,$$

where M is an absolute constant and $0^0 = 1$.

Proposition 13. *The number $c_{m,k}$ is the supremum of the absolute values of $p^{(k)}(0)$ where $p : \mathbb{R} \to \mathbb{R}$ is any polynomial satisfying*

$$|p(t)| \le (1 + |t|)^m$$

for every $t \in \mathbb{R}$.

Extremal polynomials for Proposition 13 are given to eight significant digits in Tables I and II of [7]. The corresponding extremal polynomials on $\ell^1(\mathbb{R}^2)$ for (4) are obtained from the formula

$$P(x_1, x_2) = x_1^m p\left(\frac{x_2}{x_1}\right).$$

A different approach to the justification of the computations is given in [8]. It is easy to deduce from Proposition 13 that $c_{m,k}/k! = c_{m,m-k}/(m-k)!$. Thus all values of $c_{m,k}$ for $m = 1, .., 20$ can be obtained from the table below.

No explicit or recursive formula for $c_{m,k}$ is known.

Table of values of $c_{m,k}/k!$

m	$k=1$ $k=8$	$k=2$ $k=9$	$k=3$ $k=10$	$k=4$	$k=5$	$k=6$	$k=7$
2	4.0000000						
3	6.9768501						
4	10.392305	18.000000					
5	13.889998	36.900361					
6	17.614678	63.275280	99.229356				
7	21.411366	99.404353	211.77494				
8	25.358580	144.73923	403.20602	543.47846			
9	29.365943	201.25335	692.17341	1221.0722			
10	33.485122	268.33173	1116.2572	2423.4878	3189.5441		
11	37.655346	347.75003	1697.4134	4462.2666	7235.2644		
12	41.914185	438.85701	2481.3769	7644.1709	15067.717	18508.628	
13	46.217198	543.28842	3492.7183	12477.487	28857.986	43102.021	
14	50.593432	660.37138	4784.7605	19427.245	52260.643	91432.163	111436.95
15	55.008544	791.63654	6384.1591	29245.167	89525.390	182227.81	260260.29
16	59.485946 664491.03	936.39747	8351.3837	42582.592	147415.59	341124.71	567901.37
17	63.998042 1577544.8	1096.1024	10714.754	60480.150	233318.56	610763.02	1156556.0
18	68.564278 3476297.1	1270.0561 4057566.4	13541.531	83801.360	358568.85	1045371.5	2242807.1
19	73.161829 7266132.8	1459.6402 9639233.7	16861.354	113920.08	534972.61	1730436.5	4137163.6
20	77.807218 14383919.0	1664.1539 21645133.0	20747.984 24579265.0	151935.92	780200.70	2768847.3	7354205.0

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Mathematics Department
University of Kentucky
Lexington, Kentucky 40506-0027
E-MAIL: larry@ms.uky.edu

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