

# 3 The Mathematics of Sharing

## 3.1 Fair-Division Games

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### 3.4 The Lone-Chooser Method

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# Basic Elements of a Fair-Division Game

The underlying elements of every *fair-division game* are as follows:

**The goods (or “booty”).**

This is the informal name we give to the item or items being divided. We will denote the goods with by symbol  $S$

# Basic Elements of a Fair-Division Game

## The players.

The parties with the right (or in some cases the duty) to share  $S$  are called the players.

Usually the players will be individuals, but in the players could also be institutions (ethnic groups, political parties, states, and even nations).

# Basic Elements of a Fair-Division Game

## The value systems.

We assume each player has a value system that gives the player the ability to quantify the value of the goods.

This means each player can assign a value to  $S$ , or any subset of  $S$  – either in absolute terms (“that’s worth \$147.50”) or in relative terms (“that piece is worth 30% of the total value of  $S$ ”).

# Basic Assumptions

## Rationality

Each of the players is a rational entity seeking to maximize its share of  $S$ .

In addition, we assume a player's moves are based on reason alone (we are taking emotion, psychology, mind games, and all other non-rational elements out of the picture.)

# Basic Assumptions

## Cooperation

The players accept the rules of the game as binding.

The rules are such that after a finite number of moves by the players, the game terminates with a division of  $S$ .

There are no outsiders such as judges or referees involved in these games – just the players and the rules.

# Basic Assumptions

## Privacy

Players have no useful information on the other players' value systems. Thus a player cannot predict the moves of the other players.

This assumption does not always hold in real life, especially if the players are siblings or friends.

# Basic Assumptions

## Symmetry

Players have equal rights in sharing  $S$ .

In particular, each player is entitled to a proportional share of  $S$  –

If there are two players, each is entitled to at least one-half of  $S$ ;

If there are three players, each is entitled to at least one-third of  $S$ , and so on.

# Fair Share

Given the goods  $S$  and players  $P_1, P_2, P_3, \dots, P_N$ , (each with its own value system), the goal is to divide  $S$  into  $N$  shares and *assign shares to players in such a way that each player gets a fair share.*

# Fair Share

## FAIR SHARE

Suppose that  $s$  denotes a share of the goods  $S$  and  $P$  is one of the players in a fair-division game with  $N$  players. We say that  $s$  is a fair share to player  $P$  if  $s$  is worth at least  $1/N$ th of the total value of  $S$  in the opinion of  $P$ . (Such a share is often called a proportional fair share)

# Fair Division Methods

A fair-division method is a set of rules that define how the game is to be played.

Thus, in a fair-division game we must consider not only the goods  $S$  and the players  $P_1, P_2, P_3, \dots, P_N$  (each with his or her own opinions about how  $S$  should be divided), but also a specific method by which we plan to accomplish the fair division.

# Fair Division Methods

There are many different fair-division methods known, but in this chapter we will only discuss a few of the classic ones.

Depending on the nature of the set  $S$ , a fair-division game can be classified as one of three types: continuous, discrete, or mixed,

The fair-division methods used depend on which of these types we are facing.

# Types of Fair Division Games

## Continuous

In a **continuous** fair-division game the set  $S$  is divisible in infinitely many ways, and shares can be increased or decreased by arbitrarily small amounts. Typical examples of continuous fair-division games involve the division of land, a cake, a pizza, and so forth.

We will focus primarily on continuous fair division games.

# Types of Fair Division Games

## Discrete

A fair-division game is **discrete** when the set  $S$  is made up of objects that are indivisible like paintings, houses, cars, boats, jewelry, and so on.

# Types of Fair Division Games

## Mixed

A mixed fair-division game is one in which some of the components are continuous and some are discrete. Dividing an estate consisting of jewelry, a house, and a parcel of land is a mixed fair-division game.