

ABSTRACT. Geometry and Complex Numbers

GEOMETRY AND COMPLEX NUMBERS

JERZY DYDAK

CONTENTS

1. Introduction	2
2. Solving equations	3
3. Geometric proofs	4
4. Stereographic Projection	5
5. Basic calculations with complex numbers	47

1. INTRODUCTION

2. SOLVING EQUATIONS

3. GEOMETRIC PROOFS

4. STEREOGRAPHIC PROJECTION

There are two special projections: one onto the x -axis, the other onto the y -axis. Both are well-known. Using those projections one can define functions sine and cosine. However, there is another projection, less known to students, a projection from a circle to the x -axis. It is called the **stereographic projection**. We will use it to provide geometric interpretations of multiplication, division of real numbers, the tangent function, and basic trigonometric formulae.

Problem 4.1. Find the intersection of the line joining $(0, 1)$ and $(1, 5)$ with the x -axis.

Hint(s) to 4.1: Find an equation of the line passing through the two points. How does one find its intersection with the x -axis?

Outline(s) of solution(s) to 4.1: A non-vertical line is the graph of a linear function $f(x)$. A function $f(x)$ is called **linear** if the ratio $(f(x_2) - f(x_1))/(x_2 - x_1)$ is always constant if $x_2 \neq x_1$. That ratio is the slope m of the geometric line. One can compute m from the data. Now, setting $x_2 = x$ and $x_1 = 0$ gives $(f(x) - 1)/(x - 0) = m$ as an equation of our line. Its x -intercept is the point such that $f(x) = 0$, so solve $(f(x) - 1)/(x - 0) = m$ in that case.

Answer to 4.1: $-.25$

Problem 4.2. Find the intersection of the line joining $(0, 0, 1)$ and $(1, 1, -1)$ with the xy -plane.

Hint(s) to 4.2: Find an equation of the line passing through the two points. How does one find its intersection with the xy -plane?

Outline(s) of solution(s) to 4.2: A non-vertical line on space is the graph of a linear function $f(t)$. A function $f(t)$ is called **linear** if the ratio $(f(t_2) - f(t_1))/(t_2 - t_1)$ is always constant if $t_2 \neq t_1$. Physically, that constant is the velocity v of a particle traversing our line. One can compute v from the data assuming that at $t = 0$ the particle is at point $(0, 0, 1)$ and at $t = 1$ it is at the other point. Now, setting $t_2 = t$ and $t_1 = 0$ gives $(f(t) - (0, 0, 1))/(t - 0) = v$ as an equation of our line. Its xy -intercept is the point t such that the third coordinate of $f(t)$ equals 0, so solve $(f(t) - (0, 0, 1))/(t - 0) = v$ for such t , then find $f(t)$.

Answer to 4.2: $(0.5, 0.5)$

Problem 4.3. Suppose $b \neq 1$. Show that the line joining $(0, 1)$ and (a, b) intersects the x -axis at $(\frac{a}{1-b}, 0)$.

Hint(s) to 4.3: Find an equation of the line passing through the two points. How does one find its intersection with the x -axis?

Outline(s) of solution(s) to 4.3: A non-vertical line is the graph of a linear function $f(x)$. A function $f(x)$ is called **linear** if the ratio $(f(x_2) - f(x_1))/(x_2 - x_1)$ is always constant if $x_2 \neq x_1$. That ratio is the slope m of the geometric line. One can compute m from the data. Now, setting $x_2 = x$ and $x_1 = 0$ gives $(f(x) - 1)/(x - 0) = m$ as an equation of our line. Its x -intercept is the point such that $f(x) = 0$, so solve $(f(x) - 1)/(x - 0) = m$ in that case.

Problem 4.4. Suppose $a \neq 0$ and $b \neq 1$. Show that $\frac{a}{1-b} = \frac{b}{a}$ if (a, b) is on the circle centered at $(0, 1/2)$ with radius $r = 1/2$.

Hint(s) to 4.4: What does it mean that a point (x, y) lies on the circle centered at $(0, 1/2)$ of radius $r = 1/2$? Can you write an equation of that circle? Is it equivalent to $\frac{x}{1-y} = \frac{y}{x}$ if $x \neq 0$?

Outline(s) of solution(s) to 4.4: A point (x, y) lies on the circle centered at $(0, 1/2)$ of radius $r = 1/2$ if its distance to $(0, 1/2)$ is $1/2$. Algebraically, it is the same as

$$x^2 + (y - 1/2)^2 = (1/2)^2.$$

Expanding and simplifying gives

$$x^2 + y^2 - y = 0$$

which is equivalent to $\frac{x}{1-y} = \frac{y}{x}$ if $x \neq 0$.

Problem 4.5. Consider the stereographic projection from the North Pole of the circle centered at $(0, 1/2)$ with radius $r = 1/2$ onto the x -axis. Show that the image of the point with argument $\alpha \neq \pi/2$ is $\tan(\alpha)$.

Hint(s) to 4.5:

1. (Algebraic) If a point (x, y) has argument α , how can $\tan(\alpha)$ be expressed using x and y ? How does that compare to the stereographic projection of (x, y) ?
2. (Geometric) Draw a picture of the stereographic projection from the North Pole N via $A = (x, y)$ to $B = (t, 0)$. Can you calculate all the angles on the picture? What are their tangents?

Outline(s) of solution(s) to 4.5:

1. (Algebraic) If a point (x, y) has argument α , then $\tan(\alpha) = y/x$. The stereographic projection of (x, y) is $\frac{x}{1-y}$ and that equals $\frac{y}{x} = \tan(\alpha)$ by 4.4.

2. (Geometric) Draw a picture of the stereographic projection from the North Pole N via $A = (x, y)$ to $B = (t, 0)$. If α is the argument of A , then the angle $\angle ONB$ equals α . Since $\tan(\angle ONB) = t$, we are done.

Problem 4.6. Consider the stereographic projection from the North Pole of the circle centered at $(0, 1/2)$ with radius $r = 1/2$ onto the x -axis. Show that one can construct $\frac{-1}{a}$, $a > 0$, by going straight from $(a, 0)$ to $(0, 1)$ and then turning left via 90 degrees and going straight until the x -axis is met again.

Hint(s) to 4.6: Draw a picture of the stereographic projection from the North Pole N via $B = (x, y)$ to $A = (a, 0)$. Then draw a line through N perpendicular to NA . Can you calculate all the angles on the picture? What are their tangents?

Outline(s) of solution(s) to 4.6: Draw a picture of the stereographic projection from the North Pole N via $B = (x, y)$ to $A = (a, 0)$. Then draw a line through N perpendicular to NA . Let t be the x -intercept of that line. If α is the argument of B , then the angle $\angle ONA$ equals α and the angle $\angle OCN = \alpha$, where $C = (t, 0)$. Since $\tan(\angle OCN) = -1/t$, we are done.

Problem 4.7. Consider the stereographic projection from the North Pole of the circle centered at $(0, 1/2)$ with radius $r = 1/2$ onto the x -axis. Show that one can construct $\frac{a}{b}$, $a, b > 0$, by going straight from $(a, 0)$ to the y -axis on the line forming the angle $\pi/2 + \beta$ with the x -axis, where b is the projection of the point with argument β .

Hint(s) to 4.7: Draw a picture of the stereographic projection from the North Pole N via $X = (x, y)$ to $B = (b, 0)$. Then draw a line through $A = (a, 0)$ parallel to NB . That line forms the angle $\pi/2 + \beta$ with the x -axis. Can you calculate its y -intercept?

Outline(s) of solution(s) to 4.7: Draw a picture of the stereographic projection from the North Pole N via $X = (x, y)$ to $B = (b, 0)$. Then draw a line through $A = (a, 0)$ parallel to NB . That line forms the angle $\pi/2 + \beta$ with the x -axis. Its y -intercept equals a/b .

Problem 4.8. Consider the stereographic projection from the North Pole of the circle centered at $(0, 1/2)$ with radius $r = 1/2$ onto the x -axis. Show that one can construct $a \cdot b$, $a, b > 0$, by going straight from $(0, a)$ to the x -axis on the line forming the angle $\pi - \beta$ with the y -axis, where b is the projection of the point with argument β .

Hint(s) to 4.8: Draw a picture of the stereographic projection from the North Pole N via $X = (x, y)$ to $B = (b, 0)$. Then draw a line through $A = (0, a)$ parallel to NB . That line forms the angle $\pi/2 - \beta$ with the y -axis. Can you calculate its x -intercept?

Outline(s) of solution(s) to 4.8: Draw a picture of the stereographic projection from the North Pole N via $X = (x, y)$ to $B = (b, 0)$. Then draw a line through $A = (0, a)$ parallel to NB . That line forms the angle $\pi/2 - \beta$ with the y -axis. Its x -intercept equals $a \cdot b$.

Problem 4.9. Consider the stereographic projection from the North Pole of the circle centered at $(0, 1/2)$ with radius $r = 1/2$ onto the x -axis. If $a > 0$, compute the altitudes of the triangle ABC , $A = (-a, 0)$, $B = (a, 0)$, $C = (0, 1)$, in two different ways and conclude that $\sin(2 \cdot \alpha) = 2 \cdot \sin(\alpha) \cdot \cos(\alpha)$ for positive angles $\alpha < \pi/2$.

Hint(s) to 4.9: Let $h = AD$ be the altitude of triangle ABC from vertex A onto the side CB . Consider both triangles ADC and ADB . Can you relate their angles to α , the argument of the point mapped to B by the stereographic projection?

Outline(s) of solution(s) to 4.9: Let $h = AD$ be the altitude of triangle ABC from vertex A onto the side CB . Consider both triangles ADC and ADB . Let α be the argument of the point mapped to B by the stereographic projection. The angle $\angle ACB = 2 \cdot \alpha$, the angle $\angle ABD = \pi/2 - \alpha$. Therefore $BC = 1/\cos(\alpha)$, $h = NA \cdot \sin(2 \cdot \alpha)$, and $h = AB \cdot \cos(\alpha) = 2 \tan(\alpha) \cdot \cos(\alpha) = 2 \cdot \sin(\alpha)$. Comparing the two formulae for h gives $\sin(2 \cdot \alpha) = 2 \cdot \sin(\alpha) \cdot \cos(\alpha)$.

Problem 4.10. Consider the stereographic projection from the North Pole of the circle centered at $(0, 1/2)$ with radius $r = 1/2$ onto the x -axis. If $a > 0$, compute the bases of altitudes of the triangle ABC , $A = (-a, 0)$, $B = (a, 0)$, $C = (0, 1)$, in two different ways and conclude that $\cos(2 \cdot \alpha) = 2 \cdot \cos^2(\alpha) - 1$ for positive angles $\alpha < \pi/2$.

Hint(s) to 4.10: Let $h = AD$ be the altitude of triangle ABC from vertex A onto the side CB . Consider both triangles ADC and ADB . Can you relate their angles to α , the argument of the point mapped to B by the stereographic projection? Can you compute BD and CD using both triangles?

Outline(s) of solution(s) to 4.10: Let $h = AD$ be the altitude of triangle ABC from vertex A onto the side CB . Consider both triangles ADC and ADB . Let α be the argument of the point mapped to B by the stereographic projection. The angle $\angle ACB = 2 \cdot \alpha$, the angle $\angle ABD = \pi/2 - \alpha$. Therefore $CA = 1/\cos(\alpha)$, so $CD = CA \cdot \cos(2 \cdot \alpha) = \cos(2 \cdot \alpha)/\cos(\alpha)$ and $BD = AB \cdot \sin(\alpha) = 2 \cdot \tan(\alpha) \cdot \sin(\alpha)$. Since $CB = BD + DC$, one gets $\cos(2 \cdot \alpha) = 1 - 2 \cdot \sin^2(\alpha)$.

Problem 4.11. Consider the stereographic projection from the North Pole of the circle centered at $(0, 1/2)$ with radius $r = 1/2$ onto the x -axis. If $a, b > 0$, compute the altitudes of the triangle ABC , $A = (-a, 0)$, $B = (b, 0)$, $C = (0, 1)$, in two different ways and conclude that $\sin(\alpha + \beta) = \sin(\alpha) \cdot \cos(\beta) + \cos(\alpha) \cdot \sin(\beta)$ for positive angles $\alpha, \beta < \pi/2$.

Hint(s) to 4.11: Let $h = AD$ be the altitude of triangle ABC from vertex A onto the side CB . Consider both triangles ADC and ADB . Can you relate their angles to β , the argument of the point mapped to B by the stereographic projection and α , the argument of the point mapped to $-A$ by the stereographic projection?

Outline(s) of solution(s) to 4.11: Let $h = AD$ be the altitude of triangle ABC from vertex A onto the side CB . Consider both triangles ADC and ADB . Let β be the argument of the point mapped to B by the stereographic projection. Let α be the argument of the point mapped to $-A$ by the stereographic projection. The angle $\angle ACB = \alpha + \beta$, the angle $\angle ABD = \pi/2 - \beta$. Therefore $BC = 1/\cos(\alpha)$, $h = NA \cdot \sin(\alpha + \beta)$, and $h = AB \cdot \cos(\beta) = 2 \tan(\alpha) \cdot \cos(\beta)$. Comparing the two formulae for h gives $\sin(\alpha + \beta) = \sin(\alpha) \cdot \cos(\beta) + \cos(\alpha) \cdot \sin(\beta)$.

Problem 4.12. Consider the stereographic projection from the North Pole of the circle centered at $(0, 1/2)$ with radius $r = 1/2$ onto the x -axis. If $a, b > 0$, compute the bases of altitudes of the triangle ABC , $A = (-a, 0)$, $B = (b, 0)$, $C = (0, 1)$, in two different ways and conclude that $\cos(\alpha + \beta) = \cos(\alpha) \cdot \cos(\beta) - \sin(\alpha) \cdot \sin(\beta)$ for positive angles $\alpha, \beta < \pi/2$.

Hint(s) to 4.12: Let $h = AD$ be the altitude of triangle ABC from vertex A onto the side CB . Consider both triangles ADC and ADB . Can you relate their angles to β , the argument of the point mapped to B by the stereographic projection and α , the argument of the point mapped to $-A$ by the stereographic projection?

Outline(s) of solution(s) to 4.12: Let $h = AD$ be the altitude of triangle ABC from vertex A onto the side CB . Consider both triangles ADC and ADB . Let β be the argument of the point mapped to B by the stereographic projection. Let α be the argument of the point mapped to $-A$ by the stereographic projection. The angle $\angle ACB = \alpha + \beta$, the angle $\angle ABD = \pi/2 - \beta$. Therefore $AC = 1/\cos(\alpha)$, $CD = AC \cdot \cos(\alpha + \beta)$, and $DB = AB \cdot \sin(\beta) = (\tan(\alpha) + \tan(\beta)) \cdot \sin(\beta)$. Analyzing $CB = CD + DB$ gives $\cos(\alpha + \beta) = \cos(\alpha) \cdot \cos(\beta) - \sin(\alpha) \cdot \sin(\beta)$.

Problem 4.13. Consider the stereographic projection from the North Pole of the circle centered at $(0, 1/2)$ with radius $r = 1/2$ onto the x -axis. If $b > a > 0$, consider the intersection C of the vertical line $x = b$ with the line passing through the North Pole and perpendicular to the line joining North Pole and $(a, 0)$. Show that $C = (b, 1 + a \cdot b)$ and conclude that $\tan(\beta - \alpha) = \frac{\tan(\beta) - \tan(\alpha)}{1 + \tan(\alpha) \cdot \tan(\beta)}$ for positive angles $\alpha < \beta < \pi/2$.

Hint(s) to 4.13: Draw a picture of the stereographic projection from the North Pole N via $B' = (x, y)$ to $B = (b, 0)$ and of the stereographic projection from the North Pole N via $A' = (x, y)$ to $A = (a, 0)$. Then draw a line through N perpendicular to NB and mark its intersection C with line $x = b$. What is the angle between NA and NB in terms of arguments α of A' and β of B' ? Do you see that all points N , A , B , and C lie on one circle?

Outline(s) of solution(s) to 4.13: Draw a picture of the stereographic projection from the North Pole N via $B' = (x, y)$ to $B = (b, 0)$ and of the stereographic projection from the North Pole N via $A' = (x, y)$ to $A = (a, 0)$. Then draw a line through N perpendicular to NB and mark its intersection C with line $x = b$. The angle between NA and NB can be expressed in terms of arguments α of A' and β of B' as $\beta - \alpha$. Since all points N , A , B , and C lie on one circle with diameter NB , the angle between CA and CB is also $\beta - \alpha$. Apply that information to the right triangle ABC .

5. BASIC CALCULATIONS WITH COMPLEX NUMBERS

Problem 5.1. Define complex numbers. What do we mean by $a + b \cdot i$?

Answer to 5.1: Geometrically, a complex number is a point on the plane (a plane with origin). That way of expressing complex numbers is referred to as **Argand diagram**.

Algebraically, a complex number is a pair (a, b) of real numbers. $a + b \cdot i$ is one of possible notations for (a, b) .

Problem 5.2. Define the modulus of a complex numbers z . What are possible notations for modulus of z ?

Answer to 5.2: Geometrically, the modulus of a complex number z is its distance to the origin of the plane.

Algebraically, the modulus of $|z| = a + b \cdot i$ is $\sqrt{a^2 + b^2}$. $|z|$ is one of possible notations for modulus of z .

Problem 5.3. Define the argument of a complex numbers $z \neq 0$. What are possible notations for argument of z ?

Answer to 5.3: The argument of a complex number $z \neq 0$ is the angle from the x -axis to z . $\arg(z)$ is one of possible notations for argument of z .

Problem 5.4. Define the real part of a complex numbers z . What are possible notations for real part of z ?

Answer to 5.4: The real part of a complex number z is its x -coordinate. $Re(z)$ is one of possible notations for real part of z .

Algebraically, $Re(a + b \cdot i) = a$.

Geometrically, the function $z \rightarrow Re(z)$ is one of the basic projections of the plane, the orthogonal projection onto x -axis.

Problem 5.5. Define the imaginary part of a complex numbers z . What are possible notations for imaginary part of z ?

Answer to 5.5: The imaginary part of a complex number z is its y -coordinate. $Im(z)$ is one of possible notations for imaginary part of z .

Algebraically, $Im(a + b \cdot i) = b$.

Geometrically, the function $z \rightarrow Im(z)$ is one of the basic projections of the plane, the orthogonal projection onto y -axis.

Problem 5.6. Define imaginary complex numbers

z .

Answer to 5.6: z is an imaginary complex number if it is a real multiple of i .

Algebraically, $z = b \cdot i$.

Geometrically, z lies on the y -axis.

Problem 5.7. Define the conjugate of a complex number z . What is the geometrical meaning of the conjugate of z ?

Answer to 5.7: Let $z = a + b \cdot i$.

Algebraically, $\bar{z} = a - b \cdot i$.

Geometrically, \bar{z} is the reflection of z in the real axis.

Problem 5.8. Sketch a picture illustrating addition of complex numbers.

Answer to 5.8: To draw $z_1 + z_2$ slide vector z_2 on vector z_1 .

Problem 5.9. Find $z = (1 + 2 \cdot i) \cdot (1 + 1 \cdot i)$ without using a calculator.

Answer to 5.9: $z = -1 + 3 \cdot i$

Problem 5.10. Express $z = i^{104}$ as $a + b \cdot i$ without using a calculator.

Hint(s) to 5.10: Is there any pattern in powers of i ?

Answer to 5.10: 1

Problem 5.11. Sketch $1 + i$. Find and sketch $(1 + i)^2$.

Answer to 5.11: $2 \cdot i$

Problem 5.12. Sketch $1 - i$. Find and sketch $(1 - i)^2$.

Answer to 5.12: $-2 \cdot i$

Problem 5.13. Express $z = (1+i)^{10}$ as $a+b\cdot i$ without using a calculator.

Hint(s) to 5.13: Is there any pattern in powers of $1 \pm i$?

Answer to 5.13: $z = 0 + 32 \cdot i$

Problem 5.14. Express $z = (1+i)^{103}/(1-i)^{99}$ as $a + b \cdot i$ without using a calculator.

Answer to 5.14: $z = 0 + 4 \cdot i$

Problems 5.15-5.16 form written Homework 3 due February 16, 2004.

Problem 5.15. Let $a = \sqrt{3}$ and $z = (-1 - a \cdot i)/2$. Show $z^2 = \bar{z}$ and $z^3 = 1$ without using a calculator.

Problem 5.16. Let $a = \sqrt{3}$ and $z = (1 - a \cdot i)/2$. Show $z^2 = -\bar{z}$ and $z^3 = -1$ without using a calculator.

Problem 5.17. Let $a = \sqrt{3}$ and $z = (-1 - a \cdot i)/2$. Compute z^9 without using a calculator.

Hint(s) to 5.17: Is there any pattern in powers of z ?

Answer to 5.17: 1

Problem 5.18. Let $a = \sqrt{3}$ and $z = (1 + a \cdot i)/2$. Compute z^8 without using a calculator.

Hint(s) to 5.18: Is there any pattern in powers of z ?

Answer to 5.18: $(-1 - a \cdot i)/2$

MATH DEPT, UNIVERSITY OF TENNESSEE, KNOXVILLE, TN 37996-1300, USA

E-mail address: dydak@math.utk.edu