Problems

Set $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -3 \\ -5 & 15 \\ 4 & -12 \end{pmatrix}$, and $C = \begin{pmatrix} 1 & 6 & 2 & 0 \\ 0 & 0 & 7 & 7 \\ -1 & -6 & 0 & 2 \end{pmatrix}$.

Determine the column spaces of A, B, and C.

Solution

First note that all column spaces are subspaces of $\mathbb{R}^3$. Why?

Formally, $C(A) = C_A = \left\{ \alpha_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} \;\middle|\; \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R} \right\}$. So, when is a given vector in the span of the column vectors? That is, for what vectors $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ can we find $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ so that

$$\alpha_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} ?$$

Clearly, by setting $\alpha_1 = b_1$, $\alpha_2 = \frac{b_2}{5}$, $\alpha_3 = -\frac{b_3}{2}$ we see that every vector $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \in \mathbb{R}^3$ is in the span of the column of the matrix A. Hence, $C(A) = C_A = \mathbb{R}^3$.

So, is it the case here that $C(B) = C_B = \mathbb{R}^3$ as in the above case?

We find here that

$$C(B) = C_B = \left\{ \alpha \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix} \mid \alpha \in \mathbb{R} \right\}$$

since $c_2 = (-3)\ c_1$ and $C(B)$ is a line through the origin in the direction of $\begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix}$.

What does the column space for the matrix $C = \begin{pmatrix} 1 & 6 & 2 & 0 \\ 0 & 0 & 7 & 7 \\ -1 & -6 & 0 & 2 \end{pmatrix}$ look like?

Hint: $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 6 \\ 0 \\ -6 \end{pmatrix}$ and $(-2) \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ 7 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 7 \\ 2 \end{pmatrix}$

So, formally, the column space for the matrix C is given by $\left\langle \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 7 \\ 0 \end{pmatrix} \right\rangle$.

$(\ 7x - 2y + 7z = 0\ )$