Excursions in Modern Mathematics
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Chapter 10
The Mathematics of Money

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University of Kentucky
MA 111

Fall 2011

PLEASE BRING YOUR CALCULATORS AND BOOKS TO CLASS EVERY DAY

10.1 Percentages

Percent

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What does 25% mean?

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What does 25% mean? $\frac{25}{100}$ or 0.25.

Percent

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So 32 percent or 32% means "32 per 100" or "32 out of 100" or $\frac{32}{100}$ or 0.32.

What does 25% mean? $\frac{25}{100}$ or 0.25.

What does 2% mean?

Percent

Percent means "per 100".

So 32 percent or 32% means "32 per 100" or "32 out of 100" or $\frac{32}{100}$ or 0.32.

What does 25% mean? $\frac{25}{100}$ or 0.25.

What does 2% mean? $\frac{2}{100}$ or 0.02.

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What does 25% mean? $\frac{25}{100}$ or 0.25.

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What does 325% mean?

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What does 25% mean? $\frac{25}{100}$ or 0.25.

What does 2% mean? $\frac{2}{100}$ or 0.02.

What does 325% mean? $\frac{325}{100}$ or 3.25.

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Percent means "per 100".

So 32 percent or 32% means "32 per 100" or "32 out of 100" or $\frac{32}{100}$ or 0.32.

What does 25% mean? $\frac{25}{100}$ or 0.25.

What does 2% mean? $\frac{2}{100}$ or 0.02.

What does 325% mean? $\frac{325}{100}$ or 3.25.

What does 0.07% mean?

Percent

Percent means “per 100”.

So 32 percent or 32% means “32 per 100” or “32 out of 100” or $\frac{32}{100}$ or 0.32.

What does 25% mean? $\frac{25}{100}$ or 0.25.

What does 2% mean? $\frac{2}{100}$ or 0.02.

What does 325% mean? $\frac{325}{100}$ or 3.25.

What does 0.07% mean? $\frac{0.07}{100}$ or 0.0007.

Decimals to Percents

Example 10.1 from the text. Express each score as a percent.

A quiz score of 19/25.

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A quiz score of 19/25.
$\frac{19}{25} = 0.76 = \frac{76}{100} = 76\%$.

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A quiz score of 19/25.
$\frac{19}{25} = 0.76 = \frac{76}{100} = 76\%.$

A midterm score of 49.2/60.

Decimals to Percents

Example 10.1 from the text. Express each score as a percent.

A quiz score of 19/25.
$\frac{19}{25} = 0.76 = \frac{76}{100} = 76\%.$

A midterm score of 49.2/60.
$\frac{49.2}{60} = 0.82 = \frac{82}{100} = 82\%.$

Decimals to Percents

Example 10.1 from the text. Express each score as a percent.

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$\frac{19}{25} = 0.76 = \frac{76}{100} = 76\%.$

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$\frac{49.2}{60} = 0.82 = \frac{82}{100} = 82\%.$

A final exam score of 124.8/150.

Decimals to Percents

Example 10.1 from the text. Express each score as a percent.

A quiz score of 19/25.
$\frac{19}{25} = 0.76 = \frac{76}{100} = 76\%.$

A midterm score of 49.2/60.
$\frac{49.2}{60} = 0.82 = \frac{82}{100} = 82\%.$

A final exam score of 124.8/150.
$\frac{124.8}{150} = 0.832 = \frac{83.2}{100} = 83.2\%.$

Percent

"32% of 545 is 174.4" means $\frac{32}{100} \times 545 = 174.4$.

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"32% of 545 is 174.4" means $\frac{32}{100} \times 545 = 174.4$.
It also means $\frac{32}{100} = \frac{174.4}{545}$.

"x% of Y is Z" means $\frac{x}{100} \times Y = Z$.

Percent

"32% of 545 is 174.4" means $\frac{32}{100} \times 545 = 174.4$.
It also means $\frac{32}{100} = \frac{174.4}{545}$.

$$\boxed{\text{“}x\text{% of } Y \text{ is } Z\text{” means } \frac{x}{100} \times Y = Z.}$$

REMEMBER THIS! The book may use different letters, but that doesn't really matter.

Percent

"32% of 545 is 174.4" means $\frac{32}{100} \times 545 = 174.4$.
It also means $\frac{32}{100} = \frac{174.4}{545}$.

$$\boxed{\text{"}x\% \text{ of } Y \text{ is } Z\text{" means } \frac{x}{100} \times Y = Z.}$$

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It also means $\frac{x}{100} = \frac{Z}{Y}$.

Percent

"32% of 545 is 174.4" means $\frac{32}{100} \times 545 = 174.4$.
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$$\text{"}x\text{\% of } Y \text{ is } Z\text{" means } \frac{x}{100} \times Y = Z.$$

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It also means $\frac{x}{100} = \frac{Z}{Y}$.

In the above example, 545 is the base for the percent. The base for a percent is the quantity to which the percent applies.

Percent

"32% of 545 is 174.4" means $\frac{32}{100} \times 545 = 174.4$.
It also means $\frac{32}{100} = \frac{174.4}{545}$.

$$\text{"}x\text{\% of } Y \text{ is } Z\text{" means } \frac{x}{100} \times Y = Z.$$

REMEMBER THIS! The book may use different letters, but that doesn't really matter.

It also means $\frac{x}{100} = \frac{Z}{Y}$.

In the above example, 545 is the base for the percent. The base for a percent is the quantity to which the percent applies. A very common error is to use the incorrect base for a percent.

Given x and Y, Find Z

If a county proposes charging a tax of 0.5% on a $270 purchase, what is the amount of tax paid?

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$$Z = \frac{x}{100} \times Y$$

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If a county proposes charging a tax of 0.5% on a \$270 purchase, what is the amount of tax paid?

$$Z = \frac{x}{100} \times Y$$

$$Z = \frac{0.5}{100} \times 270$$

Given x and Y, Find Z

If a county proposes charging a tax of 0.5% on a $270 purchase, what is the amount of tax paid?

$$Z = \frac{x}{100} \times Y$$

$$Z = \frac{0.5}{100} \times 270$$

So

$$Z = \$1.35.$$

Given x and Z, Find Y

Jason was transfered to a different school and now must drive 48 miles to school, which is only 75% of the previous distance. What was the previous distance?

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Jason was transfered to a different school and now must drive 48 miles to school, which is only 75% of the previous distance. What was the previous distance?

$$Z = \frac{x}{100} \times Y$$

$$48 = \frac{75}{100} \times Y$$

so

$$Y = \frac{48 \times 100}{75} = 64 \text{ miles.}$$

Given Y and Z, find x

Suppose that 25 students enrolled in MA 111 last year and 30 enrolled this year. What percentage of 25 is 30?

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Suppose that 25 students enrolled in MA 111 last year and 30 enrolled this year. What percentage of 25 is 30?

$$Z = \frac{x}{100} \times Y$$

Given Y and Z, find x

Suppose that 25 students enrolled in MA 111 last year and 30 enrolled this year. What percentage of 25 is 30?

$$Z = \frac{x}{100} \times Y$$

$$30 = \frac{x}{100} \times 25$$

Given Y and Z, find x

Suppose that 25 students enrolled in MA 111 last year and 30 enrolled this year. What percentage of 25 is 30?

$$Z = \frac{x}{100} \times Y$$

$$30 = \frac{x}{100} \times 25$$

$$\frac{x}{100} = \frac{30}{25}$$

so

$$x = \frac{30}{25} \times 100 = 120.$$

Thus the number of students enrolled this year is 120% of the number enrolled last year.

Changes Measured by Percents

If the population of a town was 1000 and it increased by 25%, what is the new population?

If a pair of shoes cost $60 and the price was reduced 30%, what is the new price?

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Method 1: Find 25% of 1000 and add it to 1000.

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If the population of a town was 1000 and it increased by 25%, what is the new population?

Method 1: Find 25% of 1000 and add it to 1000.
$\frac{25}{100} \times 1000 = 250$. So the new population is
$1000 + 250 = 1250$.

Changes Measured by Percents

If the population of a town was 1000 and it increased by 25%, what is the new population?

Method 1: Find 25% of 1000 and add it to 1000.
$\frac{25}{100} \times 1000 = 250$. So the new population is
$1000 + 250 = 1250$.

Note that this is equivalent to

$$1000 + \frac{25}{100} \times 1000$$

or

$$1000 \left(1 + \frac{25}{100}\right).$$

Changes Measured by Percents

So we have Method 2:
To increase 1000 by 25%, multiply 1000 by $(1 + \frac{25}{100})$:

$$1000 \times \left(1 + \frac{25}{100}\right) = 1250.$$

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To increase 1000 by 25%, multiply 1000 by $(1 + \frac{25}{100})$:

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LET'S PLAN TO USE METHOD 2 FROM NOW ON.

Changes Measured by Percents

If a pair of shoes cost \$60 and the price was reduced 30%, what is the new price?

Changes Measured by Percents

If a pair of shoes cost \$60 and the price was reduced 30%, what is the new price?

Method 1: Find 30% of \$60 and subtract it from \$60.

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If a pair of shoes cost \$60 and the price was reduced 30%, what is the new price?

Method 1: Find 30% of \$60 and subtract it from \$60.
$\frac{30}{100} \times 60 = 18$. So the new price is $60 - 18 = \$42$.

Changes Measured by Percents

If a pair of shoes cost \$60 and the price was reduced 30%, what is the new price?

Method 1: Find 30% of \$60 and subtract it from \$60.
$\frac{30}{100} \times 60 = 18$. So the new price is $60 - 18 = \$42$.

Note that this is equivalent to

$$60 - \frac{30}{100} \times 60$$

or

$$60\left(1 - \frac{30}{100}\right).$$

Changes Measured by Percents

So we have Method 2:
To decrease 60 by 30%, multiply 60 by $(1 - \frac{30}{100})$:

$$60 \times \left(1 - \frac{30}{100}\right) = 42.$$

Changes Measured by Percents

So we have Method 2:
To decrease 60 by 30%, multiply 60 by $(1 - \frac{30}{100})$:

$$60 \times \left(1 - \frac{30}{100}\right) = 42.$$

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General Situation:

A is increased by $x\%$ of A to get B:

$$B = A\left(1 + \frac{x}{100}\right).$$

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A is decreased by $x\%$ of A to get B:

$$\boxed{B = A\left(1 - \frac{x}{100}\right).}$$

REMEMBER THESE!

Given A and x, find B

If 75000 is increased by 20%, what is the result?

Given A and x, find B

If 75000 is increased by 20%, what is the result?

$$B = 75000\left(1 + \frac{20}{100}\right) = 90000.$$

Given A and x, find B

If 75000 is decreased by 30%, what is the result?

Given A and x, find B

If 75000 is decreased by 30%, what is the result?

$$B = 75000\left(1 - \frac{30}{100}\right) = 52500.$$

Combining Increases and Decreases

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Combining Increases and Decreases

You make \$15 per hour. Your boss comes in and says, "Congratulations! I'm giving you a 5% raise!"

One year later the boss comes in and says "We are hitting some hard economic times. I'm afraid that I must cut your wage by 5%."

So you are back to where you started, right?

Combining Increases and Decreases

If \$15 is increased by 5% and then the result is decreased by 5%, what is the final result?

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Two steps: First...

$$\$15\left(1+\frac{5}{100}\right)=\$15.75.$$

Combining Increases and Decreases

If \$15 is increased by 5% and then the result is decreased by 5%, what is the final result?

Two steps: First...

$$\$15\left(1+\frac{5}{100}\right)=\$15.75.$$

Use this result...

$$\$15.75\left(1-\frac{5}{100}\right)\approx\$14.96.$$

Thus, the final result is \$14.96, which is less than the original \$15.

Combining Increases and Decreases

Example 10.6 from the text. The price of a toy is marked up by 80%. Then that price is cut by 40%. Then that price is cut again by 25%. How does the final price compare to the original price?

Combining Increases and Decreases

Example 10.6 from the text. The price of a toy is marked up by 80%. Then that price is cut by 40%. Then that price is cut again by 25%. How does the final price compare to the original price?

Let C be the original price.
It is increased by 80%: $C \times (1 + 0.80)$.

Combining Increases and Decreases

Example 10.6 from the text. The price of a toy is marked up by 80%. Then that price is cut by 40%. Then that price is cut again by 25%. How does the final price compare to the original price?

Let C be the original price.
It is increased by 80%: $C \times (1 + 0.80)$.
The result is decreased by 40%: $C \times (1 + 0.80) \times (1 - 0.40)$.

Combining Increases and Decreases

Example 10.6 from the text. The price of a toy is marked up by 80%. Then that price is cut by 40%. Then that price is cut again by 25%. How does the final price compare to the original price?

Let C be the original price.
It is increased by 80%: $C \times (1 + 0.80)$.
The result is decreased by 40%: $C \times (1 + 0.80) \times (1 - 0.40)$.
That result is decreased by 25%:
$C \times (1 + .80) \times (1 - 0.40) \times (1 - 0.25)$.

Combining Increases and Decreases

Example 10.6 from the text. The price of a toy is marked up by 80%. Then that price is cut by 40%. Then that price is cut again by 25%. How does the final price compare to the original price?

Let C be the original price.
It is increased by 80%: $C \times (1 + 0.80)$.
The result is decreased by 40%: $C \times (1 + 0.80) \times (1 - 0.40)$.
That result is decreased by 25%:
$C \times (1 + .80) \times (1 - 0.40) \times (1 - 0.25)$.
Net effect: $C \times (0.81)$, which means a decrease of 19% from the original price, since $0.81 = 1 - 0.19$.

Given B and x, Find A

If the population of a city in 2008 was 75,870 and this was an increase of 6% since 1998, what was the population in 1998? Note that the base A of the percent is not known.

Given B and x, Find A

If the population of a city in 2008 was 75,870 and this was an increase of 6% since 1998, what was the population in 1998? Note that the base A of the percent is not known.

$$B = A\left(1 + \frac{x}{100}\right)$$

Given B and x, Find A

If the population of a city in 2008 was 75,870 and this was an increase of 6% since 1998, what was the population in 1998? Note that the base A of the percent is not known.

$$B = A\left(1 + \frac{x}{100}\right)$$

$$75870 = A\left(1 + \frac{6}{100}\right)$$

Given B and x, Find A

If the population of a city in 2008 was 75,870 and this was an increase of 6% since 1998, what was the population in 1998? Note that the base A of the percent is not known.

$$B = A\left(1 + \frac{x}{100}\right)$$

$$75870 = A\left(1 + \frac{6}{100}\right)$$

$$75870 = A(1.06)$$

Given B and x, Find A

If the population of a city in 2008 was 75,870 and this was an increase of 6% since 1998, what was the population in 1998? Note that the base A of the percent is not known.

$$B = A\left(1 + \frac{x}{100}\right)$$

$$75870 = A\left(1 + \frac{6}{100}\right)$$

$$75870 = A(1.06)$$

$$A = \frac{75870}{1.06} = 71575.$$

Given B and x, Find A

If a laptop computer sells for $1100 in 2009 and this is a decrease of 7% in the price since 2007, what was the price in 2007?

Given B and x, Find A

If a laptop computer sells for $1100 in 2009 and this is a decrease of 7% in the price since 2007, what was the price in 2007?

$$B = A\left(1 - \frac{x}{100}\right)$$

Given B and x, Find A

If a laptop computer sells for \$1100 in 2009 and this is a decrease of 7% in the price since 2007, what was the price in 2007?

$$B = A\left(1 - \frac{x}{100}\right)$$

$$1100 = A\left(1 - \frac{7}{100}\right)$$

Given B and x, Find A

If a laptop computer sells for $1100 in 2009 and this is a decrease of 7% in the price since 2007, what was the price in 2007?

$$B = A\left(1 - \frac{x}{100}\right)$$

$$1100 = A\left(1 - \frac{7}{100}\right)$$

$$1100 = A(0.93)$$

Given B and x, Find A

If a laptop computer sells for \$1100 in 2009 and this is a decrease of 7% in the price since 2007, what was the price in 2007?

$$B = A\left(1 - \frac{x}{100}\right)$$

$$1100 = A\left(1 - \frac{7}{100}\right)$$

$$1100 = A(0.93)$$

$$A = \frac{1100}{0.93} \approx \$1183.$$

Given A and B, find x

If

$$B = A\left(1 + \frac{x}{100}\right)$$

then

$$\frac{x}{100} = \frac{B}{A} - 1$$

or

$$\boxed{x = \frac{B-A}{A} \times 100 = \frac{\text{new value}-\text{old value}}{\text{old value}} \times 100\%.}$$

REMEMBER THIS FORMULA FOR PERCENT CHANGE!

Given A and B, Find x

If the cost of a gallon of gasoline increases from \$1.70 to \$1.90, what is the percent increase?

Given A and B, Find x

If the cost of a gallon of gasoline increases from \$1.70 to \$1.90, what is the percent increase?

$$\frac{\$1.90 - \$1.70}{\$1.70} \times 100\% \approx 11.76\%.$$

Given A and B, Find x

If the cost of a gallon of gasoline decreases from \$1.90 to \$1.70, what is the percent change?

Given A and B, Find x

If the cost of a gallon of gasoline decreases from \$1.90 to \$1.70, what is the percent change?

$$\frac{\$1.70 - \$1.90}{\$1.90} \times 100\% \approx -10.53\%.$$

Given A and B, Find x

By what percent is 40 increased by to get 50?

Given A and B, Find x

By what percent is 40 increased by to get 50?

$$\frac{50-40}{40} \times 100\% = 25\%.$$

Some Summary Examples

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To find 12.5% of 456:
$\frac{12.5}{100} \times 456 = 0.125 \times 456 = 57.$

Some Summary Examples

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To increase 456 by 12.5%:

Some Summary Examples

To turn 12.5% into a decimal:
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To find 12.5% of 456:
$\frac{12.5}{100} \times 456 = 0.125 \times 456 = 57.$

To increase 456 by 12.5%:
$456 \times \left(1 + \frac{12.5}{100}\right) = 456 \times (1 + 0.125) = 513.$

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To turn 12.5% into a decimal:
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To find 12.5% of 456:
$\frac{12.5}{100} \times 456 = 0.125 \times 456 = 57.$

To increase 456 by 12.5%:
$456 \times \left(1 + \frac{12.5}{100}\right) = 456 \times (1 + 0.125) = 513.$

To decrease 456 by 12.5%:

Some Summary Examples

To turn 12.5% into a decimal:
$12.5\% = \frac{12.5}{100} = 0.125.$

To find 12.5% of 456:
$\frac{12.5}{100} \times 456 = 0.125 \times 456 = 57.$

To increase 456 by 12.5%:
$456 \times \left(1 + \frac{12.5}{100}\right) = 456 \times (1 + 0.125) = 513.$

To decrease 456 by 12.5%:
$456 \times \left(1 - \frac{12.5}{100}\right) = 456 \times (1 - 0.125) = 399.$

10.2 Simple Interest

The Time Value of Money

When you deposit \$1000 into a savings account at the bank, you expect that amount to gain interest over time.
A year from now, you would have more than \$1000.

In return for having access to the present value of your money, the bank increases the future value of the money by adding interest.

The Time Value of Money

If you take out a car loan for \$10,000, you expect to pay it back with interest.
Suppose the total amount you repay over time is \$12,000.

The present value is $P = \$10,000$.
The future value is $F = \$12,000$.

What determines the future value?

The interest is the return the lender expects as a reward for the use of their money.

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The interest is the return the lender expects as a reward for the use of their money.

Since the amount of interest should depend on the amount of the loan, we consider an interest rate.

The standard way to describe an interest rate is the annual percentage rate or APR.

Simple Interest

With simple interest, only the principal (the original money invested or borrowed) generates interest over time.

The amount of interest generated each year will be the same throughout the life of the loan/investment.

Example

If you buy a \$1000 savings bond that pays 5% annual simple interest, how much is the bond worth 10 years from now?

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The present value or principal is $P = \$1000$.

Each year, the principal earns 5% interest.
How much interest will be earned in one year?

Example

If you buy a $1000 savings bond that pays 5% annual simple interest, how much is the bond worth 10 years from now?

The present value or principal is $P = \$1000$.

Each year, the principal earns 5% interest.
How much interest will be earned in one year?

$$\$1000 \cdot \frac{5}{100} = \$1000(0.05) = \$50.$$

Example

After one year, the bond will be worth

$$\$1000 + \$50 = \$1050.$$

After two years, the bond will be worth

$$\$1000 + \$50 + \$50 = \$1100.$$

How much will the bond be worth after 10 years?

Example

After one year, the bond will be worth

$$\$1000 + \$50 = \$1050.$$

After two years, the bond will be worth

$$\$1000 + \$50 + \$50 = \$1100.$$

How much will the bond be worth after 10 years?

$$\$1000 + 10(\$50) = \$1500.$$

Example

After one year, the bond will be worth

$$\$1000 + \$50 = \$1050.$$

After two years, the bond will be worth

$$\$1000 + \$50 + \$50 = \$1100.$$

How much will the bond be worth after 10 years?

$$\$1000 + 10(\$50) = \$1500.$$

How much will the bond be worth after t years?

Example

After one year, the bond will be worth

$$\$1000 + \$50 = \$1050.$$

After two years, the bond will be worth

$$\$1000 + \$50 + \$50 = \$1100.$$

How much will the bond be worth after 10 years?

$$\$1000 + 10(\$50) = \$1500.$$

How much will the bond be worth after t years?

$$\$1000 + \$50t.$$

Simple Interest Formula

Remember that the annual interest was found by multiplying $\$1000 \times \frac{5}{100}$.

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Remember that the annual interest was found by multiplying $\$1000 \times \frac{5}{100}$.

In general, if the principal is P dollars and the interest rate is R%, the amount of annual interest is

$$P\left(\frac{R}{100}\right)$$

$$\text{or } P \cdot r \text{ where } r = \frac{R}{100}.$$

Simple Interest Formula

Remember that the annual interest was found by multiplying $\$1000 \times \frac{5}{100}$.

In general, if the principal is P dollars and the interest rate is $R\%$, the amount of annual interest is

$$P\left(\frac{R}{100}\right)$$

$$\text{or } P \cdot r \text{ where } r = \frac{R}{100}.$$

Over t years, the amount of interest accrued is

$$P \cdot r \cdot t.$$

Simple Interest Formula

Thus, the total future value will be

$$P + P \cdot r \cdot t$$

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$$P + P \cdot r \cdot t$$

If P dollars is invested under simple interest for t years at an APR of $R\%$, then the future value is:

$$\boxed{F = P\,(1 + r \cdot t)}$$

where r is the decimal form of $R\%$.

Using the Simple Interest Formula

Suppose you want to buy a government bond that will be worth \$2500 in 8 years. If there is 5.75% annual simple interest on the bond, how much do you need to pay now?

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We know the future value $F = \$2500$ and we want to find the present value or principal P.

Using the Simple Interest Formula

Suppose you want to buy a government bond that will be worth \$2500 in 8 years. If there is 5.75% annual simple interest on the bond, how much do you need to pay now?

We know the future value $F = \$2500$ and we want to find the present value or principal P.

Solve for P:

$$\begin{aligned} 2500 &= P\,(1 + (0.0575)(8)) \\ 2500 &= P(1.46) \\ P &= \frac{2500}{1.46} \\ P &= \$1712.33. \end{aligned}$$

Using the Simple Interest Formula

Page 393, #27: A loan of $5400 collects simple interest each year for eight years. At the end of that time, a total of $8316 is paid back. Find the APR for the loan.

Using the Simple Interest Formula

Solution: \$5400 is the present value P, and \$8316 is the future value F. Solve for r:

$$8316 = 5400(1 + 8r)$$

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Using the Simple Interest Formula

Solution: \$5400 is the present value P, and \$8316 is the future value F. Solve for r:

$$\begin{aligned} 8316 &= 5400(1+8r) \\ 8316 &= 5400 + 5400 \cdot 8r \\ 2916 &= 43200r \\ r &= 0.0675. \end{aligned}$$

The APR is 6.75%.

10.3 Compound Interest

Compound Interest

With compound interest, the interest rate applies to the principal *and* the previously accumulated interest.

Money collecting compound interest will grow faster than that collecting simple interest. Over time, the difference between compound and simple interest becomes greater and greater.

Example

If you invest \$2000 in a fund with a 6% APR, how much is the investment worth after one year?

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After two years? The interest rate will be applied to the previous amount, 2000(1.06).

$$2000(1.06)(1.06) = 2000(1.06)^2.$$

After three years?

$$2000(1.06)^2(1.06) = 2000(1.06)^3.$$

Example

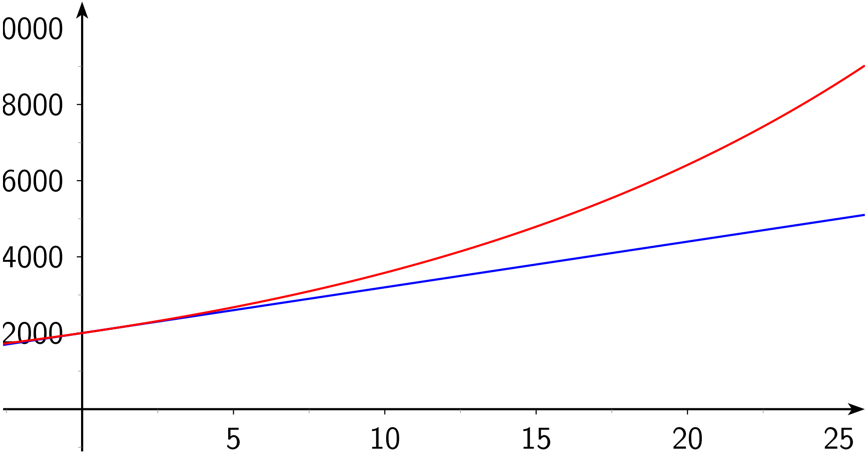
If you invest $2000 in a fund with a 6% APR, how much is the investment worth after fifteen years?

Example

If you invest \$2000 in a fund with a 6% APR, how much is the investment worth after fifteen years?

$$2000(1.06)^{15} = 2000(2.3966) = \$4793.20.$$

Compound vs. Simple Interest



Blue line: 6% annual simple interest
Red line: 6% annual compound interest

Compound Interest Formula

If P dollars is compounded annually for t years at an APR of $R\%$, then the future value is

$$\boxed{F = P\left(1 + r\right)^t}$$

where r is the decimal form of $R\%$.

Example

Suppose you invest \$2000 in a fund with a 6% APR that is *compounded monthly*. That is, interest is applied at the end of each month (instead of just the end of each year).

Since the interest rate is 6% annually (APR), it must be

$$\frac{6\%}{12} = 0.5\% \text{ per month.}$$

After one month, you'll have $\$2000(1.005) = \$2010$.
After one year (twelve months), you'll have $\$2000(1.005)^{12} = \$2123.36$.

Example

If you invest \$2000 in a fund with a 6% APR compounded monthly, how much is the investment worth after fifteen years?

Example

If you invest \$2000 in a fund with a 6% APR compounded monthly, how much is the investment worth after fifteen years?

$$2000\,(1.005)^{15\cdot 12} = 2000(1.005)^{180} = \$4908.19.$$

Compound Interest Formula

If P is invested at an APR of $R\%$ compounded n times per year, for t years, then the future value F is:

$$\boxed{F = P\left(1 + \frac{r}{n}\right)^{nt}}$$

where r is the decimal form of $R\%$. Here is another equivalent form:

$$\boxed{F = P\left(1 + p\right)^{T}}$$

where p is the period interest rate expressed as a decimal, $p = \frac{r}{n}$, and T is the total number of times the interest is compounded.

Using the Compound Interest Formula

You put \$800 in a bank account that offers a 4.5% APR compounded weekly. How much is in the account in 5 years?

Since there are 52 weeks in a year, the interest rate is $\frac{r}{n} = \frac{4.5\%}{52} = 0.086538\%$ or 0.00086538.
In 5 years, interest will be compounded $nt = 52 \cdot 5 = 260$ times.

$$\begin{aligned} 800(1+0.00086538)^{260} &= 800(1.00086538)^{260} \\ &= 800(1.252076) \\ &= \$1001.76. \end{aligned}$$

Using the Compound Interest Formula

You want to save up \$1500. If you can buy a 3 year CD (certificate of deposit) from the bank that pays an APR of 5% compounded biannually, how much should you invest now?

Note: Biannually means two times per year.

Using the Compound Interest Formula

You want to save up \$1500. If you can buy a 3 year CD (certificate of deposit) from the bank that pays an APR of 5% compounded biannually, how much should you invest now?

Note: Biannually means two times per year.
Solve for P:

$$\begin{aligned} 1500 &= P\left(1+\frac{.05}{2}\right)^{2\cdot 3} \\ 1500 &= P(1.025)^6 \\ 1500 &= P(1.159693) \\ P &= \$1293.45. \end{aligned}$$

Compounding Continuously

You invest \$100 for three years at 5% interest. How much do you end up with if it is compounded:

- Yearly?
- Monthly?
- Weekly?
- Daily (365 days/year)?

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You invest \$100 for three years at 5% interest. How much do you end up with if it is compounded:

- Yearly? $100(1+\frac{.05}{1})^3 \approx \$115.76$

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You invest \$100 for three years at 5% interest. How much do you end up with if it is compounded:

- Yearly? $100(1 + \frac{.05}{1})^3 \approx \$115.76$
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- Weekly?

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- Monthly? $100(1+\frac{.05}{12})^{(12\cdot 3)} \approx \$116.15$
- Weekly? $100(1+\frac{.05}{52})^{(52\cdot 3)} \approx \$116.18$

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- Weekly? $100(1+\frac{.05}{52})^{(52\cdot 3)} \approx \$116.18$
- Daily (365 days/year)?

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- Weekly? $100(1 + \frac{.05}{52})^{(52\cdot 3)} \approx \$116.18$
- Daily (365 days/year)? $100(1 + \frac{.05}{365})^{(365\cdot 3)} \approx \$116.18$

Compounding Continuously

Financial institutions frequently calculate interest as compounded continuously, which means taking this process of subdividing the time intervals to the limit. There is a formula to calculate this:

$$\boxed{F = P \times e^{rt}}$$

where r is the annual interest rate expressed as decimal, and t is the number of years.

Compounding Continuously

e is a special number approximately equal to 2.718281828. Look for the *e* button on your calculator, or possibly the exp button.

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In our case, $F = 100 \times e^{(0.05 \cdot 3)} \approx \$116.18$.

Compounding Continuously

e is a special number approximately equal to 2.718281828. Look for the e button on your calculator, or possibly the exp button.

In our case, $F = 100 \times e^{(0.05 \cdot 3)} \approx \$116.18$.

Lending institutions often choose to charge interest on loans up to the instant that you make payments to maximize their receipts.

Annual Percentage Yield

The annual percentage yield or APY of an investment is the percentage of profit that is generated in a one-year period.

The APY is essentially the same as the *percent increase* in the investment over one year.

Example: APY

If an investment of \$575 is worth \$630 after one year, what is the APY?

Example: APY

If an investment of \$575 is worth \$630 after one year, what is the APY?

The profit made is $\$630 - \$575 = \$55$. Thus the annual percentage yield is:

$$\frac{\$630 - \$575}{\$575} = \frac{\$55}{\$575} = 0.096 = 9.6\%.$$

Comparing Investments

The APY allows us to compare different investments.

Use the APY to compare an investment at 3.5% compounded monthly with an investment at 3.55% compounded annually.

The amount of principal is unimportant. Pick $P = \$100$ to make our lives easier.

Comparing Investments

For the first loan, after one year we have

$$100\left(1+\frac{.035}{12}\right)^{12}=103.5571.$$

So the APY is $\frac{103.5571-100}{100}=0.035571\approx 3.56\%$.

For the second loan, after one year we have

$$100(1+.0355)=103.55.$$

So the APY is $\frac{103.55-100}{100}=0.0355=3.55\%$.
The first loan is better because it has a higher APY.

10.5 Deferred Annuities

Compound Interest Reminder

If P is invested at an APR of $R\%$ compounded n times per year, for t years, then the future value F is:

$$F = P\left(1 + \frac{r}{n}\right)^{nt}$$

where r is the decimal form of $R\%$. Here is another equivalent form:

$$F = P\left(1 + p\right)^{T}$$

where p is the period interest rate expressed as a decimal, $p = \frac{r}{n}$, and T is the total number of times the interest is compounded.

Fixed Annuities

A fixed annuity is a sequence of equal payments made or received over regular time intervals.

Examples:

- making regular payments on a car or home loan
- making regular deposits into a college fund
- receiving regular payments from an retirement fund or inheritance

Two Types of Fixed Annuities

A deferred annuity is one in which regular payments are made first, so as to produce a lump-sum payment at a later date.

- Example: Making regular payments to save up for college.

An installment loan is an annuity in which a lump sum is paid first, and then regular payments are made against it later.

- Example: Receiving a car loan, and paying it back with monthly payments.

Deferred Annuities

Small Example. You deposit $1000 on January 1 each year for 4 years in an investment that earns 3% interest compounded annually, with the interest added on December 31 of each year. How much will you have at the end of four years?

Start of Year	Payment 1	Payment 2	Payment 3	Payment 4
1	1000			
2	⋮	1000		
3	⋮	⋮	1000	
4	⋮	⋮	⋮	1000
5				

Deferred Annuities

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Start of Year	Payment 1	Payment 2	Payment 3	Payment 4
1	1000			
2	⋮	1000		
3	⋮	⋮	1000	
4	⋮	⋮	⋮	1000
5	1125.51			

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1	1000			
2	⋮	1000		
3	⋮	⋮	1000	
4	⋮	⋮	⋮	1000
5	1125.51	1092.73		

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Start of Year	Payment 1	Payment 2	Payment 3	Payment 4
1	1000			
2	⋮	1000		
3	⋮	⋮	1000	
4	⋮	⋮	⋮	1000
5	1125.51	1092.73	1060.90	

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2	⋮	1000		
3	⋮	⋮	1000	
4	⋮	⋮	⋮	1000
5	1125.51	1092.73	1060.90	1030.00

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Start of Year	Payment 1	Payment 2	Payment 3	Payment 4
1	1000			
2	⋮	1000		
3	⋮	⋮	1000	
4	⋮	⋮	⋮	1000
5	1125.51	1092.73	1060.90	1030.00

The total at the end of year 4 is $4309.14.

Deferred Annuities

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Start of Year	Payment 1	Payment 2	Payment 3	Payment 4
1	1000			
2	⋮	1000		
3	⋮	⋮	1000	
4	⋮	⋮	⋮	1000
5	1125.51	1092.73	1060.90	1030.00

The total at the end of year 4 is $4309.14.

Spreadsheets are great this!

Deferred Annuities

Bigger Example: A newborn's parents set up a college fund. They plan to invest \$100 each month. If the fund pays 6% annual interest, compounded monthly, what is the future value of the fund in 18 years?

Notice that each monthly installment has a different "lifespan":

- The first installment will generate interest for all $18 \cdot 12 = 216$ months.
- The second installment will generate interest for 215 months.

  $\vdots$
- The last installment will generate interest for only one month.

Deferred Annuities: Example

The first installment will generate interest for all $18 \cdot 12 = 216$ months. Using the compound interest formula, after 18 years the installment is worth:

$$100\left(1+\frac{.06}{12}\right)^{18\cdot 12} = 100(1.005)^{216}.$$

The future value of the second installment is:

$$100\left(1+\frac{.06}{12}\right)^{215} = 100(1.005)^{215}.$$

$$\vdots$$

The future value of the final installment is:

$$100(1.005)^{1}.$$

Deferred Annuities: Example

The total future value is the sum of all of these future values:

$$\begin{aligned} F &= 100(1.005) + 100(1.005)^2 + \cdots + 100(1.005)^{215} + 100(1.005)^{216} \\ &= 100(1.005)\left[1 + 1.005 + \cdots 1.005^{214} + 1.005^{215}\right]. \end{aligned}$$

Deferred Annuities: Example

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What do we do with the long sum (let's call it S) in square brackets? Nice idea: Multiply it by 1.005 and subract the result from S! Lots of cancellation!

Deferred Annuities: Example

$$S = 1 + 1.005 + \cdots 1.005^{214} + 1.005^{215}$$

Deferred Annuities: Example

$$\begin{aligned} S &= 1 + 1.005 + \cdots 1.005^{214} + 1.005^{215} \\ (1.005)S &= 1.005 + 1.005^2 + \cdots 1.005^{215} + 1.005^{216} \end{aligned}$$

Deferred Annuities: Example

$$\begin{aligned} S &= 1 + 1.005 + \cdots 1.005^{214} + 1.005^{215} \\ (1.005)S &= 1.005 + 1.005^2 + \cdots 1.005^{215} + 1.005^{216} \\ 1.005S - S &= 1.005^{216} - 1 \end{aligned}$$

Deferred Annuities: Example

$$
\begin{aligned}
S &= 1 + 1.005 + \cdots 1.005^{214} + 1.005^{215} \\
(1.005)S &= 1.005 + 1.005^{2} + \cdots 1.005^{215} + 1.005^{216} \\
1.005S - S &= 1.005^{216} - 1 \\
S(1.005 - 1) &= 1.005^{216} - 1
\end{aligned}
$$

Deferred Annuities: Example

$$
\begin{aligned}
S &= 1 + 1.005 + \cdots 1.005^{214} + 1.005^{215} \\
(1.005)S &= 1.005 + 1.005^{2} + \cdots 1.005^{215} + 1.005^{216} \\
1.005S - S &= 1.005^{216} - 1 \\
S(1.005 - 1) &= 1.005^{216} - 1 \\
S &= \frac{1.005^{216} - 1}{1.005 - 1} = \frac{1.005^{216} - 1}{0.005}
\end{aligned}
$$

Deferred Annuities: Example

$$\begin{aligned} S &= 1 + 1.005 + \cdots 1.005^{214} + 1.005^{215} \\ (1.005)S &= 1.005 + 1.005^2 + \cdots 1.005^{215} + 1.005^{216} \\ 1.005S - S &= 1.005^{216} - 1 \\ S(1.005 - 1) &= 1.005^{216} - 1 \\ S &= \frac{1.005^{216}-1}{1.005-1} = \frac{1.005^{216}-1}{0.005} \end{aligned}$$

So

$S = [1 + 1.005 + \cdots 1.005^{214} + 1.005^{215}] = \frac{1.005^{216}-1}{0.005}$

Now, to finish up finding F:

$$F = 100(1.005) \times \left[\frac{1.005^{216} - 1}{0.005}\right] = \$38,929.00$$

The Fixed Deferred Annuity Formula

The future value F of a fixed deferred annuity consisting of T payments of \$$P$ each, having a periodic interest rate p (in decimal form) is:

$$\boxed{F = L\left(\frac{(1+p)^T - 1}{p}\right)}$$

where L denotes the future value of the last payment.

Note that the periodic interest rate $p = \frac{r}{n}$ where r is the APR in decimal form and the interest is compounded n times per year. *Pay attention to what quantities the various variables stand for!*

Example

Page 395, #63:
Starting at age 25, Markus invests $2000 at the beginning of each year in an IRA (individual retirement account) with an APR of 7.5% compounded annually. How much money will there be in Markus's retirement account when he retires at age 65?

Example

Page 395, #63:
Starting at age 25, Markus invests \$2000 at the beginning of each year in an IRA (individual retirement account) with an APR of 7.5% compounded annually. How much money will there be in Markus's retirement account when he retires at age 65?

Notice that the periodic interest rate p is $p = 0.075$ and $T = 65 - 25 = 40$.
The future value of the last payment is $L = 2000(1.075)$ because the final payment will accumulate interest for one year.

Example

So the future value is:

$$
\begin{aligned}
F &= 2000(1.075)\left(\frac{1.075^{40}-1}{0.075}\right) \\
&= 2000(1.075)\left(\frac{18.044239-1}{0.075}\right) \\
&= 2000(1.075)\left(\frac{17.044239}{0.075}\right) \\
&= 2000(1.075)(227.25652) \\
&= 488601.52.
\end{aligned}
$$

Markus will have $488,601.52 in his account when he retires.

Example

Same example as before, except that Markus invests the money at the *end* of each year, *after* the interest for that year has been added to the account.

Example

This time, $L = 2000$. So the future value is:

$$\begin{aligned} F &= 2000\left(\frac{1.075^{40}-1}{0.075}\right) \\ &= 2000\left(\frac{18.044239-1}{0.075}\right) \\ &= 2000\left(\frac{17.044239}{0.075}\right) \\ &= 2000(227.25652) \\ &= 454513.04. \end{aligned}$$

Markus will have $454,513.04 in his account when he retires.

Example

Suppose you want to set up an 18-year annuity at an APR of 6% compounded monthly, if your goal is to have $50,000 at the end of 18 years. How much should the monthly payments be?

Example

Suppose you want to set up an 18-year annuity at an APR of 6% compounded monthly, if your goal is to have $50,000 at the end of 18 years. How much should the monthly payments be?

Remember

$$F = L\left(\frac{(1+p)^T - 1}{p}\right).$$

We know $F = 50,000$, $p = \frac{.06}{12} = 0.005$, and $T = 18 \times 12 = 216$. Let P be the unknown monthly payment. Then we know $L = P(1.005)$.

Example

Substitute:

$$50,000 = P(1.005)\left(\frac{(1.005)^{216}-1}{0.005}\right) = P(389.29).$$

So

$$P = \frac{50,000}{389.29} = \$128.44.$$

Example

Suppose you want to have \$2000 at the end of 7.5 years. You already have \$875 in the bank, invested at a 6.75% APR compounded monthly. You want to put more money each month into the bank to end up with the \$2000 goal. What should your monthly deposit be?

Example

Suppose you want to have \$2000 at the end of 7.5 years. You already have \$875 in the bank, invested at a 6.75% APR compounded monthly. You want to put more money each month into the bank to end up with the \$2000 goal. What should your monthly deposit be?

First, the \$875 in the bank will grow to $875(1 + \frac{0.0675}{12})^{7.5(12)} = \$1449.62$. So you only need $\$2000 - 1449.62 = \$550.38$ more.

Example

So we have $F = 550.38$, $p = \frac{0.0675}{12} = 0.005625$, $T = 7.5(12) = 90$, and $L = P(1.005625)$. Substituting,

$$550.38 = P(1.005625)\left(\frac{(1.005625)^{90} - 1}{0.005625}\right) = P(117.404).$$

Therefore

$$P = \frac{550.38}{117.404} = \$4.69.$$

10.6 Installment Loans

Installment Loans

Small Example. You get take out a loan at the beginning of year 1 at 3% interest compounded annually. You pay back the loan by making payments of $1000 at the end of the next four years (including year 1). How much did you borrow?

	Payment 1	Payment 2	Payment 3	Payment 4
Loan				
End of Year 1	1000	⋮	⋮	⋮
2		1000	⋮	⋮
3			1000	⋮
4				1000

The amount of the loan will be present value of each of the future payments.

Installment Loans

	Payment 1	Payment 2	Payment 3	Payment 4
Present Value				
End of Year 1	1000	⋮	⋮	⋮
2		1000	⋮	⋮
3			1000	⋮
4				1000

Installment Loans

	Payment 1	Payment 2	Payment 3	Payment 4
Present Value	970.87			
End of Year 1	1000	⋮	⋮	⋮
2		1000	⋮	⋮
3			1000	⋮
4				1000

Installment Loans

	Payment 1	Payment 2	Payment 3	Payment 4
Present Value	970.87	942.60		
End of Year 1	1000	⋮	⋮	⋮
2		1000	⋮	⋮
3			1000	⋮
4				1000

Installment Loans

	Payment 1	Payment 2	Payment 3	Payment 4
Present Value	970.87	942.60	915.14	
End of Year 1	1000	⋮	⋮	⋮
2		1000	⋮	⋮
3			1000	⋮
4				1000

Installment Loans

	Payment 1	Payment 2	Payment 3	Payment 4
Present Value	970.87	942.60	915.14	888.49
End of Year 1	1000	⋮	⋮	⋮
2		1000	⋮	⋮
3			1000	⋮
4				1000

Installment Loans

	Payment 1	Payment 2	Payment 3	Payment 4
Present Value	970.87	942.60	915.14	888.49
End of Year 1	1000	⋮	⋮	⋮
2		1000	⋮	⋮
3			1000	⋮
4				1000

So the amount of the loan must have been $3717.10. Note that you paid back $4000, so you paid $282.90 in interest.

Installment Loans

We already know that the future value F of a payment P made today is

$$F = P(1+p)^T,$$

where p is the periodic interest rate $p = \frac{r}{n}$, and T is the total number of time periods.

Installment Loans

We already know that the future value F of a payment P made today is

$$F = P(1+p)^T,$$

where p is the periodic interest rate $p = \frac{r}{n}$, and T is the total number of time periods.

For example, if we have an annual interest rate of 6% compounded monthly, $200 today is worth $200(1+\frac{.06}{12})^{36} = 200(1.005)^{36} = \$239.34$ in 36 months.

Installment Loans

Thinking backwards in time and solving for P, we also know that if we want a future value of F in T months, then we must invest

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today.

Installment Loans

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For example, if we have an annual interest rate of 6% compounded monthly, \$300 in 36 months is worth $\frac{300}{(1.005)^{36}} = \$250.69$ today.

Installment Loans

This is how we can figure out the payments to pay back an installment loan. You will receive a certain loan amount today (in the present), and make periodic payments on into the future. The *present* values of all of these future payments must add up to the *present* amount of the loan.

Installment Loans

Example. You receive a loan of $25,000, at an annual interest rate of 6% compounded monthly. You will repay the loan by making monthly payments over 36 months. How much will each payment be? Here, $p = \frac{.06}{12} = 0.005$.

Installment Loans

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Since the payments occur in the future, we denote the amount of each payment by F.
Present value of payment 1: $\frac{F}{(1.005)^1}$,
Present value of payment 2: $\frac{F}{(1.005)^2}$,
Present value of payment 3: $\frac{F}{(1.005)^3}, \ldots$
Present value of payment 36: $\frac{F}{(1.005)^{36}}$.

Installment Loans

These must all sum up to the present value of the loan, $25,000.

$$25,000 = \frac{F}{(1.005)^1} + \frac{F}{(1.005)^2} + \frac{F}{(1.005)^3} + \cdots + \frac{F}{(1.005)^{36}}$$

$$25,000 = \frac{F}{(1.005)} \left[1 + \frac{1}{(1.005)^1} + \frac{1}{(1.005)^2} + \cdots + \frac{1}{(1.005)^{35}} \right]$$

Installment Loans

We can use a method similar to the one in the previous section to determine a formula for the sum (but this time I am skipping the details).

$$S = 1 + \frac{1}{(1.005)^1} + \frac{1}{(1.005)^2} + \cdots + \frac{1}{(1.005)^{35}}$$

$$= \frac{\left(\frac{1}{(1.005)^{36}-1}\right)}{\left(\frac{1}{1.005} - 1\right)}.$$

This equals 33.035371.

Installment Loans

Finishing up by solving for F:

$$25,000 = \frac{F}{1.005} \times 33.035371 = F \times 32.87101.$$

$$F = \frac{25,000}{32.87101} = \$760.55.$$

So you will pay \$760.55 each month. Over 36 months, this amounts to a total of \$27,379.80.

Installment Loans

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So you will pay \$760.55 each month. Over 36 months, this amounts to a total of \$27,379.80.
(Remember that you borrowed \$25,000.)

Amortization Formula

This procedure leads to the general Amortization Formula: If an installment loan of P dollars is paid off in T payments of F dollars at a periodic interest of p (written in decimal form), then

$$\boxed{P = Fq\left[\frac{q^T - 1}{q - 1}\right]}$$

where $\boxed{q = \frac{1}{1+p}}$.

Example: Car Loan

You want to buy a car for \$23,995 for which you have \$5000 for a down payment, and the dealer offers you two choices:

1. Cash rebate of \$2000, and financing for 6.48% annual interest for 60 months.
2. Financing for 0% APR for 60 months.

Which is better?

Example: Car Loan

First option: Finance $P = \$16,995$. $p = \frac{0.0648}{12} = 0.0054$.

$$16,995 = \left(\frac{F}{1.0054}\right)\left[\frac{\left(\frac{1}{1.0054}\right)^{60} - 1}{\left(\frac{1}{1.0054}\right) - 1}\right].$$

Solve for F to get $F = \$332.37$ for your monthly payment.

Example: Car Loan

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Solve for F to get $F = \$332.37$ for your monthly payment.
Second option: Finance $P = \$18,995$, with monthly payment $\frac{18,995}{60} = \$316.59$.
So the second option has a lower monthly payment.

Example: Lottery

Suppose you win \$9 million in the lottery, and that after taxes this amounts to \$6.8 million. You are offered two choices:

1. Annuity option: Receive 25 annual installments of \$272,000 per year.
2. Lump sum option: Receive an immediate lump sum of \$3.75 million.

Which is better?

Example: Lottery

We can calculate the present value P of this sequence of payments, and compare to the value of the lump sum. We should pick an interest rate that is reasonable for the current market.

If we try 5%, then $p = \frac{0.05}{1} = 0.05$ and

$$P = 272,000 \left[\frac{\left(\frac{1}{1.05}\right)^{25} - 1}{\left(\frac{1}{1.05}\right) - 1} \right] = \$4,025,230.$$

Note that the first payment comes immediately, so we do not have to divide F by 1.05.

Example: Lottery

If we try 6%, then $p = \frac{0.06}{1} = 0.06$ and

$$P = 272,000 \left[\frac{\left(\frac{1}{1.06}\right)^{25} - 1}{\left(\frac{1}{1.06}\right) - 1} \right] = \$3,685,700.$$

So if we are conservative about interest rates, the annuity option appears better, but if we are less conservative about interest rates, the lump sum option appears better.

Example: Mortgage

This problem is more complicated, but more realistic!
You take out a mortgage on your home, borrowing $180,000 for 30 years at an annual rate of 6.75% with monthly payments.

1. What is your monthly payment?
2. What is the balance on your mortgage after you have made 30 payments?
3. How much interest will you pay over the life of the loan?

Example: Mortgage

Note that $p = \frac{0.0675}{12} = 0.005625$.

$$180,000 = \left(\frac{F}{1.005625}\right)\left[\frac{\left(\frac{1}{1.005625}\right)^{360} - 1}{\left(\frac{1}{1.005625}\right) - 1}\right].$$

Solve for F to get $F = \$1167.48$.

Example: Mortgage

For the second question, calculate the present value of the loan when only 330 payments remain:

$$P = \left(\frac{1167.48}{1.005625}\right)\left[\frac{\left(\frac{1}{1.005625}\right)^{330} - 1}{\left(\frac{1}{1.005625}\right) - 1}\right] = \$174,951.$$

So even though you have made 30 payments of \$1167.48, you have only reduced your loan by \$5049!

Example: Mortgage

For the second question, calculate the present value of the loan when only 330 payments remain:

$$P = \left(\frac{1167.48}{1.005625}\right)\left[\frac{\left(\frac{1}{1.005625}\right)^{330} - 1}{\left(\frac{1}{1.005625}\right) - 1}\right] = \$174,951.$$

So even though you have made 30 payments of \$1167.48, you have only reduced your loan by \$5049!
Over the life of the loan you will make 360 payments of \$1167.48, for a total of \$420,293, so your total interest will be \$420,293-180,000=\$240,293.