Our initial definition of a line is a bit complicated but will be improved soon: lines are either vertical and have an equation $x = c$ or are non-vertical and have an equation $y = m \cdot x + b$.

***Problem* 13.1.** (H)

Suppose a, b, c are real and both a and b are not zero at the same time. Show that $a \cdot x + b \cdot y + c = 0$ represents a line.

Hint(s) to 13.1: Can you solve for y? What if you cannot solve for y?

***Problem* 13.2.** (H)

Show that each line on the plane has equation of the form $a \cdot x + b \cdot y + c = 0$, where a, b, c are real and both a and b are not zero at the same time.

Hint(s) to 13.2: First consider vertical lines, then non-vertical ones.

***Problem* 13.3.** (H)

Suppose a line on the plane has two equations: $a \cdot x + b \cdot y + c = 0$ and $a' \cdot x + b' \cdot y + c' = 0$. Show that there is a non-zero real number k such that $a' = k \cdot a$, $b' = k \cdot b$, and $c' = k \cdot c$.

Hint(s) to 13.3: Can you solve for x or y from both equations? If you can solve for x what kind of a line is that?

***Problem* 13.4.** Show that each line on the plane has equation of the form $a \cdot z + b \cdot \bar{z} + c = 0$, where a is the conjugate of b, $b \neq 0$, and c is real.

***Problem* 13.5.** (H)

Suppose a line on the plane has two equations: $a\cdot z+b\cdot\bar{z}+c=0$ and $a'\cdot z+b'\cdot\bar{z}+c'=0$. Show that there is a non-zero complex number k such that $a'=k\cdot a$, $b'=k\cdot b$, and $c'=k\cdot c$. Conclude that, if $c\neq 0$ is real, then b is the conjugate of a.

Hint(s) to 13.5: Can you solve for z or $\bar{z}$? What does it mean if you can solve for z (is that a line)?

***Problem* 13.6.** (A) The line passing through $1+2\cdot i$ and $4+5\cdot i$ has equation $a\cdot z+b\cdot \bar{z}+c=0$, where $c=2$. Find a.

Answer to 13.6: $1 + 1i$

***Problem* 13.7.** Let a, b, c be three complex numbers. Show that if c is the algebraic average of a and b (i.e., $c = (a + b)/2$), then c is the geometric midpoint between a and b (i.e., $|a-c| = |b-c| = |a - b|/2$).

***Problem* 13.8.** Let a, b, c be three complex numbers. Show that if c is the geometric midpoint between a and b (i.e., $|a - c| = |b - c| = |a - b|/2$), then it is the algebraic average of a and b (i.e., $c = (a + b)/2$).

***Problem* 13.9.** Suppose z and w are two different complex numbers and t is between 0 and 1. Prove that $v = t \cdot z + (1-t) \cdot w$ lies on the segment between z and w.

***Problem* 13.10.** Suppose a, b, and c are three different complex numbers such that $|a - c| + |c - b| = |a - b|$. Prove that there is t between 0 and 1 so that $c = t \cdot a + (1 - t) \cdot b$.

***Problem* 13.11.** Suppose z_1 and z_2 are two different points of plane. Prove that z lies on the line joining z_1 and z_2 if and only if there is a real number t such that $z = t \cdot z_1 + (1-t) \cdot z_2$.

***Problem* 13.12.** Suppose z_1 and z_2 are two different points of plane. Prove that the line joining z_1 and z_2 contains z if and only if $(z - z_2)/(z_1 - z_2)$ is a real number.

***Problem* 13.13.** Suppose z_1 and z_2 are two different points of plane. Suppose z_3 and z_4 are two different points of plane. Prove that the line joining z_1 and z_2 is parallel to the line joining z_3 and z_4 if and only if $(z_1 - z_2)/(z_3 - z_4)$ is a real number.

***Problem* 13.14.** Suppose z_1 and z_2 are two different points of plane. Suppose z_3 and z_4 are two different points of plane. Prove that the line joining z_1 and z_2 is perpendicular to the line joining z_3 and z_4 if and only if $(z_1 - z_2)/(z_3 - z_4)$ is an imaginary number.

***Problem* 13.15.** Suppose z_1, z_2, and z_3 are three different points of plane. Prove that they form a right triangle at z_1 if and only if $|z_2 - z_3|^2 = |z_1 - z_2|^2 + |z_1 - z_3|^2$. Do not use Pythagoras Theorem. This is Pythagoras Theorem.

***Problem* 13.16.** Prove that the segment joining midpoints of two sides of a triangle is parallel to the third side and equal to half its length.

***Problem* 13.17.** Prove that any two medians of a triangle cut each other into segments whose lengths have ratio 2:1.

***Problem* 13.18.** Prove that the three medians of any triangle are concurrent.

Math Dept, University of Tennessee, Knoxville, TN 37996-1300, USA
E-mail address: dydak@math.utk.edu