Regular Polytopes

To begin, a *polytope* is an object which occupies some region of n-space and is closed and bounded by $(n-1)$-dimensional hyperplanes. Each of these "faces" or "facets" is itself a polytope of $(n-1)$ dimensions, bounded by $(n-2)$-dimensional hyperplanes. This process continues, so that ultimately, polytopes of any dimension can be broken down into points and line-segments.

1 Polygons

To make this clearer, let us begin with the 2-D case. In 2 dimensions, polytopes are commonly referred to as polygons. Polygons are objects which lie in a plane and are bounded by line-segments, called *edges*. Each edge is a 1-D object bounded by two 0-D end-points, called *vertices*. Any straight-line closed figure drawn on a page represents a polygon. The word "closed" is a necessary part of the definition. Strictly speaking, it says that a polygon divides the plane into two parts, one bounded—called the *interior*—and one unbounded—called the *exterior*. Basically, this means that the line drawings start and stop in the same place.

Though this definition appears confining, it allows for an infinite number of polygons to be constructed. They can be drawn with any number of edges connecting any number of vertices in any way imaginable. The edges can simply go around a loop, coming back to the starting point, or they can cross over each other any number of times, forming stars like the familiar pentagram—a star with five vertices and five edges. When two non-adjacent edges of a polygon intersect one another, the polygon is called *reflex*. Otherwise, we say the polygon is *simple*.

Furthermore, each of the corners of a polygon can be "turned out" or "in". Imagine the outline of a bow-tie: six vertices simply connected by six edges so that the two corners in the middle are turned out, and the other four are turned in (see Figure 1). Another way to say this is that the interior angles at these two corners are greater than $180°$. Note that if you pick a point in the upper-left part of the figure and another point in the upper-right part

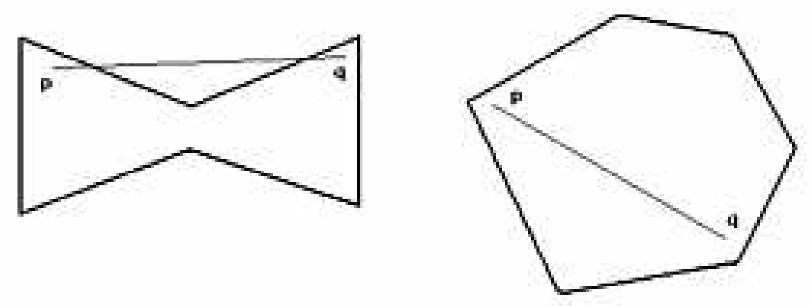

Figure 1: Concave and Convex Hexagons

and connected them with a straight line, that line would go outside of the polygon. This point of view allows us to define convex.

A polytope is *convex* if given any two points in its interior, the line-segment joining them will lie entirely in the interior of the polytope. If a polytope is not convex, it is said to be *concave*. That is, if you can draw any straight line from one part of the figure to another and cross the boundaries of the figure, it is concave. For the 2-dimensional case, saying an object is convex is equivalent to saying that all interior angles are less than $180°$. Another simple way to define convexity is to say that, as one moves around a polygon clockwise, only right turns are taken at the vertices. The more general definition, however, not only implies that the figure is simple, but allows us to work in higher dimensions without a need to change it in any way.

Limiting ourselves to the discussion of convex polygons allows us to talk about triangles, rectangles, trapezoids, kites, darts, and a slew of other polygons. For our purposes, we need be concerned only with *regular* polygons. Regular polygons are easily recognizable figures, and most of us already know the names of many of them. To say that a polygon is regular first requires that it be simple and convex. Next, we must require that all edges be of the same length and all interior angles be congruent. In the case of a triangle, it is immediately simple and convex. Generating a triangle with equal edges yields one with congruent angles, and vice versa. However, for polygons of more than three sides, both requirements are necessary. Quadrilaterals with equal angle measures are called rectangles, and those with equal edges are called rhombi (or sometimes rhombuses). Satisfying both conditions yields a square, a regular polygon familiar to all of us.

So we have now limited ourselves to a class of polygons easily defined and recognizable.

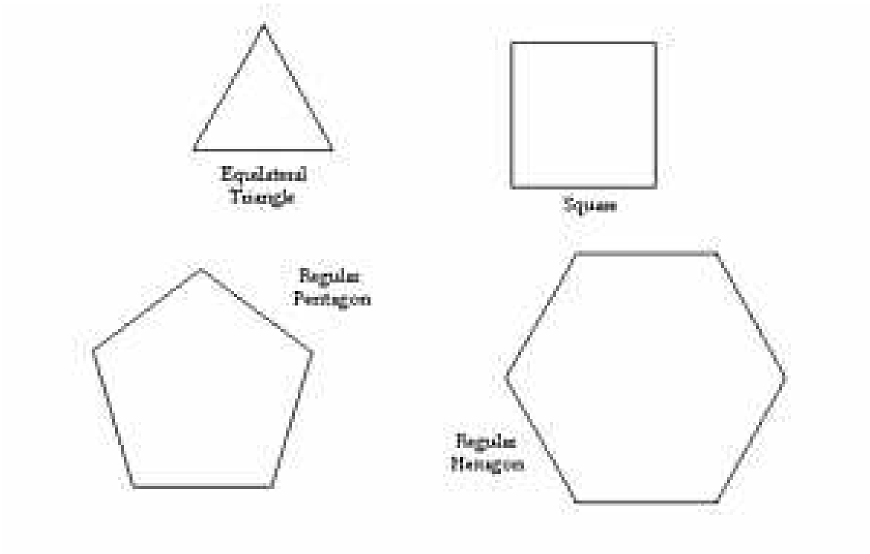

Figure 2: Regular Polygons

Yet there are still infinitely many polygons which can be called regular. These can be constructed by simply taking a number of vertices evenly spaced on a circle and connected each to the two adjacent to it. Or one could think of simply "opening" the polygon at one vertex and inserting another segment of equal length between the two. Clearly, either of these processes can be repeated *ad nauseum.* See Figure 2.

The first seven regular polygons have common names: triangle, square, (regular) pentagon, (regular) hexagon, (regular) octagon, (regular) nonagon, and (regular) decagon. Some other regular polygons also have common names, like the 12-sided dodecagon, but generally, polygons with a large number of sides are referred to as n-gons (i.e., 11-gon, 13-gon, 27-gon, etc.).

2 Polyhedra

Before we go any further into the study of polygons, let us now turn to the three dimensional case. The 3-D polytopes are called *polyhedra.* Each polyhedron is bounded by polygons, with two polygons meeting at an edge. Each of these polygons is called a *face* of the polyhedron. As seen from the discussion above, polyhedra are ultimately bounded by vertices and edges. Each planar group of vertices and edges is a polygon, though only the bounding (i.e. external)

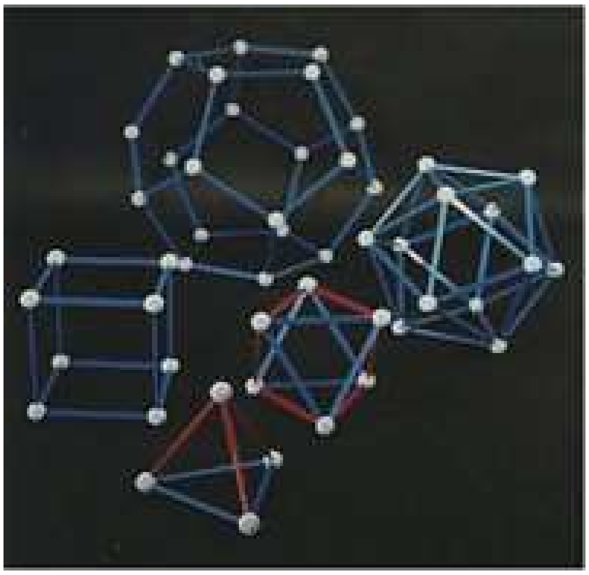

Figure 3: The Platonic Solids

polygons are considered faces.

The study of polyhedra dates back to the ancient Greeks, like Pythagoras, and there are an uncountable number of them. So we will again restrict our discussion to those which are considered regular, as often the Greeks did. The five regular polyhedra are commonly known as the Platonic solids. They are named for the Greek philosopher Plato who, in his work *Timaeus*, immortalized them by associating them with the "elements" of nature. The cube was considered representative of the element Earth. The icosahedron stood for Water. Air was believed to be in the form of an octahedron. And Fire was associated with the tetrahedron. The fifth and largest Platonic solid, the dodecahedron, was believed to be the Cosmos; each of its twelve pentagonal faces stood for a sign of the Zodiac. See Figure 3. Before we talk about them in detail, we must again define *regular*.

To define regular in three dimensions requires only slightly more than the definition already offered. First, we must require that the polyhedra be convex. Concave polyhedra are easy to picture, and there are polyhedra whose non-adjacent faces intersect. Next we require that each face of the polyhedron be the same regular polygon. As in the case of equal edges above, this is not enough to generate only regular polyhedra. There are, in fact, eight "deltahedra", each of which is a convex polyhedron with equilateral triangles as faces. Of these, only three are regular.

2.1 Vertex Figures

The analogy to the requirement that all interior angles of a polygon be equal is this: In a regular polyhedron, the same number of faces must meet at every vertex. Another way of stating this is to say that every *vertex figure* must be the same regular polygon. A vertex figure is the figure generated when a corner is chopped off, or *truncated*, halfway down the edges coming out of it. To get a better picture of this, imagine a square pyramid, like those of ancient Egypt. If one were to cut through the pyramid half-way to its apex, the figure left on top would be a square. Since there are four faces meeting at the apex, the vertex figure has four sides. The reason for using the vertex figure instead of simply the number of faces that meet at a point will become apparent when we take up the discussion of the 4-D polytopes.

The most familiar regular polyhedron is undoubtedly the cube. It is bounded by eight vertices, twelve edges, and six square faces, three of which meet at a vertex. (Examine any box close at hand and imagine all the sides are squares.) Now imagine truncating a corner of the cube (or box) and picturing the edges of the hole as forming a new face. Each edge is generated by a face that was cut, and each vertex is along a previous edge. In the case of the cube, the vertex figure is a triangle.

Though there are infinitely many regular polygons one can construct in the plane, it may seem peculiar that there are only five regular polyhedra: the tetrahedron, the cube, the octahedron, the dodecahedron, and the icosahedron. Except for the cube, each of these figures gets its name from the number of faces (*tetra*= 4, *octa*= 8, *dodeca*= 12, and *icosa*= 20).

There is a simple demonstration that only five regular polyhedra are possible by looking at the ways regular polygons can meet at a vertex. To better understand the following description, imagine a circular piece of paper with a wedge cut out of it. The two exposed radii where the cuts were made could be brought together by bending the page out of the plane. This forms a circular cone, with the center of the circle becoming the apex of the cone. Any size wedge will allow the paper to be folded this way; the key is that the number of degrees around the center of the circle must be less than 360° for the paper to be folded down. Imagine if the paper somehow came around to overlap itself, so that the number of degrees around the center was greater than 360°. If the edges were brought together, the paper would buckle.

Polyhedra are objects composed of polygons, and not circles or cones. So let us start with equilateral triangles, with vertex angles of 60°. If we imagine three triangles in contact with one another so that they share a point, then we can think of folding the triangles along their edges until the left and right triangles meet. This forms the corner of a tetrahedron.

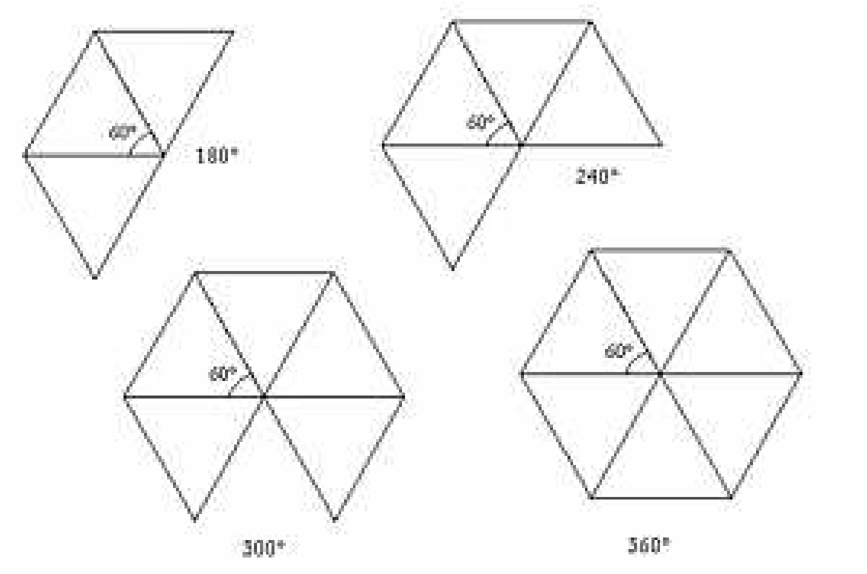

Figure 4: Construction of Vertices with Triangles

If we tried this with four, the triangles would wrap around the point $4 \times 60° = 240°$. These triangles would fold together to form the vertex of an octahedron. With five triangles, the total degree measure is 300°, and they fold together to form an icosahedron. When six triangles are laid down in this way, the degree measure reaches $360^{c}irc$, and they lie flat in the plane, in the shape of a hexagon. See Figure 4.

Next, we might try squares around a point, each having an angle of $90^{c}irc$. Using three squares, there are $3 \times 90^{c}irc = 270°$ around the center. These fold together to form the vertex of a cube. With four squares, 360° is reached, and they lie flat in the plane. Pentagons have a vertex angle of 108°. Three together almost touch each other in the plane at 324°. When folded down, three pentagons form the vertex of a dodecahedron. Four is impossible (without causing the paper to buckle, which would lead to a concave polyhedron). And with hexagons, three together at a point lie in a plane (which is a common bathroom tiling). This says that it is impossible to generate a polyhedron with only hexagons as sides. See Figure 5.

The proof that all five of the possibilities above generate closed figures is more complicated. It is independent of the construction given. In a sense, it is a remarkable coincidence that the five possible constructions of regular polyhedra all exist.

The simplest of the regulars is the tetrahedron. It is essentially a triangular pyramid (a pyramid with a triangle as a base) with all of its sides congruent. If you turn it on any side,

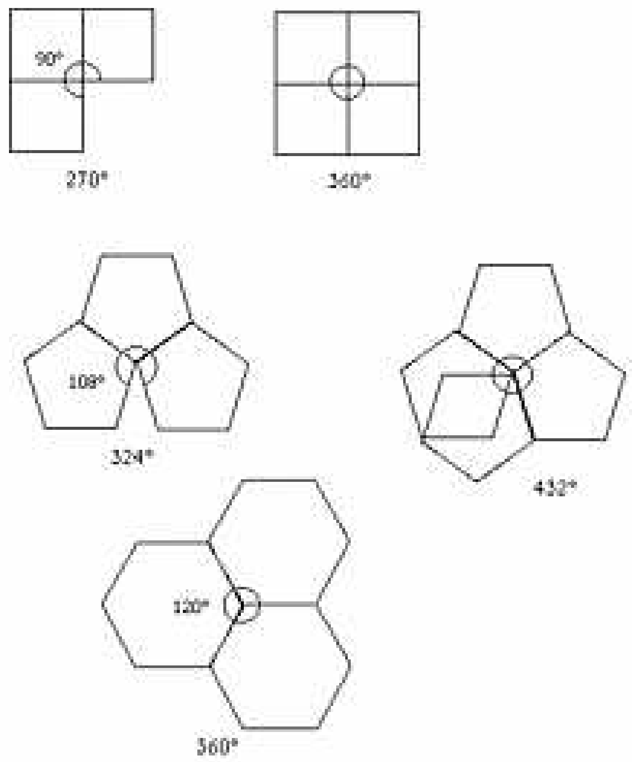

Figure 5: Construction of Vertices with Various Polygons

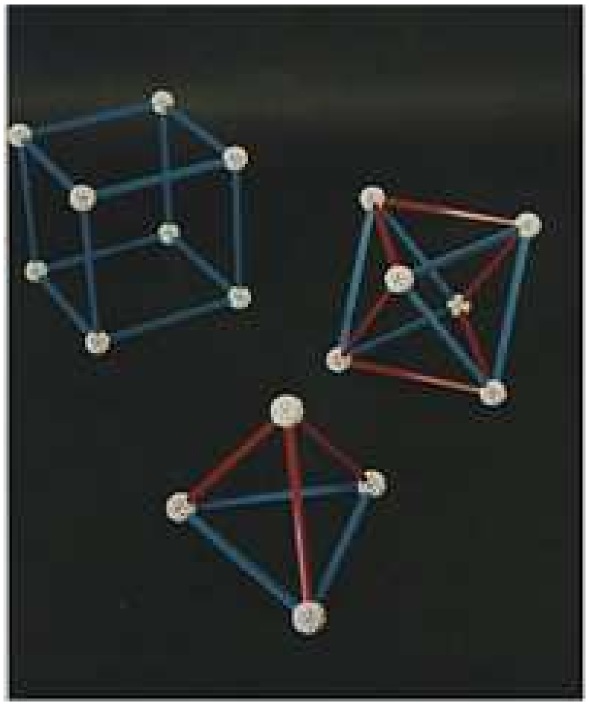

Figure 6: Tetrahedron, Octahedron and Cube

it looks the same, as is true of all of the regulars. It is composed of four triangular faces, meeting at a total of four vertices (three around the bottom triangle, and one above), and bounded by six edges. Its vertex figure is a triangle, as three faces meet at each vertex. See Figure 6.

The cube is described above, and is undoubtedly already familiar to us. It has eight vertices, twelve edges, and six square faces. See Figure 6.

An octahedron is formed when equilateral triangles meet four at a vertex. It is a regular square dipyramid, that is, two pyramids with congruent bases stuck together at the base. (As such, the square base is internal and not actually part of the octahedron.) It has a total of six vertices, twelve edges, and eight triangular faces. Its vertex figure is a square. See Figure 6.

The icosahedron is composed of 20 equilateral triangles, meeting five at a vertex. Its vertex figure is a pentagon, and it has twelve vertices and 30 edges. The dodecahedron is composed of twelve pentagons, meeting three at a vertex. Its vertex figure is a triangle, and it has 20 vertices and 30 edges. (See Figure 7.)

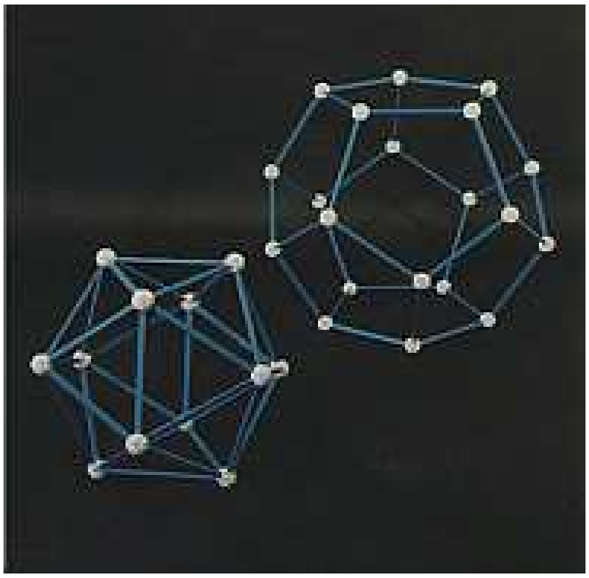

Figure 7: Icosahedron and Dodecahedron

A relation between the number of vertices, edges, and faces was discovered and proven by the famous mathematician Leonhard Euler. He noticed that the sum of the vertices and faces of any of the regular polyhedra is always 2 more than the number of edges. In mathematical notation, we would write Euler's formula as $V - E + F - 2 = 0$.

2.2 Duals

You may have already noticed another very important relation between the polyhedra above. Look, for instance, at the icosahedron and the dodecahedron. They have the same number of edges: 30. Furthermore, the icosahedron has as many vertices as the dodecahedron has faces and vice versa. The same relationship holds for the cube and the octahedron. Note also that the tetrahedron has the same number of vertices as faces (and perhaps one could say it has the same number of edges as edges).

One central idea to the theory of polytopes is that of dualism. Dualism gives us a way of generating one polytope from another, given the relationship above. If a point is placed at the center of each of the six faces of a cube, these form the vertices of a regular octahedron. If a point is placed at the center of each of the eight faces of the octahedron, a cube is formed. Thus, the octahedron and the cube are called *duals*. When one is inscribed in another, being formed by its face-centers, it is clearly smaller. If the octahedron inside the cube expanded uniformly, remaining centered in the cube, at some point its twelve edges will each intersect an edge of the cube. These edges intersect simultaneously, perpendicular to each other, and at their midpoints. Finally, note that the vertex figure of one is the face polygon of the other. Precisely the same relationship holds for the dodecahedron and the icosahedron, and for the tetrahedron and itself.

All of this only brushes the surface of the incredible amount of mathematics done with these five regular polyhedra. New, more remarkable relationships have been discovered continually since Plato first expounded on them over 2000 years ago. For more information on regular polyhedra, several good texts are listed in the bibliography, the most technical and in-depth of which is Coxeter's *Regular Polytopes*.

3 4-Polytopes

Now we are finally ready to turn to a discussion of 4-D polytopes, henceforth referred to simply as polytopes. As polyhedra are three-dimensional objects composed of polygons called faces, polytopes are four-dimensional objects composed of polyhedra called *facets* or *cells*. Each of these facets lies in its own hyperplane, which bounds the polytope. The polyhedra can themselves be decomposed into faces, which are decomposed into edges and vertices. Simple, isn't it?

We are again going to discuss only regulars, so we must define what this means. As always, a regular polytope must be convex. It must be bounded by congruent regular polyhedra, called *facets* or *cells*, and it must have the same number of polyhedra meeting at an edge. Another way of saying this is that the vertex figure formed by slicing the polytope with a hyperplane must be the same at every vertex. Here, the vertex figure is itself a polyhedron.

The most accepted definition of "regularity" for polytopes is such that it can be used for any number of dimensions. It is stated in terms of symmetry groups—motions which can in some way move objects "rigidly" (i.e., without distorting the dimensions of the object)—and "flags". In four dimensions, a flag is any vertex, plus any edge from that vertex, plus any face containing that edge, plus any cell containing that face. For a polytope to be regular, for any two flags there must be rigid motion moving the polytope onto itself that matches up the two flags. In other words, by rotating or reflecting any flag, it will match up with any other flag, cell for cell, face for face, edge for edge, and vertex for vertex. Under these conditions, we say that the symmetry groups are flag-transitive.

But what does all that mean? To start, the idea of convexity is not difficult to extend to four dimensions. But how are polytopes bounded by polyhedra? And what is the vertex figure? Let's begin our discussion with analogy. Picture three squares lying in a plane such that the second square is joined to the first and the third at adjacent edges. Now imagine folding this ''L" so that the first and third squares meet at an edge. The most natural thing to do is to fold each of these up along the edges of the middle square until they meet, forming a corner. But what if you were restricted to the plane? Either the squares would tear away from the first, or their angles would have to be distorted, increasing from 90° to 120°, so that three rhombi are formed. If you draw those three rhombi, and use a little imagination and perspective, you could "see" a corner, though this is only a trick of your mind. As they exist in the plane, these three quadrilaterals are no longer squares.

Now imagine three cubes such that cube one is against a face of cube two and cube three is against an adjacent face. Try to imagine how you might fold the two outer cubes so that the two faces between them meet. Likely, you see that either the cubes will be ripped away from the middle one, or they will be distorted in some way, becoming parallelepipeds (a box with parallelograms for sides). If we had any experience with the fourth dimension, that is, a fourth direction in which we could fold these cubes, the solution would be straightforward: simply fold the cubes "up" until they meet and form the edge of a hypercube. (If this is not a problem for you, you need not read any further.)

So here the difficulty begins. Let us continue discussing the analogy between two, three, and four dimensions. When two polygons meet, they share an edge. Folding them along that edge sends one or both out of their plane. When two polyhedra meet, they share a face. "Folding" them "along that face" sends one or both of them out of their hyperplane. (I use the word "hyperplane", as it contributes more to a mental image than "space".)

In the same way that regular polyhedra can be constructed by folding together regular polygons, regular polytopes can be constructed by "folding" together regular polyhedra. For example, when one folds together eight cubes in 4-space, a hypercube is formed—the same way four folded lines make a square and six folded squares make a cube.

It turns out there are six regular polytopes which exist in 4-space that can be made from the five regular polyhedra. The first is called a simplex, or more specifically a 4-simplex, and is made up of 5 tetrahedra. Next is the hyperoctahedron, or cross-polytope, which is made up of 16 tetrahedra. The third is the hypercube, composed of 8 cubes. Twenty-four octahedra make up the so-called 24-cell; 120 dodecahedra make up the 120-cell; and there are 600 tetrahedra in the 600-cell. H.S.M Coxeter, the world's foremost geometer alive today, offers proofs for the existence of each of these polytopes in his *Introduction to Geometry*. These six are the true objects of our study.

3.1 Vertex Figures

When we discuss regular polyhedra, we need to know two things to determine which of the five regulars one might be: what the polygonal faces are and how many meet at a vertex. The second number gives us the number of sides in the vertex figure, produced by slicing off a corner with a plane.

In 4-space, there is also a notion of truncation, but with a hyperplane rather than a plane. The figure exposed in this process is three-dimensional, not flat. It is itself a regular polyhedron, as the vertex figures of regular polyhedra are regular polygons. These regular polyhedra have as faces the vertex figures of each of the polyhedral cells which meet at the truncated vertex. There are as many faces of the vertex polyhedron as cells that come together. This says that the number of cells meeting at a vertex for regular polytopes is either four, six, eight, 12, or 20.

There is an argument for the construction of regular polytopes by folding cells together about an edge, which runs parallel to the argument above for the construction of polyhedra by folding faces together around a point. The argument requires that the dihedral angle of the polyhedral cells, the angle between the faces at an edge, be such that at least three can come together around an edge. Since the icosahedron has a dihedral angle of greater that $120°$, three icosahedra cannot fit around an edge. The tetrahedron has a dihedral angle of about $71°$, so three, four, or five can be folded together to form a polytope. The cube's faces meet at right angles, so like squares, three can meet at an edge and be folded. And three dodecahedra can also meet, as its face-angle is less than $120°$.

3.2 Duals of Polytopes

For polytopes in 4-space, the concept of dual also has meaning. Here, instead of the vertices coming from the face-centers of the polyhedra, they come from the cell-centers of the polytope. That is, the centers of each of the polyhedra that make up the polytope become vertices of the dual, and vertices become the centers of polyhedra. Edges are now carried to faces, and the faces each become edges which are orthogonal to them. After appropriate scaling, Each face of one polytope divides an edge of its dual at the midpoint, and each edge of a polytope passes through the center of a face of its dual.

The simplex is its own dual. The hypercube and hyperoctahedron are duals. The 24-cell is its own dual. And the 600-cell and 120-cell are duals. The 600 cells of the 600-cell generate the 600 vertices of the 120-cell, and its 720 edges and 1200 faces become the 120-cell's 1200 edges and 720 faces.

There is a relationship which holds for polytopes analogous to Euler's formula relating the number of vertices, edges, and faces of a polyhedron. It is often referred to as merely an extension of Euler's formula. The relation is that the sum of the vertices and faces of a polytope will always equal the sum of the edges and cells. This is usually written $V - E + F - C = 0$.

4 Schläfli Symbols

At this point, we require a shorthand notation to facilitate our work further. Ludwig Schläfli, a school teacher from Thun, Switzerland, developed a notation for all regular polytopes while investigating those of the fourth dimension in the mid-19th century. It was Schläfli who first described the regular polytopes, having discovered all six by 1855. His notation is straightforward and easy to remember, and gives us immediate access to the properties of the polytope, without the need to memorize tables.

The Schlfli symbol for a regular polygon with n sides is $\{n\}$. Thus, an equilateral triangle is given by $\{3\}$, a square by $\{4\}$, and a regular pentagon by $\{5\}$. For regular polyhedra, two numbers are needed. In the Schläfli symbol $\{p, q\}$, p is the number of sides of the regular polygons which make up the faces of the polyhedron, and q is the number of faces which meet at a vertex. That is, the faces of $\{p, q\}$ are $\{p\}$, and the vertex figure is $\{q\}$. Thus, a tetrahedron is denoted $\{3, 3\}$, a cube $\{4, 3\}$, and an octahedron $\{3, 4\}$. The icosahedron is made up of 20 triangles, five of which meet at a vertex, so its symbol is $\{3, 5\}$. The dodecahedron has 12 pentagons meeting 3 at a vertex, and is denoted $\{5, 3\}$. (See Figure 7.)

Notice that the Schläfli symbols for the duals are reversed. The reason for this is simple. Polygons always have the same number of vertices as edges. When a vertex of a polyhedron is truncated, the vertex figure has one edge along every face. So the vertex figure $\{q\}$ has one vertex for every face. When the dual of a polyhedron is formed, it has one vertex for every face. Edges are drawn between the centers of adjacent faces. If q faces meet at a vertex, the q points at their centers will be connected to the centers of adjacent faces, forming a $\{q\}$ polygon as a face of the dual polyhedron.

For polytopes in four dimensions, three numbers are needed in the Schläfli scheme. In the symbol $\{p, q, r\}$, $\{p, q\}$ represents the polyhedral cells which make up the polytope, and r is the number of polyhedra which meet at an edge. Each face $\{p\}$ is shared by two cells, and each edge is in r cells.

Since there are r cells coming together along an edge, there are at least $r + 1$ cells which meet at a vertex. These cells each have a vertex figure $\{q\}$. When the vertex is truncated by a hyperplane, every edge becomes a vertex, as occurs when a polyhedron is truncated by a plane. This exposes the vertex figure of each cell, r of which meet at the vertex generated by each edge. Hence, $\{q, r\}$ represents the vertex polyhedron of the polytope.

It can also be shown that the dual of the polytope $\{p, q, r\}$ is represented by reversing the order of the Schläfli symbol. Since r cells meet at an edge, their adjacent centers will be joined by edges passing through each of their faces. So the dual will have polygonal faces $\{r\}$. The polyhedral cells $\{p, q\}$ get transformed into the vertex figures $\{q, p\}$ of the dual, and the vertex figures $\{q, r\}$ become the $\{r, q\}$ cells of the dual. Thus, the dual of $\{p, q, r\}$ is given by $\{r, q, p\}$.

5 Shadows and Projections

So far, we've talked a fair amount about the properties of 4-D objects by simply extending the properties of 3-D objects in a logical fashion. Yet, without any experience with the fourth dimension, how are we going to see any of these objects? For the moment, let us think about shadows. From our everyday experience, we recognize a shadow as the "outline" of the object which is casting it, and it is often shaped by the surface onto which it is cast. For simplicity, let us say we are standing in a room with flat walls and a bright light. Given any three-dimensional object in the room, its shadow appears on the wall as a two-dimensional shape.

Now say we are lying in a plane in a "room" (say a square) with straight walls and a bright light. Given any two-dimensional object in the room, its shadow appears as a shaded line-segment on the wall, a one-dimensional object. The idea here is that an object will cast a shadow onto a "plane" which is one dimension less than the object. Hence, we could say that fourth-dimensional objects cast three-dimensional shadows on hyperplanes.

Imagine you have a wire-frame cube and are holding it up to the light in the (3-D) room. Since the light passes through the cube, the shadow is cast by its vertices and edges, so that the shadow has as many vertices and edges as the object does. The shadow can change as the cube is held in different ways, and will appear as any of the ways we have of drawing a cube on a piece of paper. The precise mathematical term for this process is *projection*. See Figure 8.

A plane in 3-space passing through the origin is determined by an equation of the form $ax+by+cz = 0$, just as the equation for any line in 2-space passing through the origin can be expressed in the form $ax + by = 0$. For example, the equation $-x + y = 0$ is the same as the line $y = x$. The constants a and b can be expressed as an ordered pair $(a, b) = (-1, 1)$. This ordered pair can be thought of as representing a vector from the origin $(0, 0)$ to the point $(-1, 1)$, which is an arrow pointing "northwest" in the plane. This vector is perpendicular to the line $y = x$, which runs southwest to northeast.

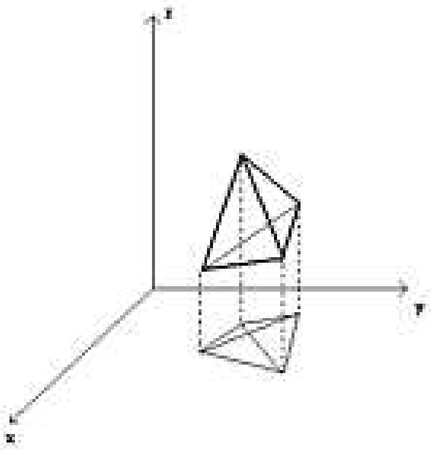

Figure 8: Projection of Tetrahedron into the XY-Plane

To get back to the more interesting 3-space case, $ax + by + cz = 0$. That plane can be determined by the vector $v = (a, b, c)$. The set of all lines perpendicular to the segment from the origin $(0, 0, 0)$ to (a, b, c) and containing the origin is a plane. All of the points (x, y, z) in that plane satisfy the relation $ax + by + cz = 0$.

In order to "see the shadow" that an image would make on a plane, we need to project all of its points in 3-space in the direction of the vector (a, b, c), and determine where these projections intersect the plane. Think of a light bulb as being positioned in such a way that it shines light in the direction (or opposite to the direction) of the vector. Points can make "shadows" in the plane from either side of it. Think of the figure as being squeezed flat into the plane in a direction perpendicular to it.

For the case of a hyperplane in 4-space passing through the origin, it is determined by an equation of the form $aw + bx + cy + dz = 0$. If we project all of the points of a polytope in 4-space along a vector of the form $v = (a, b, c, d)$, they will be mapped to points which lie in the hyperplane orthogonal (i.e., perpendicular) to v. We can then generate an image in 3-space with the projected points and see the "shadow" of a 4-D object. Just as holding a cube in different ways generates different images on the wall, the images in 3-space from a 4-D polytope will change as the vector v changes.

The pages which follow display 3-D images (which are actually projected by the computer into the 2-D plane of the screen) which are the "shadows" of 4-D objects. These images can be viewed from any angle within the hyperplane. In order to perform this projection, we first find three mutually orthogonal vectors that are also orthogonal to $v = (a, b, c, d)$. This set of vectors can be chosen to be continuous for the set of nonzero v (this cannot be done in 3-space, since any vector there determines only a plane and not all of space.) Then for any point in 4-space, its 3-space coordinates are obtained by determining the components of this point with respect to the three orthogonal vectors. This process basically uses the three orthogonal vectors as bases, that is, the x, y, and z directions. One way to picture it is this: the vector v and the three orthogonal vectors are rotated together to coincide with the w-, x-, y-, and z-axes. The point which is being projected undergoes the same rotation, and then the w coordinate is dropped, so the vector falls into the xyz-hyperplane. Simple, isn't it? The specifics of this process are discussed in more detail in Appendix A, which gives detailed explanations of all procedures used in the worksheets.

When this is done for each of the vertices of a polytope in the form (w, x, y, z), points of the form (x', y', z') are generated. Then Maple or some other mathematical software can plot the points in 3-space and display an image. Or the coordinates can be used to build models which can be handled to get a "flattened" view of higher-dimensional objects. Furthermore, points in 4-space can be projected onto a plane (which is what is actually happening when 3-space shadows are displayed on a screen or printed on a page). Or 5-space or higher objects can be projected to 3-space. However, in most cases, the shadow images can become very complicated and confusing. The more dimensions through which a figure is projected, the more congested and distorted its image becomes.

It is possible to construct 3-D models of various projections of the 4-D regulars. We used the mathematical model kit called Zometool to construct these models. Pictures of some of the models possible with Zometool can be found in Appendix B.

6 Rotations

Now that we have a way of displaying 3-D images of 4-D objects, it would be nice if we could spin them around and look at them from different angles. Even better, we could rotate the object in four dimensions and continuously project it onto a hyperplane and view the image as it changes. In order to do this, we have to understand the mathematics behind rotations. As usual, we should begin with a discussion of the familiar cases before generalizing the concepts to higher dimensions.

When we talk about rotating an object in the plane, that rotation is thought of as spinning the object in the plane about a point called the *center of rotation*. Essentially, for any point p in the plane, we imagine a radius connecting it to the center of rotation p_0 and then rotate the radius through a given angle measure θ. If we talk about rotating a point about another

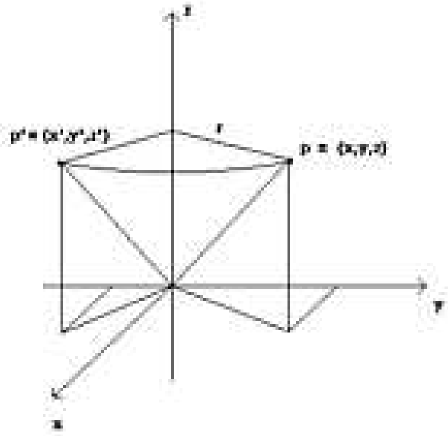

Figure 9: Rotation About the z-Axis

point in a line, it does not make sense. A point separates a line into two distinct rays. In order for a point to pass from one side of the given point to the other, it must either pass through that point or leave the line. Likewise, we cannot talk about rotating a point about a line in a plane. The line divides the plane into two half-planes. In order to pass from one side to the other, the point must leave the plane.

Thus, we can talk about rotation in 3-space about a line, called the *axis of rotation.* In this case, we can imagine a radius which connects a point to the axis of rotation and is orthogonal to the axis. This radius can then be rotated about the axis through a given angle measure θ. As an example, say we wish to rotate the point $p_1 = (1, 5, 2)$ $30°$ about the z-axis in a counterclockwise direction, taking it to the point $p_2 = (x_2, y_2, z_2)$. Note first that $z_2 = z_1$, since our rotation is about this axis. Next, with some use of trigonometry and a good diagram, we should be able to determine the relationships

$$x_2 = x_1(\cos\theta) - y_1(\sin\theta) = 1(\sqrt{3}/2) - 5(1/2) = (\sqrt{3} - 5)/2$$

and

$$y_2 = x_1(\sin\theta) + y_1(\cos\theta) = 1(1/2) + 5(\sqrt{3}/2) = (1 + 5\sqrt{3})/2.$$

These relationships are the same as they would be for rotating a point in the plane about the origin (by simply letting $z = 0$). See Figure 9.

But what if we wish to rotate an object about an arbitrary axis determined by the vector $v = (a, b, c)$? We do this by doing a series of rotations which we can handle. Essentially, what we do is bring the axis of rotation up to the z-axis (and the given point with it), perform the rotation, and then put the axis back where it was. This results in the given point being rotated about this axis. So the first rotation is about the x-axis, and we bring the axis of rotation into the xz-plane. Then we rotate about the y-axis until the axis is brought up to the z-axis. Now we perform the rotation as above. After this is done, we perform the reverse rotation about the y-axis, and the reverse rotation about the x-axis, and the axis of rotation is back where it started, but the given point has been rotated about it to its new position.

When performing these rotations, it is convenient to think of them as multiplication by a matrix. For instance, to perform the rotation with the equations above, we can think of the point p_1 as a column matrix $[x_1, y_1, z_1]$ and the rotation as a matrix R given as:

$$R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so that $(R)p_1 = p_2$. Notice that the determinant $|R| = 1$. This is true for any rotation with constant radius. Now we can think of the process given above as a product of rotation matrices $(R_1^{-1}R_2^{-1}R_0R_2R_1)p_1$ where R_0 is the rotation we wish to perform (about the z-axis), R_1 and R_1^{-1} are rotations about the x-axis, and R_2 and R_2^{-1} are rotations about the y-axis. The product of the 3×3 matrices on the left can be written as one 3×3 matrix R which, when multiplied with p_1, performs the desired rotation. Notice that if each of these rotations has determinant 1, the rotation R will have determinant 1.

In order to understand a 4-space rotation, we must think about what we'd rotate about. If one tries to rotate about a point in 3-space, there are two many directions in which one can move. Essentially, the same would be true if we try to rotate about a line in 4-space. Since we rotate about a point in the plane and about a line in 3-space, it seems natural we would rotate about a plane in 4-space. But does that seem natural? How does one rotate about a plane? We must remember that, like a line in 2-space, a plane divides all of 3-space into two separate regions. To get from one to the other, a point must pass through the plane—or leave the space entirely! Thus, a point can be rotated about a plane only in 4-space. The process is completely analogous to that in three space, so that we basically multiply the point as a column matrix by a number of rotation matrices, or simply by one matrix, in order to rotate the point about the plane. The specifics of this operation for the figures in this package are discussed in Appendix A.

7 Notes

At this point, we should have enough background and introductory information to begin looking at the six regular polytopes in 4-space. Perhaps for some readers, the discussion above was over-simplified, but I suspect there are many who did not follow every part. That is perfectly okay; there isn't going to be a test. The pages which follow are detailed discussions of the properties of each of the regular polytopes, along with explanations of the images generated in the terminology used above. Some understanding is certainly necessary to appreciate these images.

When looking at the displays, try to keep in mind that they are merely shadows, "flattened" into three dimensions. In order to fully appreciate them, 3-D models are extremely helpful. The models in the pictures associated with the polytope information were constructed using a mathematical model kit called Zometool, which is discussed in Appendix B.

Finally, the pages which follow are Maple worksheets. You need Maple V Release 4 to view them. For optimal use, download the files and run them on your computer. Essentially all you need to know about Maple is this: Pressing enter when the cursor is somewhere in red executes all the commands in that execution group and takes the cursor to the next command line. After opening a polytope worksheet, execute the first command line. This will completely define the polytope in Maple. From there, the cursor will move to the first command which displays the polytope. Each execution group creates one display. Occasionally, a lot of text will be skipped when Maple moves the cursor. Simply scroll back up the page to the end of the last set of commands where the text begins.

Also, think of these worksheets as frameworks in which you can generate your own images by adjusting various parameters. Detailed explanations of each procedure are given in Appendix A.