Introduction

1 Introduction

Assuming the reader has a basic knowledge and good intuitive feel for Euclidean geometry, let us talk a little about the mathematical computation that we can do in two or three dimensions. We have a good sense of what a point, a line, and a plane are, and how they differ in the aspect of degrees of freedom, or dimensions. We also have a pretty good idea of what space is: it is simply every direction one can move in. We generally do computations in each of these dimensions, with one variable for each separate direction of motion, or more precisely, for every orthonormal base in the vector-space.

In other words, when we talk about distance along a straight line, like the distance between two points, we need speak only of one variable, such as x or t. Every point on the line can be represented by a real number (a). When we measure in two directions at once, we have a notion of area, with variables like x and y. Every point in the plane can then be specified by an ordered pair (a, b). Finally, we can speak of spatial size in three dimensions, measuring volume in the x, y, and z directions. Each point in space is expressed as an ordered triple (a, b, c). In 3-space, we have a notion of distance and relations between points, lines, and planes.

But now we should realize an important distinction between the mathematics we can do and the reality it represents: they are entirely separable. For example, we can easily imagine train tracks as one-dimensional objects, as in problems like "Train A leaves Los Angeles at 4 p.m. . . ." Or we picture fields as 2-dimensional objects with area and no volume. This is clear because the other one or two dimensions are superfluous to the problem at hand. In fact, no line-segment exists in a way that one could hold it up and show it off. Even the pages on which these words are printed have some thickness, and hence are not truly planar.

When someone objects to studying the fourth dimension, or any higher dimensions, the first argument is that no such objects exist, or we have no experience with them and never could. Yet if we accept that we can do mathematics using one or two dimensions, which have no *real* analogue, can we not accept that the number of dimensions that we work with is entirely arbitrary? Why not four, five, or twenty-six dimensions, if it solves the kind of problems we are talking about?

The next argument is that some real systems have n "degrees of freedom" or aspects that can change independent of one another, like temperature, pressure, density, as well as length, width, and height. But here, we are talking about dimensions which represent actual "spatial" measurements, that is distance in n independent directions. Furthermore, this excludes the idea of time as a fourth dimension, in the same way it excludes temperature. We will be dealing with objects that exist at any time and at any temperature and pressure, but that occupy space in more than three mutually perpendicular directions.

In four-dimensional space, or 4-space, we think of these directions as represented by four variables, instead of three, and call them w, x, y, and z. Every point in 4-space can then easily be represented by an ordered quadruple (a, b, c, d).

Now, we can extend our definition of distance to include all four variables. We should also note that this fourth dimension is Euclidean, for the purposes of our discussion. As such, we can still speak of relationships between points, lines, and planes. But now we must add an idea of measurement in 4-space, sometimes referred to as "hyper-volume" or more generally "measure", "content", etc. We must also include a study of the relationships between "spaces", or "hyperplanes"—3-D spaces which can intersect each other or be parallel to one another. At this point, many people have trouble linking the geometric ideas with a picture or mental image.

When we first study solid geometry, the geometry of three dimensions, we begin by defining the dimension as a line through the origin which is perpendicular to each of the axes in the plane. As such, this line is perpendicular to every line in that plane. The same is done when we begin discussing 4-space. We must define a line which is "perpendicular" to the three axes. As such, this line is perpendicular to *every* line in space.

Let us imagine that two hyperplanes intersect. What would their intersection be? For a straightforward understanding, it is quite easy to use an analogy from spaces of lesser degree. When two straight lines intersect, the intersection is a point (or the lines coincide and intersect at every point). The intersection of two planes is a line. One would then induce that the intersection of two 3-D hyperplanes is a 2-D plane, and correctly so.

Many of the characteristics of 4-space can best be understood through analogy. By comparing how we see 2-space from 3-space, we can speculate how a 4-D creature might picture 3-space. As an example, imagine looking at a square in the plane. If we were lying in the plane, we could only see the outside edge of the square. But by rising above it (i.e., moving in the third dimension), we can see the interior of the square. In fact, we could touch any part of the square's interior without passing through its edges. This would be impossible in the plane.

Now imagine for a moment you are a 4-D creature lying "flat" in our 3-space. You would see us on the outside like we see one another. Now if you were to rise up (in the fourth dimension), you would at once be able to see everything in our universe as if it were laid out flat in front of you. You could see directly inside our bodies and could perform surgery by going straight to the part on which you wish to operate, without ever cutting through the skin.

Here it is important to realize that analogy is only a tool to aid our understanding and visualization. It should not be considered a proof, in any strict mathematical sense. However, the level of this paper is such that analogy will be relied upon heavily in the explanations of the topics, in order to make the material more accessible to a general audience.

2 Purpose

The purpose of this project is to offer an explanation of fourth-dimensional Euclidean geometry, particularly the study of regular 4-polytopes, objects which exist in 4-space and are analogous to regular polygons in 2-space and regular polyhedra in 3-space. These polytopes will be strictly defined mathematically, and their structure will be discussed in terms of both their coordinates in 4-space and their composition of points, line-segments, polygons, and polyhedra. All of these strict definitions will be supplemented by explanations in laymen's terms to make the material more accessible to a general audience. Most importantly, visual aids will be offered and visualization discussed extensively. All of this is meant to give the reader the clearest possible image of these objects so that the intuitive feel for geometry can be extended beyond our everyday 3-space existence.

Though in most cases, pictures, diagrams, or computer-generated images supplement the verbal description, it would likely be very helpful to have a pen and pad of paper at hand to do your own sketches of images. For precision, a ruler, protractor, or compass may also be used. Though the rigors of geometry are generally tackled through algebra, visualization is the best tool for understanding any of its ideas. Several of the books listed in the references offer clear explanations for the construction of most of the objects discussed here.