9. Triangle Inequality for complex numbers

***Problem* 9.1.** Suppose z_1 and z_2 are complex numbers such that $z_1 + z_2$ is real. Prove that $|z_1 + z_2|$ is not greater than $|z_1| + |z_2|$ by using similar inequality for real numbers.

***Problem* 9.2.** Suppose z_1 and z_2 are non-zero complex numbers such that z_1+z_2 is real. Prove that if $|z_1+z_2| = |z_1|+|z_2|$, then z_1 and z_2 are real numbers whose ratio is positive.

***Problem* 9.3.** Prove algebraically that $|z_1 + z_2|$ is not greater than $|z_1| + |z_2|$. Do not use Triangle Inequality. This is Triangle Inequality.

***Problem* 9.4.** Suppose z_1 and z_2 are non-zero complex numbers. Prove that if $|z_1+z_2| = |z_1|+|z_2|$, then z_1/z_2 is a positive real number.

10. Multiplication of complex numbers

***Problem* 10.1.** Prove geometrically that $i \cdot z$ is z rotated counterclockwise by 90 degrees.

***Problem* 10.2.** $[1, 1]$ is rotated clockwise by 90 degrees. Find the resulting vector.

Hint(s) to 10.2: What is the result of rotating e_1 and e_2 ? What is the result of rotating $a \cdot e_1 + b \cdot e_2$?

Answer to 10.2: $[1, -1]$

***Problem* 10.3.** Prove geometrically that $(\cos(\theta)+i\cdot\sin(\theta))\cdot z$ is z rotated counterclockwise by θ radians.

***Problem* 10.4.** The geometrical interpretation of $z \cdot (\cos(\alpha) + i \cdot \sin(\alpha))$ is z rotated counterclockwise by angle α. $[3, 4]$ is rotated **clockwise** by 6 degrees. Find the resulting vector.

Hint(s) to 10.4: Switch to complex numbers and stick into a calculator.

Answer to 10.4: $[3.4, 3.66]$

***Problem* 10.5.** Sketch a picture illustrating multiplication of unit complex numbers.

***Problem* 10.6.** Show that $(1+2\cdot i)^{2001}+(1-2\cdot i)^{2001}$ is a real number.

Hint(s) to 10.6: Show that the number equals its conjugate.

***Problem* 10.7.** Prove that $z/(z^2+1)$ is a real number if z lies on the unit circle.

Hint(s) to 10.7: Show that the number equals its conjugate.

***Problem* 10.8.** Prove that $z/(z+1)^2$ is a real number if z lies on the unit circle.

Hint(s) to 10.8: Show that the number equals its conjugate.

Problem **10.9.** Show that $i\cdot(1+z)/(1-z)$ is a real number if $z\neq 1$ is a unit complex number.

Hint(s) to 10.9: Show that the number equals its conjugate.

***Problem* 10.10.** Show that $(1-z^2)\cdot w/(w^2-z^2)$ is a real number if $z \neq w$ are unit complex numbers.

Hint(s) to 10.10: Show that the number equals its conjugate.

Math Dept, University of Tennessee, Knoxville, TN 37996-1300, USA
E-mail address: dydak@math.utk.edu