11. Scalar product and vector product for complex numbers

***Problem* 11.1.** Given two complex numbers z and w, the scalar product $S(z, w)$ is defined as $(z \cdot \bar{w} + \bar{z} \cdot w)/2$. Show that $S(z, w) = Re(\bar{z} \cdot w)$. Conclude that $S(z, w) = |z| \cdot |w| \cdot \cos(\alpha)$, where α is the angle from z to w measured in counterclockwise direction.

***Problem* 11.2.** Given two complex numbers z and w, the scalar product $S(z, w)$ is defined as $(z \cdot \bar{w} + \bar{z} \cdot w)/2$. Show that $|z + w|^2 = |z|^2 + |w|^2 + 2 \cdot S(z, w)$. Derive the Cosine Theorem from that equality.

***Problem* 11.3.** Given two complex numbers z and w, the scalar product $S(z, w)$ is defined as $(z \cdot \bar{w} + \bar{z} \cdot w)/2$. Show algebraically that $S(z, w) = S(w, z)$.

***Problem* 11.4.** Given two complex numbers z and w, the scalar product $S(z, w)$ is defined as $(z \cdot \bar{w} + \bar{z} \cdot w)/2$. Show algebraically that $S(z, a \cdot w + b \cdot v) = a \cdot S(z, w) + b \cdot S(z, v)$ provided a and b are real.

***Problem* 11.5.** Given two complex numbers z and w, the vector product $V(z,w)$ is defined as $i\cdot(z\cdot\bar{w}-\bar{z}\cdot w)/2$. Show that $V(z,w) = Im(\bar{z}\cdot w)$. Conclude that $V(z,w) = |z|\cdot|w|\cdot\sin(\alpha)$, where α is the angle from z to w measured in counterclockwise direction. Conclude that $|V(z,w)|$ is the area of parallelogram formed by z and w.

***Problem* 11.6.** Given two complex numbers z and w, the vector product $V(z,w)$ is defined as $i \cdot (z \cdot \bar{w} - \bar{z} \cdot w)/2$. Show algebraically that $V(z,w) = -V(w,z)$.

***Problem* 11.7.** Given two complex numbers z and w, the vector product $V(z, w)$ is defined as $i \cdot (z \cdot \bar{w} - \bar{z} \cdot w)/2$. Show algebraically that $V(z, a \cdot w + b \cdot v) = a \cdot V(z, w) + b \cdot V(z, v)$ provided a and b are real.

***Problem* 11.8.** Given two complex numbers z and w, the scalar product $S(z, w)$ is defined as $(z\cdot\bar{w}+\bar{z}\cdot w)/2$. If $z = x_1+y_1\cdot i$ and $w = x_2+y_2\cdot i$, show that $S(z, w) = x_1 \cdot x_2 + y_1 \cdot y_2$.

***Problem* 11.9.** Given two complex numbers z and w, the vector product $V(z,w)$ is defined as $i \cdot (z \cdot \bar{w} - \bar{z} \cdot w)/2$. If $z = x_1 + y_1 \cdot i$ and $w = x_2 + y_2 \cdot i$, show that $V(z,w) = x_1 \cdot y_2 - x_2 \cdot y_1$, the determinant of the matrix $[[x_1, y_1], [x_2, y_2]]$.

***Problem* 11.10.** Find the area of the triangle with vertices $P(-1,1)$, $Q(1,-1)$ and $R(1,1)$.

Hint(s) to 11.10: How are triangles related to parallelograms ? How do we compute areas of parallelograms ?

Answer to 11.10: 2

***Problem* 11.11.** Find the remaining two vertices Q and S of the square whose diagonal joins points $P = (1, -1)$ and $R = (3, 1)$.

Hint(s) to 11.11: Can you find the center C of the square ? How does one get vector CQ from the vector CP ?

Answer to 11.11: $Q = (1, 1), S = (3, -1)$

***Problem* 11.12.** If $z_1/z_2 = a+b\cdot i$, then $z_1 = a\cdot z_2+b\cdot(i\cdot z_2)$, $a\cdot z_2$ is parallel to z_2, and $b\cdot(i\cdot z_2)$ is perpendicular to z_2. Express vector $\vec{v} = [-2, 4]$ as $\vec{v} = \vec{v}_1 + \vec{v}_2$, where $\vec{v}_1$ is parallel to $[1, 1]$ and $\vec{v}_2$ is perpendicular to $[1, 1]$. Report $\vec{v}_2$.

Answer to 11.12: $[-3, 3]$

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